Rook Placements and Partition Varieties $P_\gamma \backslash M_\lambda$

Kequan Ding [1]
Institute for Advanced Study, Princeton, NJ 08540
DIMACS, Rutgers University, New Brunswick, NJ 08855
(ABSTRACT)

1. Introduction

Let M be the set of all m by n matrices of rank m over the complex field. Consider a partition $\lambda = (\lambda_1, \cdots, \lambda_m)$. Let F_λ be the right justified Ferrers board corresponding to λ as in [2], i.e., there are λ_i positions in the i-th row. Define $M_\lambda := \{a \in M \mid a_{i,j} = 0, (i,j) \notin F_\lambda\}$. An r-rook placement on F_λ is a placement of r non-attacking rooks on F_λ. Let R_λ^r be the set of all r-rook placements on F_λ. In the last thirty years, there have been many interesting progresses on rook placements. For example, see Foata and Schützenberger [4], Gouldman, Joichi, Reiner and White [6, 7, 8, 9, 10], Gould [11], Wachs and White [19], Garsia and Remmel [5], Sagan [17]. But the geometric aspect of the rook placements was only explored until recently [2]. In [1], we introduced the idea of invisible permutations, a *length* function for rook placements on a Ferrers board and rook length polynomial. The major result in [1] is an explicit formula for rook length polynomials. As a consequence, we also obtained an explicit formula for Garsia-Remmel polynomials. In [2], we introduced the notion of a *partition variety* as a quotient space $B \backslash M_\lambda$ where B is the Borel subgroup of upper triangular matrices of $GL_m(\mathbb{C})$. We proved that $B \backslash M_\lambda$ is a projective subvariety which has the structure of a CW-complex. The Poincaré polynomials for cohomology of partition varieties with coefficients in real field $\mathbb{R}$ are proved to be rook length polynomials as introduced in [1] with $r = m$.

Definition 0.1 *A composition of m is a tuple $(m_1, \cdots, m_s)$ of positive integers such that $\sum_{i=1}^{s} m_i = m$. Let $\gamma = (\gamma_1, \cdots, \gamma_k)$ be a composition of some*

[1] The author wish to thank Professor Louis Solomon for his encouragements during this research. This research is partially supported by Alfred P. Sloan Foundation Grant 93-6-6 and NSF Grant STC 91-19999.

positive integer. Let P_γ denote the parabolic subgroup of G_m of the shape

$$P_\gamma = \begin{pmatrix} G_{\gamma_1} & * & \cdots & * \\ 0 & G_{\gamma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & G_{\gamma_k} \end{pmatrix}$$

where $G_{\gamma_i} = GL_{\gamma_i}(\mathbf{C})$, and the $*$'s are arbitrary matrices of the appropriate sizes.

In this paper we continue our study on the connection between the combinatorics of rook placements and the geometry of partition varieties in [2]. Here we consider the partition varieties in their most general setting $P_\gamma \backslash M_\lambda$ where P_γ is a parabolic subgroup. Our main results are Theorem 0.18, 0.19 and 0.20. Here, we have a unified treatment to the homology cohomology for both the flag manifold and the Grassmann manifold.

2. Preliminaries

Let $[n]$ denote the set $\{1, \cdots, n\}$. Sometimes, we write a partition λ as $\lambda = (1^{\mu_1} 2^{\mu_2} \cdots n^{\mu_n})$ where μ_i is the number of λ_j's which are equal to i. View a Ferrers board F_λ of shape λ as a subarray of an m by n matrix, where $n = \lambda_1$ and the k-th row has length λ_k for $1 \le k \le m$. If $r \ge 1$ let $\nu_r = \sum_{i=1}^{r} E_{i,n-r+1}$. where $E_{i,j}$ is the matrix with 1 at (i,j) and 0's elsewhere. Let W_n be the symmetric group on $[n] = \{1, \cdots, n\}$. Let $S(n)$ be the set of distinguished generators of W_n: $S(n) = \{(12), (23), \cdots, (n-1\ n)\}$.

Definition 0.2 *Let λ be a partition such that $\nu_r \in F_\lambda$. When $r = 0$, let $\nu_0 = 0$. For $\sigma \in R_\lambda^r$, the length function $l(\sigma)$ is defined by*

$$l(\sigma) = min\{k + h \mid \sigma = s_k \cdots s_1 \nu_r s_1' \cdots s_h'\}$$

where $s_i \in S(m)$ and $s_j' \in S(n)$ and

$$s_p \cdots s_1\, \nu_r\, s_1' \cdots s_q' \in R_\lambda^r$$

for each $1 \le p \le k$ and $1 \le q \le h$.

