A Computationally Intractable Problem on Simplicial Complexes

*Ömer Eğecioğlu** and *Teofilo F. Gonzalez*

Department of Computer Science

University of California Santa Barbara, CA 93106

Abstract

We analyze the problem of computing the minimum number $er(\mathcal{C})$ of internal simplexes that need to be removed from a simplicial 2-complex $\mathcal{C}$ so that the remaining complex can be nulled by deleting a sequence of external simplexes. This is equivalent to requiring that the resulting complex be collapsible to a 1-complex. By reducing a restricted version of the Satisfiability problem SAT, we show that this problem is $\mathcal{NP}$-complete. This implies that there is no simple formula for $er(\mathcal{C})$ terms of the Betti numbers of the complex. The problem can be solved in linear time for graphs.

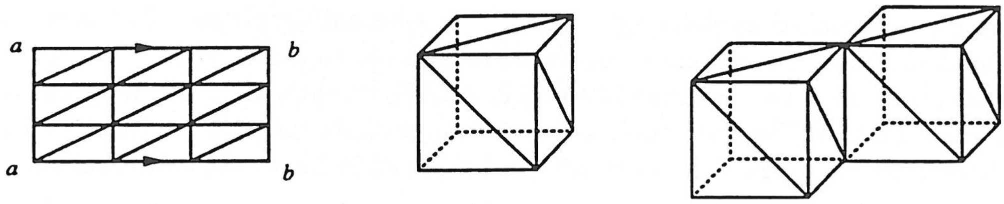
On analyze le probleme de calculer le nombre minimum $er(\mathcal{C})$ de simplexes internes que doivent etre enlever d'un 2-complexe simplicial C de sorte que le complex resultant peut etre annuler en effaçant une sequence de simplexes internes. Ceci est equivalent à demander que le complexe resultant est reduisible à un 1-complexe. En reduisant une version restricteé du probleme Satisfiability, SAT, on demontre que ce probleme est $\mathcal{NP}$-Complete. Ceci implique q'il n'y-as pas de formule simple pour trouver $er(\mathcal{C})$ en fonction de les nombres Betti du complexe. Le probleme peut etre resolu en temps linear pour les graphs.

Keywords: Simplicial complex, collapsing, Betti number, algorithmic complexity, intractable problem.

1 Introduction

We consider finite connected simplicial 2-complexes, all of whose maximal simplexes are 2-dimensional. Such a complex can be viewed as a collection of 2-simplexes $\mathcal{C} = \{s_1, s_2, \ldots, s_n\}$ modulo an equivalence relation that identifies pairs of simplexes s_i and s_j with $i \neq j$ along a common edge or a vertex. It is known that a simplicial 2-complex $\mathcal{C}$ has a geometric realization as a subset of the Euclidean 5-space in which each s_i is a closed triangular plane region. We refer the reader to the texts [3], [4], [6], [7], and [8] for more information.

We study the properties of complexes in terms of the subcomplexes obtained by removal of a subset of its 2-simplexes. A 2-simplex $s \in \mathcal{C}$ is called an *external* simplex of $\mathcal{C}$ if s has at least one proper face which not shared with any other simplex in $\mathcal{C}$; otherwise s is called *internal*. Given a 2-complex $\mathcal{C}$ and a 2-simplex $s_i \in \mathcal{C}$, we denote by $\mathcal{C} - s_i$ the 2-complex obtained by restricting the given identifications defining $\mathcal{C}$ to $\{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\}$. We say that $\mathcal{C} - s_i$ is obtained from $\mathcal{C}$ by *removing* (*erasing*) the internal (external) simplex s_i. If $\mathcal{C}'$ is obtained from $\mathcal{C}$ by erasing an external simplex of $\mathcal{C}$, then we denote this by $\mathcal{C} \rightsquigarrow \mathcal{C}'$. More generally if two complexes $\mathcal{C}$ and $\mathcal{C}_m$ are related by a sequence of erasures of external simplexes $\mathcal{C} \rightsquigarrow \mathcal{C}_1 \rightsquigarrow \cdots \rightsquigarrow \mathcal{C}_m$, then we denote this also by $\mathcal{C} \rightsquigarrow \mathcal{C}_m$. We say that the complex $\mathcal{C}$ is *erasable* (or *nullable*) if $\mathcal{C} \rightsquigarrow \phi$. As examples, the segment of a *pipe* in Figure 1 (a) is erasable. However the triangulation of the 2-dimensional sphere S^2 in Figure 1 (b), and the complex (c) are not erasable since these have no external simplexes. Note that the operation $\rightsquigarrow$ is not a topological invariant, since it can destroy the fundamental group.



