Reduced decompositions in affine Coxeter groups

Kimmo Eriksson
Department of Mathematics, KTH
S-100 44 Stockholm, Sweden
kimmo@nada.kth.se

Abstract

Let $r(w)$ denote the number of reduced words for an element w in a Coxeter group W. Stanley proved a formula for $r(w)$ when W is the symmetric group A_n, and he suggested looking at $r(w)$ for the affine group $\tilde{A}_n$. We prove that for any affine Coxeter group $\tilde{X}_n$ there is a finite number of *types* of elements in $\tilde{X}_n$, such that to every element w can be associated a type t, an element v in the finite group X_n, and an n-tuple $(m_1, m_2, \ldots, m_n)$ of integers $m_i \geq 0$. Then $r(w) = r_t^v(m_1, \ldots, m_n)$, and for every r_t^v and for large enough m_i, a homogeneous linear n-dimensional recurrence holds. For $\tilde{A}_n$, this takes a nice combinatorial form. We also discuss a canonical reduced word for w associated to its n-tuple.

Résumé

Soit $r(w)$ le nombre de mots réduits pour un élément w d'un groupe de Coxeter. Stanley a démontré une formule pour $r(w)$ au cas du groupe symétrique A_n, et il a posé le problème d'analyser $r(w)$ pour le groupe affine $\tilde{A}_n$. Nous montrons qu'il y a, pour tout groupe de Coxeter affine $\tilde{X}_n$, un nombre fini de *types* d'éléments tels qu'on peut associer à chaque élément w un type t, un élément v du groupe fini X_n et une suite $(m_1, m_2, \ldots, m_n)$ d'entiers $m_i \geq 0$. Alors, $r(w) = r_t^v(m_1, \ldots, m_n)$ et pour m_i assez grands, r_t^v satisfait à une récurrence homogène linéaire n-dimensionelle. Pour $\tilde{A}_n$, cela prend une forme combinatoire plaisante. Nous présentons aussi une décomposition réduite canonique pour w, associé à la suite des m_i.

1 Introduction

For an element w of a Coxeter group (W, S), a *reduced decomposition* of w is obtained by writing w as a minimal product of generators. Let $r(w)$ denote the number of reduced decompositions of w. For example, in A_2, the symmetric group on three elements generated by the adjacent transpositions $s_1 = (12)$ and $s_2 = (23)$, the transposition (13) has two reduced decompositions: $s_1s_2s_1$ and $s_2s_1s_2$.

Stanley [5] and Greene-Edelman [1] studied the number of reduced decompositions of elements in A_n, showing an intimate relationship with standard tableaux of the corresponding shape. Haiman [2] generalized their work to include the finite Coxeter group B_n as well. Thus, for most finite Coxeter groups, the combinatorics of $r(w)$ is very well understood. In his paper [5], Stanley also suggested that one should study the number of reduced decompositions in the affine group $\tilde{A}_n$. This is our purpose here.

We will mainly do the following. By working in the Coxeter complex, we shall show that for any affine Coxeter group $\tilde{X}_n$ corresponding to a finite Coxeter group X_n, there is a finite number of *types* of elements in $\tilde{X}_n$, such that to every element w can be associated a type t, an element v from the finite group X_n, and an n-tuple $(m_1, m_2, \ldots, m_n)$ of integers $m_i \geq 0$. Thus, for any type t and element $v \in X_n$, we can define the more concrete function $r_t^v(m_1, \ldots, m_n) \overset{\text{def}}{=} r(w)$. Every r_t^v is then shown to satisfy a homogeneous linear n-dimensional recurrence. For any fixed type t, the same recurrence will hold for every v but with different start values depending on the element v.

We will also discuss a canonical reduced word for w related to v, t and $(m_1, \ldots, m_n)$.