Thus $l(\sigma)$ is the minimum number of adjacent row and/or column transpositions required to get σ from $\nu_r = \sum_{i=1}^{r} E_{i,n-r+i}$. This length function was first used as the length of rook matrices by Solomon [18].

Proposition 0.3 (Local Formula) *Let $\sigma \in R_\lambda^r$. Write $\sigma = \sum_{i=1}^{r} E_{c_i,b_i}$ where $c_1 < c_2 < \cdots < c_r$. Let α_i be the number of zero rows above the c_i-th row in σ, γ_i the number of zero columns to the right of the b_i-th column in σ, and β_i the number of 1's to the "northeast" of the i-th 1 (not including the i-th 1). Then,*

$$l(\sigma) = \sum_{i=1}^{r} (\alpha_i + \beta_i + \gamma_i). \qquad (1)$$

Definition 0.4 *Let*

$$RL_r(\lambda, q) = \sum_{\sigma \in R_\lambda^r} q^{l(\sigma)}.$$

We call this a rook length polynomial (see [1] for the formula).

One of the main ideas in [1] is to extend a placement σ of r rooks on an m by n board to a placement $P(\sigma)$ of $m+n-r$ rooks on an $m+n-r$ by $m+n-r$ board. We identify $P(\sigma)$ with the corresponding permutation of $[m+n-r]$ and call $P(\sigma)$ the *invisible permutation* corresponding to σ.

Definition 0.5 *Let $\sigma \in R_{nm}^r$. Let $\sigma = \sum_{i=1}^{r} E_{c_i,b_i}$ with $c_1 < c_2 < \cdots < c_r$. Define a permutation $P(\sigma) \in W_{m+n-r}$ by*

$$P(\sigma) = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & n-r & n-r+c_1 & \cdots & n-r+c_r & d_1 & \cdots & d_{m-r} \\ a_1 & \cdots & a_{n-r} & b_1 & \cdots & b_r & n+1 & \cdots & n+m-r \end{array} \right)$$

where $\{a_1, a_2, \ldots, a_{n-r}\}$ is the complement of $\{b_1, b_2, \ldots, b_r\}$ in $[n]$ with $a_1 < a_2 < \cdots < a_{n-r}$, and $\{d_1, d_2, \ldots, d_{m-r}\}$ is the complement of $\{n-r+c_1, \cdots, n-r+c_r\}$ in $\{n-r+1, \ldots, n-r+m\}$ with $d_1 < d_2 < \cdots < d_{m-r}$.

Definition 0.6 (Projective Variety) *A subset of the projective space $\mathbf{P}^n$ is called a projective variety if it is the set of common zeros of a collection of homogeneous polynomials. These homogeneous polynomials are called the defining polynomials of the projective variety.*

If X is a topological space and A is a subset of X then $\overline{A}$ denotes the closure of A.

Definition 0.7 (Finite CW-complex,[15]) *A Hausdorff space X is called a finite CW-complex if X has a partition $X = \bigsqcup_{i \in I} e_i$ into disjoint subsets $\{e_i\}_{i \in I}$ satisfying the following conditions:*

- *The index set I is finite.*
- *Each e_i (called a cell) is homeomorphic to an open ball in some $\mathbf{C}^n$.*
- *If $x \in \overline{e_i} - e_i$, then there is some e_j of lower dimension such that $x \in e_j$.*

Theorem 0.8 (Chow's Theorem) *If $X \subseteq \mathbf{P}^n$ is an analytic subvariety then X is an projective subvariety.*

Recall that M_λ is a subset of the affine space M. We give it the subspace topology. In the next section we introduce the notion of γ-compatible partition λ such that $P_\lambda M_\lambda \subseteq M_\lambda$. Thus we may consider the geometry in the quotient space $P_\lambda \backslash M_\lambda$. We give $P_\lambda \backslash M_\lambda$ the quotient topology. In this paper we use some known results on Grassmannians. There is an exposition in [3] on this which is a complex version of that in Milnor and Stasheff [15].

3. γ-Compatible Partitions and Partition Variety $P_\gamma \backslash M_\lambda$

Definition 0.9 *Let $\gamma = (\gamma_1, \cdots, \gamma_t)$ be a composition of m. A partition $\lambda = (\lambda_1, \cdots, \lambda_m)$ is called a γ-compatible partition if $\lambda = (k_1^{\gamma_1}, \cdots, k_t^{\gamma_t})$ where $k_1 \geq k_2 \geq \cdots \geq k_t > 0$. (In this paper we use γ-compatible partitions only).*

In the Ferrers board F_λ, we call the rows corresponding to $k_i^{\gamma_i}$ the i-th γ-block.

