METRIC GEOMETRY: CONNECTIONS WITH COMBINATORICS

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ABSTRACT. First introduced by Alexsandrov in the 1950's, metric geometry allows one to apply techniques of differential geometry to a very general class of metric spaces. Recent work of Gromov explores the notion of "nonpositive curvature" in metric geometry. We discuss these ideas and show how they connect to problems in combinatorics involving the cd-index, the lower bound theorem, and arrangements of hyperplanes.

Metric geometry dates back to the work of Alexsandrov in the 1950's. Recent work of Gromov ([10], [11]) applying techniques of metric geometry to problems in infinite group theory have led to an explosion of new activity in this area with applications to topology, group theory, and combinatorics. In this talk I will introduce the fundamental ideas of metric geometry and discuss connections with several problems in combinatorics.

In differential geometry, the fundamental objects of study are smooth manifolds with Riemannian metrics. Metric geometry, while borrowing many ideas from differential geometry, applies to a much more general class of metric spaces, known as "geodesic metric spaces". A *geodesic segment* in a metric space X is an isometric embedding of an interval into X. A metric space is a *geodesic space* if any two points are connected by a geodesic segment. The fundamental tool in differential geometry is the notion of curvature. It is possible to characterize (sectional) curvature in a Riemannian manifold by the shape of its triangles. (Vaguely, the more positive the curvature, the "fatter" the triangles, the more negative the curvature, the "thinner" the triangles.) Using this idea, it is possible to define a notion of curvature in the context of geodesic metric spaces: a geodesic space X has ***curvature*** $\leq c$ if, locally, triangles in X are "at least as thin" as those in a Riemannian manifold of constant curvature c.

It turns out that this notion of curvature, though seemingly very general and very simple, is surprisingly powerful, especially in the case of spaces of nonpositive curvature. For example, any geodesic metric space X with curvature ≤ 0 is aspherical (that is, its universal covering space is contractible), and if X is compact, its fundamental group satisfies a host of interesting group theoretic properties (see [10], [9], [1]).

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What has this got to do with combinatorics? The connection begins when we ask how we can find metrics of nonpositive curvature. One way to do this is to begin with an abstract cubical complex X (the cubical analogue of a simplicial complex) and assign to each abstract n-cube the metric of a regular Euclidean n-cube. This induces a geodesic metric on X. While each cube clearly has curvature ≤ 0, the "thin triangle" condition may fail for this metric at a point where several cubes come together. It turns out that any such failure can be detected in the combinatorics of the links of vertices in X. (Recall that the link of a vertex v is the simplicial complex L_v formed by the poset of faces of X containing v.) We say that a simplicial complex is a *flag complex* if any collection of vertices which are pairwise joined by edges, span a simplex.

Theorem (Gromov [10]). *A cubical metric on X has nonpositive curvature if and only if L_v is a flag complex for every vertex v in X.*

If we construct our space X out of Euclidean simplicies (of varying shapes) instead of Euclidean cubes, we still have a link condition to determine whether the induced geodesic metric on X has nonpositive curvature, but the condition is not purely combinatorial. The links, in this case, come equipped with a natural piecewise spherical geodesic metric. We say that L_v is *large* if any two points in L_v of distance $\leq \pi$ are connected by a *unique* geodesic segment.

Theorem (Gromov [10], Bridson [3]). *A piecewise Euclidean metric on X has nonpositive curvature if and only if L_v is large for every vertex v.*

I will discuss three combinatorial problems that arise in connection with these link conditions. What follows is joint work with Michael Davis.

<u>The Hopf Conjecture.</u> An old conjecture of H. Hopf states that if M is a $2n$-dimensional manifold of nonpositive curvature, then it's Euler characteristic, $\chi(M)$, should satisfy

$$(-1)^n \chi(M) \geq 0.$$

The conjecture was originally stated for Riemannian manifolds, but makes sense in our more general context. Suppose the metric on M is the geodesic metric induced by a cubation of M as described above. In this case, the hypothesis that M has nonpositive curvature is equivalent to the condition that links of vertices in M are flag complexes. Since M is a manifold, these links are triangulations of the $(2n-1)$-sphere. On the other hand, one can break up $\chi(M)$ into a sum of local contributions at the vertices,

$$\chi(M) = \sum_v \kappa(L_v)$$

where

$$\kappa(L_v) = 1 - \tfrac{1}{2} f_0 + \tfrac{1}{4} f_1 - \cdots + (\tfrac{-1}{2})^{2n} f_{2n-1}$$

$$f_i = \text{number of } i\text{-simplicies in } L_v.$$

Thus we obtain a "local Hopf conjecture".

Conjecture [5]. *If S^{2n-1} is triangulated as a flag complex, and f_i is the number of i-simplicies in the triangulation, then*

$$(-1)(1 - \tfrac{1}{2}f_0 + \tfrac{1}{4}f_1 - \cdots + (\tfrac{-1}{2})^{2n} f_{2n-1}) \geq 0$$

Relations between the f_i's have been studied by Bayer and Billera, Stanley, and others by means of the *cd-index.* (See survey article [13].) It follows from a theorem of Stanley, that the local Hopf conjecture is true for the face-lattice of a convex polytope (which is always a flag complex). The general conjecture is still open.

The Lower Bound Theorem. Let f_i be the number of i-dimensional faces of a simple, n-dimensional polytope X. The lower bound theorem, proved by Barnette in [2], gives lower bounds on the f_i's in terms of f_{n-1}. For example,

$$(*) \qquad f_{n-2} \geq n f_{n-1} - \binom{n+1}{n-1}.$$

Metric geometry can be used to give a simple proof of this inequality. In [6] we show that any hyperbolic structure on X (i.e. a realization of X as a convex polytope in $\mathbb{H}^n$) gives rise to a large, piecewise spherical structure on the dual simplicial complex, X^*. This gives an embedding of the space of hyperbolic structures on X into the space of piecewise spherical structures on X^*. The dimensions of these spaces are easily computable and give the inequality (*) above. It is likely that other such inequalities can be obtained by similar means.

Arrangements of Hyperplanes. There are many interesting combinatorial and topological questions concerning arrangements of affine hyperplanes in a vector space (see [12]). One question which has been widely studied is when the space Y obtained by removing a collection of complex hyperplanes from $\mathbb{C}^n$ is aspherical. (This is known as the $K(\pi, 1)$-problem.) It was proved by Deligne [8] and Brieskorn [4] in the 1970's that for arrangements of hyperplanes arising as fixed planes of a finite reflection group acting on $\mathbb{R}^n$ (complexified), this is always the case. It was conjectured that the same should hold for infinite reflection groups. In [7], we prove that this conjecture holds for a large class of infinite reflection groups. This is accomplished by finding a cubical complex X, homotopy equivalent to a covering space of the hyperplane complement Y, and proving that all links in X are flag complexes. It follows that the cubical metric on X has nonpositive curvature and hence, X is aspherical.

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