Chip-firing and Coxeter elements

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Abstract

For a Coxeter group (W, S), the product of all generators $s_1, \ldots, s_n$ in any order is called a Coxeter element. We show that these elements correspond to acyclic orientations of the edges of the Coxeter graph, and that these in turn correspond to minimal recurring positions in the chip-firing game of Björner, Lovász and Shor.

Winding this correspondence backwards, we find that firing a legal node in the game means reversing all edge directions leading into a sink, which for the n-letter Coxeter word means moving the first letter to the end. Reachability is an equivalence relation on these game positions and it corresponds to rotation equivalence of Coxeter words.

These equivalence classes are, in fact, the conjugacy classes of the Coxeter elements. Several enumerative results about the conjugacy classes follow.

Résumé

Pour un système de Coxeter (W, S), on appelle élément de Coxeter tout produit $s_1, \ldots, s_n$ de tous les générateurs en ordre quelconque. Nous montrons que ces éléments correspondent aux orientations acycliques des arêtes du graph de Coxeter, ceux qui correspondent aussi aux positions recurrents minimaux dans le "chip-firing game" de Björner, Lovász et Shor.

En réversant cette correspondance, nous trouvons que le "firing" d'un sommet légal en ce jeu signifie l'inversion du sens de tous les arêtes terminants en une sortie, aussi correspondant au déplacement de la dernière lettre d'une mot de Coxeter jusqu'au debut du mot. Accessibilité est une relation d'équivalence pour ces positions du jeu et correspond à l'équivalence relatif à rotation des mots de Coxeter.

Ces classes d'équivalence sont, en fait, les classes des élements de Coxeter conjugués. Plusieurs resultats énumeratifs en résultes.

1 Introduction

For the theory of finite reflection groups, Coxeter elements play an important role. A Coxeter element is a product $w = s_1 s_2 \cdots s_n$ of all the generating reflections s_i, taken in any order. All Coxeter elements are conjugate and therefore have the same eigenvalues. It turns out that these eigenvalues immediately determine the exponents of the group, and this is probably the simplest way of computing these numbers.

For infinite Coxeter groups, much less is known about the Coxeter elements. A'Campo [1] showed that they have infinite order, Howlett [6] that they have a real eigenvalue ≥ 1. The affine case has been treated in more detail by Steinberg [9] and by Berman, Lee and Moody [2].

Our interest in the matter is the enumerative aspect. Surprisingly, the combinatorics turns out to be a special case of the chip-firing game by Björner, Lovász and Shor [4]. The connection is as follows. Given a Coxeter element $w = s_1 s_2 \cdots s_n$, we put a certain number of chips on each vertex s_i in the Coxeter graph, namely the number of neighbours s_j that succeed s_i in w. Every Coxeter element gives rise to a well-defined distribution of chips and the legal play sequences correspond to rotations of the n-letter words. As a second surprise, the reachability relation partitions these game positions precisely according to conjugacy classes.

2 Edge orientations and chip-firing

Let G be a connected, undirected graph. An *acyclic edge orientation* is an assignment of directions to all edges, such that the resulting digraph is acyclic. This is always possible. A simple observation is that the resulting digraph contains at least one *sink*, i.e. a vertex with no out-going edges.

We go on to explain the connection with the *chip-firing game* of Björner, Lovász and Shor, introduced in [4]. If each arrow-head is detached and pronounced a chip, we get a distribution of chips on the vertices. This distribution contains all information, as stated by the following result.

Proposition 1. *An acyclic edge orientation can be retrieved from its distribution of chips, i.e., the in-degrees determine all edge directions.*

PROOF. It is well-known that an acyclic digraph must have a sink, so for some vertex, the number of chips equals the degree. That reveals the orientation of all edges at that vertex. But after removing these edges and their chips, we still have a distribution corresponding to an acyclic edge orientation, so the procedure can be continued until all edge orientations have been revealed. □

A legal move in the game consists of choosing a vertex with at least as many chips as the degree and then moving one chip to each neighbour. Translated into edge orientations, a legal move means choosing a sink and reversing its edges. Since neither sinks nor sources belong to any cycles, the graph will still be acyclic and contain a sink, so the game goes on forever. The following fact is crucial.