Definition 0.10 *A rook placement σ is said to be γ-compatible if $\sigma(i) < \sigma(i+1)$ whenever $\sum_{j=1}^{j'} \gamma_j < i < i+1 \leq \sum_{j=1}^{j'+1} \gamma_j$, for some j'. Let $R_\lambda^r(\gamma)$ denote the set of all γ-compatible rook placements on the board F_λ with r rooks, i.e., in every block of the Ferrers board, the column indices of rooks increase as the row indices increase.*

Definition 0.11 *Let λ be a γ-compatible partition. Define the γ-rook length polynomial*

$$RL_r(\lambda,\gamma,q) := \sum_{\sigma\in R^r_\lambda(\gamma)} q^{l(\sigma)}. \tag{2}$$

When $\gamma = (1^m)$, this is the rook length polynomial in Definition 0.4. By the Local Formula of the length function and induction, we have

Theorem 0.12 *Let λ be a γ-compatible partition. Then*

$$RL_m(\lambda,\gamma,q) = \prod_{i=1}^{t} \begin{bmatrix} k_i - \gamma_{i+1} - \gamma_{i+2} - \cdots - \gamma_t \\ \gamma_i \end{bmatrix}_q \tag{3}$$

where $\left[{s \atop t}\right]_q = \frac{(s)_q!}{(t)_q!(s-t)_q!}$ *and* $(s)_q! = (s)_q(s-1)_q\cdots(1)_q$.

Corollary 0.13 *If $t = 1$, then $\lambda_i = n$ for each i and*

$$RL_m(\lambda,\gamma,q) = \begin{bmatrix} n \\ m \end{bmatrix}_q. \tag{4}$$

Corollary 0.14 *If $\gamma_i = 1$ for all i, then every partition is γ-compatible.*

$$RL_m(\lambda,\gamma,q) = RL_m(\lambda,q) \tag{5}$$

$$= \prod_{i=1}^{m}(\lambda_i - m + i)_q \tag{6}$$

Using Plücker coordinates and Chow's Theorem, we get

Theorem 0.15 *$P_\gamma\backslash M_\lambda$ is a projective variety if $\lambda_i \geq m - i + 1$, $1 \leq i \leq m$ and λ is γ-compatible.*

4. Cellular Decomposition of $P_\gamma\backslash M_\lambda$

In this section we give a cellular decomposition of the partition variety $P_\gamma\backslash M_\lambda$ and the topological implication of this cellular decomposition. Let $I(m,n) = R^m_{(n^m)}$. To state our results we need to construct a transversal for $P_\gamma\backslash M$ of the form $\bigsqcup_{\sigma\in I(m,n)} Y_\gamma(\sigma)$ where the sets $Y_\gamma(\sigma)$ are defined as follows: Let

$$H(\sigma) = H_1(\sigma) \cup H_2(\sigma) \tag{7}$$

where

$$
\begin{aligned}
H_1(\sigma) &= \{(i,j)\,|\,\sigma(i) < j, \text{ for } j \notin J(\sigma)\} && (8)\\
H_2(\sigma) &= \{(i,j)\,|\,\sigma(i) < j, \text{ for } j \in J(\sigma) \text{ and } \sigma^{-1}(j) < i\} && (9)\\
J(\sigma) &= \{\sigma(i)\,|\,1 \le i \le m\} && (10)
\end{aligned}
$$

Define

$$Y(\sigma) = \sigma + \sum_{(i,j)\in H(\sigma)} CE_{i,j}. \qquad (11)$$

Lemma 0.16 *$Y_\gamma(\sigma)$ is an affine space of dimension $l(\sigma) = |H(\sigma)|$.*

For $\sigma \in I(m,n)$ define

$$X_\gamma(\sigma) = \{P_\gamma\, y \,|\, y \in Y_\gamma(\sigma)\} \subseteq P_\gamma \backslash M. \qquad (12)$$

The space $X_\gamma(\sigma)$ has the subspace topology from $P_\gamma\backslash M$ and $Y(\sigma)$ has the subspace topology from M. There is a natural homeomorphism from $X_\gamma(\sigma)$ onto $Y_\gamma(\sigma)$ defined by $P_\gamma\, y \to y$. By row eliminations according to a parabolic matrix, we have

Theorem 0.17 (Cellular Decomposition of $P_\gamma\backslash M_\lambda$)
Let λ be a γ-compatible partition. Then the partition variety $P_\gamma\backslash M_\lambda$ has the following cellular decomposition:

$$P_\gamma\backslash M_\lambda = \bigsqcup_{\sigma\in R^m_\lambda(\gamma)} X_\gamma(\sigma)$$

where $\bigsqcup$ denotes the disjoint union.