(a) (b) (c)

Figure 1: (a) A pipe segment, (b) The sphere S^2, (c) Two spheres with a common edge.

Given a 2-complex $\mathcal{C}$ we define $er(\mathcal{C})$ to be the minimum number of internal 2-simplexes that need to be removed from $\mathcal{C}$ so that the resulting complex is erasable. For example, for the complexes in Figure 1 (a), (b), and (c), we have $er(\mathcal{C}) = 0, 1$, and 2, respectively. The quantity $er(\mathcal{C})$ also gives the minimum number of internal 2-simplexes that need to be removed from $\mathcal{C}$ so that the resulting complex can be *collapsed* to a 1-dimensional subcomplex. If $\mathcal{C}$ collapses to a d or lower dimensional subcomplex, this is denoted by $\mathcal{C} \searrow d$.

In this paper we show that the problem of computing $er(\mathcal{C})$ for a given 2-complex is intractable:

Erasability Problem :
INSTANCE : A pair $(\mathcal{C}, k)$ where $\mathcal{C}$ is a 2-complex and k is a non-negative integer.
QUESTION : Is $er(\mathcal{C}) = k$? i.e., does $\mathcal{C}$ contain a subset $\mathcal{K}$ of 2-simplexes of cardinality k such that $\mathcal{C} - \mathcal{K} \rightsquigarrow \phi$?

The Erasability Problem can be paraphrased as the following decision problem involving collapsibility

Collapsibility Problem : *INSTANCE :* A pair $(\mathcal{C}, k)$ where $\mathcal{C}$ is a 2-complex and k is a non-negative integer. *QUESTION :* Is $er(\mathcal{C}) = k$? i.e., does $\mathcal{C}$ contain a subset $\mathcal{K}$ of 2-simplexes of cardinality k such that $\mathcal{C} - \mathcal{K} \searrow 1$?

The main result is

Theorem 1 *The Erasability and the Collapsibility Problems are $\mathcal{NP}$-complete.*

As basic building blocks, we make use of properties of certain special complexes. The simplest of these are the Klein bottle Figure 2 (a), and the complex called an AND gate shown in Figure 2 (b). The Klein bottle is a nonorientable surface which has no embedding in 3-dimensional space. It is necessary and sufficient to remove a single 2-simplex from it to make it erasable. Thus $er(\textit{Klein bottle}) = 1$. The complex in Figure 2 (b) can be used as a logical AND gate in the following sense: The portion C of the complex connected to the top via the circle Z can be erased (without removing a simplex from C) only when both portions A and B connected to X and Y, respectively, are erased. However if only A or only B is erased, C is left intact. Note that the interior of the common circle that is shared by the pipes containing X, Y, and Z is not part of the AND gate in Figure 2 (b), just as the interior of the top circle that appears in the Klein bottle in Figure 2 (a) is not part of the Klein bottle.

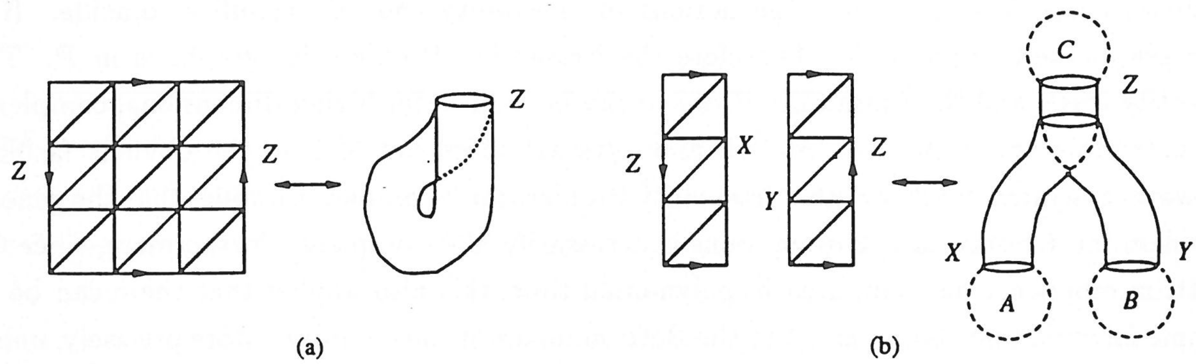


Figure 2: (a) The Klein bottle, (b) An AND gate: C is erasable if both A and B are erasable.