Proposition 2. *Let u and v be two acyclic edge orientations. Then there is a legal game from u to v if and only if there is a legal game from v to u.*

PROOF. If a single move can be inverted, so can a sequence of moves. Thus, it is sufficient to consider the case when v is the result of firing a single vertex in position u. In order to prove this by induction, we first strengthen the property like this: For every vertex x that can be fired in position u, there is a continuation in which *every other vertex is fired exactly once.* Clearly, such a firing sequence leads back to u.

The statement is true for the case $\bullet\!\leftarrow\!\!\!-\!\bullet$ and induction over the number of vertices proves the proposition: Fire some vertex x, to reach some position u_1. Let $u_1 - x$ be the acyclic edge orientation obtained by deleting x and its edges. By the induction assumption, it is clear that there exists a firing sequence from $u_1 - x$ in which all vertices are fired exactly once, and this is still legal after reinserting x and all its edges, since these edges are directed out from x. □

Remark 1. According to this result, *reachability* constitutes an equivalence relation that partitions acyclic edge orientations into *reachability classes.*

Remark 2. The proposition is not generally true for chip-firing games. The simplest counterexample is $u = \overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ and $v = \overset{1}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$. The position u can never reappear, although the game is infinite.

By Theorem 3.3 in [4], the total number of chips in an infinite game must be at least equal to the total number of edges. The distributions considered by us have exactly one chip for each edge, so they are minimal among infinite game positions. This minimality, together with the recurrence property in the last proposition, characterizes these positions.

Definition. A position is *recurrent* if there is some game in which it occurs twice. It is *minimal recurrent* if no chip can be removed without destroying the recurrency.

Proposition 3. *Minimal recurrent chip-firing positions are precisely positions corresponding to acyclic edge orientations.*

PROOF. By Theorem 4.1 in [3], for any recurrent position u, there is a recurrent game from u to u such that each vertex is fired exactly once. So along each edge a chip is fired in each direction. Let us always use the same chip on the return route! After the game, remove all chips that were not used. The result is a position corresponding to an edge orientation. Further, it must be acyclic, for all vertices are fired and vertices in a circuit can never be fired. □

For many graphs, it is now a rather simple matter to enumerate acyclic edge orientations and reachability classes. Two basic cases are covered by our next proposition.

Proposition 4. *For a tree with n nodes, there are 2^{n-1} acyclic edge orientations but only one reachability class. For an n-cycle, there are $2^n - 2$ acyclic edge orientations and $n - 1$ reachability classes of sizes $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$.*

PROOF. An n-vertex tree has got $n-1$ edges with no restrictions on orientations. The statement that all are reachable from each other is obvious for a two-vertex tree $\bullet\!\leftarrow\!\bullet$. Assume that it is true for all n-vertex trees and consider an $(n+1)$-vertex tree $T_{n+1} =$ x—T_n (where x is a leaf vertex) and two acyclic edge orientations on T_{n+1}, u and v. By assumption, their restrictions to T_n can be connected by a game and if x is fired whenever possible, this also defines a game on T_{n+1}, say from u' to v'. Now, either $u' = u$ or u' is the result of firing x in u. The same argument for v' confirms that u and v are in the same reachability class.