By the Ferrers board characterization of Schubert cells in Grassmannians, we have

Theorem 0.18 (Ehresmann Argument) *a) There is a one to one correspondence Γ between the set $\mathcal{C}_\gamma = \{\overline{X_\gamma(\sigma)} \mid \sigma \in R^m_{nm}(\gamma)\}$ and the set of sequences of Ferrers boards $\mathcal{B}_\gamma = \{F^{(\mu_t)} \subset \cdots \subset F^{(\mu_1)}\}$ where $F^{(\mu_k)}$ is a Ferrers board of μ_k rows which is obtained by deleting γ_{k-1} rows from $F^{\mu_{(k-1)}}$ for $2 \le k \le t$. The correspondence Γ is defined by*

$$\Gamma(\overline{X_\gamma(\sigma)}) = (F_{\lambda(\sigma_{\mu_t})}, \cdots, F_{\lambda(\sigma_{\mu_1})}),$$

where $\sigma_k = \sum_{i=m-k+1}^{m} E_{i,\sigma(i)}$ *and* $\lambda(\sigma_k)$ *is a partition obtained by rearranging the numbers in* $\{n - \sigma(i) + 1\}_{i=m-k+1}^{m}$ *into non-increasing order.*

b) Let $\sigma \in R_{n^m}^{m}(\gamma)$. *Then* $X(\sigma) \subseteq \overline{X(\sigma')}$ *if and only if* $F_{\lambda(\sigma_{\mu_i})} \subseteq F_{\lambda(\sigma'_{\mu_i})}$, *for* $1 \leq i \leq t$.

Theorem 0.19 *For any given composition* γ *of* m, *let* λ *be a* γ*-compatible partition. For* σ *and* $\sigma' \in R_{\lambda}^{m}(\gamma)$, *let* $P(\sigma)$ *and* $P(\sigma')$ *be the invisible permutations of* σ *and* σ', *respectively.*

a) $X(\sigma) \subset \overline{X(\sigma')}$ *if and only if* $X * (P(\sigma)) \subset \overline{X * (P(\sigma'))}$.

b) $X * (P(\sigma)) \subset \overline{X * (P(\sigma'))}$ *if and only if* $P(\sigma) \leq P(\sigma')$

where

- *the order "*$\leq$*" is the Bruhat order of the symmetric group* W_n.
- *we use* $X * (P(\sigma))$ *to denote the Schubert cell of* $P(\sigma)$ *in the homogeneous space* $P_{(1^{n-m},\gamma)}\backslash G_n$ *where* $(1^{n-m}, \gamma) := (\underbrace{1, \cdots, 1}_{n-m}, \gamma_1, \cdots, \gamma_t)$.

Note that the boundary operator $\partial = 0$ over the complex field. We have

Theorem 0.20 (Main Theorem) *Let* λ *be a* γ*-compatible partition. Then*

a) the partition variety $P_\gamma\backslash M_\lambda$ *is a CW-complex consisting of all the cells* $X(\sigma)$ *of* $P_\gamma\backslash M$ *which fit into the board* F_λ

$$P_\gamma\backslash M_\lambda = \bigsqcup\nolimits_{\sigma \in R_{\lambda}^{m}(\gamma)} X_\gamma(\sigma) \quad . \tag{13}$$

b) the Poincaré polynomial for cohomology of the partition variety $P_\gamma\backslash M_\lambda$ *with real coefficients is*

$$\begin{aligned} Poin(P_\gamma\backslash M_\lambda, \mathbf{R}) &= \sum \dim H^i(P_\gamma\backslash M_\lambda, \mathbf{R}) q^i \\ &= RL_m(\lambda,\, \gamma,\, q^2). \end{aligned} \tag{14}$$