2 Remarks

The Betti numbers β_i for $0 \leq i \leq d$ of a d-dimensional simplicial complex $\mathcal{C}$ are topological invariants related to high dimensional connectivity properties of $\mathcal{C}$. The Betti number β_0 is the number of connected components of $\mathcal{C}$, and intuitively, β_i is the number of "i-dimensional holes" in $\mathcal{C}$ (see [4]). Since $er(\mathcal{C})$ appears to count the number of three dimensional regions enclosed by $\mathcal{C}$, we may expect some relationship between $er(\mathcal{C})$ and the Betti numbers of $\mathcal{C}$. To this end, we first consider 1-dimensional simplicial complexes, also referred to as *graphs*. For a connected graph $\mathcal{G}$ with n vertices and e edges (1-simplexes), the 1-dimensional Betti number has a simple interpretation: β_1 is the maximum number of linearly independent elementary cycles in $\mathcal{G}$ ([6] p. 71). Equivalently, β_1 is the dimension of the circuit space of $\mathcal{G}$. For graphs, β_1 and the rank r of the incidence matrix of $\mathcal{G}$ are related by $\beta_1 = e - r$, and thus β_1 can be found by a rank computation in polynomial time. Actually the 1-dimensional Betti number for graphs can be expressed explicitly by $\beta_1 = e - n + 1$ as a consequence of the the Euler-Poincaré relation: for an arbitrary d-dimensional complex, this relation is

$$\chi(\mathcal{C}) = \sum_{i=0}^{n}(-1)^i\alpha_i = \sum_{i=0}^{n}(-1)^i\beta_i ,$$

where α_i be the number of i-simplexes of $\mathcal{C}$. This common value is the Euler characteristic of $\mathcal{C}$.

It is known that all of the Betti numbers $\beta_0, \beta_1, \ldots, \beta_d$ of a general d-dimensional complex $\mathcal{C}$ can be computed from the quantities α_i, and the ranks of the incidence matrices relating the i-dimensional simplexes of $\mathcal{C}$ to its $(i-1)$-dimensional simplexes, $1 \le i \le n$. Consequently these invariants can be computed in polynomial time in the total number of simplexes of $\mathcal{C}$ (see [8], [4], [6]).

It is interesting to note that when we restrict the operation of removal and erasure to 1-dimensional simplexes, then the notions of erasability and collapsibility coincide. If $\mathcal{G}$ is a graph then $er(\mathcal{G}) = \beta_1$. Therefore the Erasability Problem for graphs is in $\mathcal{P}$. The quantity $er(\mathcal{C})$ and the Erasability Problem can be defined for higher dimensional complexes by extending the notions of internal and external simplexes of $\mathcal{C}$ in the obvious fashion. However the intractability of the Erasability Problem for 2-complexes implies that the general problem for arbitrary $d \geq 2$ dimensions is necessarily $\mathcal{NP}$-complete. Furthermore, since the Betti numbers can be computed in polynomial time, this also implies that there can be no simple formula that relates $er(\mathcal{C})$ to the Betti numbers of the complex. More precisely, unless $\mathcal{P} = \mathcal{NP}$, there can be no polynomial time computable function $f(x_0, x_1, \ldots, x_d)$ for which $er(\mathcal{C}) = f(\beta_0, \beta_1, \ldots, \beta_d)$.

Also note that the Erasability Problem is solvable in polynomial time for constant k, since it suffices to generate all k-element subsets $\mathcal{K}$ of internal simplexes of $\mathcal{C}$ and for each subset $\mathcal{K}$ check in polynomial time whether $\mathcal{C} - \mathcal{K} \rightsquigarrow \phi$.

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