For an n-cycle, exactly two orientations are forbidden, namely all n clockwise or all n anti-clockwise. Consider the $\binom{n}{k}$ orientations with k anti-clockwise edges. Firing a node may be seen as moving the anti-clockwise arrow one step forward, e.g. $\bullet\!\leftarrow\!\bullet\!\rightarrow\!\bullet\!\leftarrow\!\bullet\!\leftarrow\!\bullet$ to $\bullet\!\leftarrow\!\bullet\!\leftarrow\!\bullet\!\rightarrow\!\bullet\!\leftarrow\!\bullet$

It is obvious that any position with k anti-clockwise arrows can be reached in this way.
$\square$

3 Coxeter elements

An *irreducible Coxeter group* (W,S) is defined by a connected, edge labelled graph G with vertex set S and labels in $\{3,4,\ldots\}$. The group W is generated by the $s_i \in S$ modulo the relations $s_i^2 = e$ (generators are involutions) and

$$\begin{aligned} s_i s_j &= s_j s_i && \text{when there is no edge between } s_i \text{ and } s_j, \\ (s_i s_j)^m &= e && \text{when there is an } m\text{-labelled edge between them.} \end{aligned}$$

A product of all n generators, in any order, is called a *Coxeter element.* Two permutations of $s_1,\ldots,s_n$ define the same Coxeter element if and only if one can be transformed into the other by repeated application of the commutation rule $s_i s_j = s_j s_i$ for nonconnected vertices. This is a consequence of Tits's Word Theorem (see Brown [5]). Because of this, the edge labels are not so important in our line of investigation. In most cases, we shall not even mention them in our statements.

Every permutation of $s_1,\ldots,s_n$ induces an *acyclic edge orientation* on G by directing the edge $s_i \leftarrow s_j$ if s_i precedes s_j. Bu the above, we have the following simple result.

Proposition 5. *There is a bijective correspondence between Coxeter elements and acyclic edge orientations of the Coxeter graph.*

We can choose a slightly different outlook and regard the acyclic edge orientation corresponding to a certain Coxeter element w as a *poset.* A specific n-letter word in the s_i representing w can be viewed as a *linear extension* of the partial order and there is of course a wealth of enumerative results to be applied. We confine ourselves to the following useful observation.

Proposition 6. *The sinks of the acyclic edge orientation corresponding to a Coxeter element w are the s_i that appear as the first letter of some n-letter word representing w. The* *number of such words starting with s_i can be expressed as*

$$\binom{n-1}{n_1 \dots n_k} e(G_1)e(G_2)\cdots e(G_k),$$

where the G_j are the components of $G - s_i$, $n_j = |G_j|$ and $e(G_j)$ denotes the number of linear extensions of the poset G_j. This formula gives a recursion for the computation of $e(G)$.

PROOF. In any n-letter word representing w, each sink s_i has all its vertex neighbours to the right, so it can be freely moved to the left end of the word.

The n_j letters in component G_j may come in $e(G_j)$ different relative orders. The factor $\binom{n-1}{n_1 \dots n_k}$ reflects the number of ways that the $n-1$ positions after the first letter may be distributed over the G_j. □

If the first letter of a Coxeter word is moved last, the correponding vertex obviously changes from sink to source. Conversely, every sink is a first letter of some Coxeter word corresponding to the edge orientation and therefore, any chip-firing play corresponds to rotation of the word.

Proposition 7. *Rotation of Coxeter words induces an equivalence relation on the set of Coxeter elements, that corresponds precisely to the reachability relation on the set of acyclic edge orientations.*

If $w = s_1 s_2 \cdots s_n$, then $s_1 w s_1^{-1} = s_2 \cdots s_n s_1$ so rotation equivalent elements are conjugate. The converse is also true.

Proposition 8. *Coxeter elements belong to the same conjugacy class if and only if they are rotation equivalent.*

PROOF. What we have to prove is that acyclic edge orientations in different reachability classes correspond to nonconjugate elements. According to Proposition 4, there is nothing to be proved when the graph is a tree. We refer to [5] for a proof in the general case. The principal idea of this proof appears in the proof of our next proposition. □

For an important class of Coxeter groups, including all finite and affine groups, propositions 4, 7 and 8 enumerate conjugacy classes of Coxeter elements. For the tree case, this is an old result (see [7], 8.4) but the cycle case may be new. Recall that an n-cycle with all edges labeled by 3 is the graph of the affine group denoted by $\tilde{A}_{n-1}$.