Corollary 0.21 (Homogeneous Space $P_\gamma\backslash G_m$) *Let* γ *be any decomposition of* m. *Then*

a) The homogeneous space $P_\gamma\backslash G_m$ *is a CW-complex.*

b) The homology (and the cohomology) groups of the homogenous space $P_\gamma \backslash G_m$ *are given by*

$$\begin{aligned} Poin(P_\gamma \backslash G_m, \mathbf{R}) &= RL_m((m^m), \gamma, q^2) \\ &= \prod_{i=1}^{t} \begin{bmatrix} m - \gamma_{i+1} - \gamma_{i+1} - \cdots - \gamma_t \\ \gamma_i \end{bmatrix}_{q^2} \end{aligned}$$

Corollary 0.22 (Flag Manifold)

a) The flag manifold $B_m \backslash G_m$ *is a CW-complex consisting of Bruhat cells.*

b) The homology (and the cohomology) groups of the flag manifold $B_m \backslash G_m$ *are given by*

$$\begin{aligned} Poin(B_m \backslash G_m, \mathbf{R}) &= RL_m((m^m), (1^m), q^2) \\ &= (m)_{q^2}! \end{aligned}$$

Corollary 0.23 (Grassmann Manifold)

a) The Grassmann Manifold $G_m \backslash M_{n^m}$ *is a CW-complex consisting of Schubert cells.*

b) The homology (and the cohomology) groups of the Grassmann manifold are given by

$$\begin{aligned} Poin(G_m \backslash M_{n^m}, \mathbf{R}) &= RL_m((n^m), (m^m), q^2) \\ &= \begin{bmatrix} n \\ m \end{bmatrix}_{q^2}. \end{aligned}$$

References

[1] K. Ding, Invisible Permutations and Rook Placements on a Ferrers Board, Proceedings of the Fourth Conference on Formal Series and Algebraic Combinatorics, Publications du LACIM, Université du Québec à Montréal, Québec, 1992, p.137-154. To appear in Discrete Mathematics.

[2] K. Ding, Rook Placements and Cellular Decomposition of Partition Varieties $B \backslash M_\lambda$, preprint.

[3] K. Ding, Rook Placements and Cellular Decomposition of Partition Varieties, Ph D thesis, University of Wisconsin-Madison, 1993.

[4] D. Foata and M. Schützenberger, On the rook polynomials of Ferrers relations, Colloq. Math. Soc. János Bolyai, 4, Combinatorial Theory and its Applications, Vol. 2, (P. Erdős et al. eds.) North-Holland, Armsterdam, 1970, p. 413-436.

[5] A. M. Garsia and J. B. Remmel, q-counting rook configurations and a formula of Frobenius, J. Combin. Theory Ser. A 41(1986) 246-275.

[6] J. R. Goldman, J. T. Joichi, and D. E. White, Rook theory I, Rook equivalence of Ferrers boards, Proc. Amer. Math. Soc. 52(1975), 485-492.

[7] J. R. Goldman, J. T. Joichi, D. L. Reiner, and D. E. White, Rook theory II, Boards of binomial type, SIAM J. Appl. Math. 31 (1976), 618-633.

[8] J. R. Goldman, J. T. Joichi, and D. E. White, Rook theory III, Rook polynomials and the chromatic structure of graphs, J. Combin. Theory, Ser. b 25 (1978) 135-142.

[9] J. R. Goldman, J. T. Joichi, and D. E. White, Rook theory IV, Orthogonal sequences of rook polynomials, Studies in Appl. Math., 56(1977) 267-272.

[10] J. R. Goldman, J. T. Joichi, and D. E. White, Rook theory V, Rook polynomials, Mõbius inversion and the umbral calculus, J. Combin. Theory, Ser. A 21(1976), 230-239.

[11] H. Gould, The q-Stirling numbers of the first and second kinds, Duke Math. J. , 28(1961), 281-289.

[12] P. Griffiths and J. Harris, Principles of Algebraic Geometry, John Wiley & Sons, Inc., New York , 1978.

[13] H. Hiller, Geometry of Coxeter Groups, Pitman Books Limited, 1982.

[14] J. E. Humphreys, Linear Algebraic Groups, Springer-Verlag, New York, 1975.

[15] J. Milnor and J. Stasheff, Characteristic Classes, Annals of Mathematics Studies, no. 76, Princeton University Press, 1974.

[16] J. Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958.

[17] B. Sagan, A maj statistic for set partitions, Europ. J. Combin., 12(1991), 69-79.

[18] L. Solomon, The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field, Geometriae Dedicata 36(1990) 15-49.

[19] M. Wachs and D. White, p, q-Stirling numbers and set partition statistics, J. Combin. Theory Ser. A 56:1(1991) 27-46.