Proposition 9. *In $\tilde{A}_{n-1}$ (and in all groups with n-cycle Coxeter graphs) the Coxeter elements fall into $n-1$ different conjugacy classes of sizes $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}$. A representative of the k-th class is $w_k = s_1 s_2 \cdots s_k s_n s_{n-1} \cdots s_{k+1}$. A Coxeter element $w = s_{i_1} \cdots s_{i_n}$ belongs to class k if exactly k of the indices precede their numerical successors. The numerical successor of i is defined as $i+1$, unless $i = n$ in which case it is 1.*

PROOF. In the proof of Proposition 4, class k is characterized as having k anti-clockwise oriented edges $s_i \leftarrow s_{i+1}$, and that is also the number of s_i that precede s_{i+1} in w_k. What remains to be shown is that w_k and $w_{k'}$ are nonconjugate if $k \neq k'$. Instead of referring to the previous proposition, we shall give a direct argument.

If two group elements are conjugate, then so are their m-th powers. Let us study elements of the form $s_i w_k^m s_i$, where $m = n!$ (or at least divisible by $1, 2, \ldots, n-1$). When s_i is a sink or a source, this is of course a one-step rotation of w_k^m. We shall prove that in all other cases, it is equal to w_k^m itself! Then, we can iterate and draw the conclusion that every conjugate of w_k^m is a rotation, whence the statement about w_k can be deduced.

In fact, $s_i w_k = w_k s_{i-1}$ if $3 \leq i \leq k$ and $s_i w_k = w_k s_{i+1}$ if $k+2 \leq i \leq n-1$, as can be verified directly. The double-step relations $s_2 w_k^2 = w_k^2 s_k$ and $s_n w_k^2 = w_k^2 s_{k+2}$ have more complicated, but still trivial, verification. We conclude that, unless s_i is the first or last letter in w_k, we have $s_i w_k^m = w_k^m s_i$ if m is divisible by k and by $n-k$. The remaining details are easy. □

Example. Consider $\tilde{A}_5$. In the Coxeter element $w = s_3 s_5 s_1 s_2 s_4$, 1, 3 and 5 precede their numerical successors, so w is conjugate to $w_3 = s_1 s_2 s_3 s_5 s_4$. In fact, we have $uwu^{-1} = w_3$ with $u = s_4 s_5 s_3$.

In $\tilde{A}_{n-1}$ (as in all Coxeter groups with n generators), there are $n!$ Coxeter words and one may ask how many of these that fall into each conjugacy class. We know that rotating the word does not alter its conjugacy class, so it is enough to consider permutations where n comes last. In that case, n does not precede its numerical successor, so the conjugacy class number k is $(n-1)$ minus the number of inversions among numerically adjacent pairs $(1,2), (2,3), \ldots, (n-2, n-1)$. Ignoring the n we get a permutation $\pi \in S_{n-1}$. Now, $r+1$ precedes r in π if and only if r is a *descent* in π^{-1}, so we can write $k = n-1-d(\pi^{-1})$. But it is known that the *Eulerian numbers* count permutations with given number of descents. The definition is

$$A(n-1, m) = |\{\pi \in S_{n-1} : d(\pi) = m-1\}|,$$

see Stanley's book [8]. Putting $m = n-k$ we get the following formula. Recall that there are n rotations of every word obtained above.

Proposition 10. *In $\tilde{A}_{n-1}$ (and in all groups with n-cycle Coxeter graphs), the $n!$ words representing Coxeter elements are partitioned by conjugacy into $n-1$ classes of sizes $n\,A(n-1, n-k)$ for $k = 1, \ldots, n-1$.*

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