Example. For $\tilde{A}_2$ there are two types, and there are six elements v in the finite group A_2. In this case the recurrence is in fact independent also of the type, so that for both types t we have

$$r_t^v(m_1, m_2) = r_t^v(m_1 - 1, m_2) + r_t^v(m_1, m_2 - 1).$$

Also, in this case the start conditions are independent of v; for every v we have $r_t^v(0, m) = r_t^v(m, 0) = 1$ for all $m \geq 0$. Thus, in this case (dropping v and t) we get the very simple explicit formula

$$r(m_1, m_2) = \binom{m_1 + m_2}{m_1}.$$

2 A geometric construction

Let X_n denote an arbitrary finite reflection group in $\mathbf{R}^n$, generated by n Coxeter generators $\sigma_1, \ldots, \sigma_n$, and let $\mathcal{H}$ be the arrangement of reflecting hyperplanes, splitting $\mathbf{R}^n$ into cones. Then the group elements correspond bijectively to the cones. Every cone is bounded by n walls, and they can be canonically labeled by σ_1 through σ_n, such that when one cone is mapped to another via a sequence of reflections, the labeling of the walls is invariant.

Example. We will present a running example with B_2 as the Coxeter group "X_n". B_2 is the group of 8 elements generated by reflections in two lines with an angle between them of 45°.

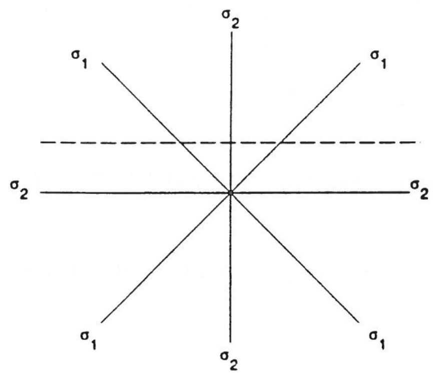

Figure 1: The hyperplane arrangement of Coxeter group B_2. The affine hyperplane used to generate $\tilde{B}_2$ is indicated.

The affine group $\tilde{X}_n$ corresponding to the finite group X_n is obtained by adding to the set of generators a reflection in an affine hyperplane parallel to one of the hyperplanes in $\mathcal{H}$. Let $\tilde{\mathcal{H}}$ denote the infinite affine hyperplane arrangement; thus $\mathcal{H} \subset \tilde{\mathcal{H}}$.

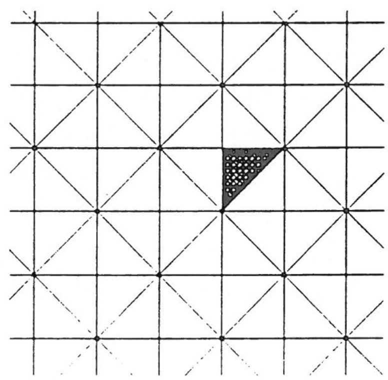

Figure 2: The affine hyperplane arrangement of affine group $\tilde{B}_2$, with the fundamental alcove C_0 painted.

Let $\mathcal{C}$ be the alcove complex defined by $\tilde{\mathcal{H}}$, and let C_0 be the fundamental alcove. A *gallery* is a walk in the complex, and a minimal gallery is a shortest possible gallery between two alcoves. Any alcove $C \in \mathcal{C}$ defines an interval $[C_0, C]$, which is the subset of $\mathcal{C}$ consisting of all alcoves that you can visit by walking minimal galleries from C_0 to C, which is equivalent to walks that cross only hyperplanes that separate C_0 from C. If $w_C \in \tilde{X}_n$ is the group element corresponding to alcove C, the interval $[C_0, C]$ is isomorphic to the interval $[e, w_C]$ in the weak order of $\tilde{X}_n$.

What we are going to do is covering the complex $\mathcal{C}$ by a finite set of congruent cones, $\mathcal{V} = \{V_x : x \in X_n\}$, where each $V_x \in \mathcal{V}$ is bounded by some hyperplanes in $\tilde{\mathcal{H}}$. We identify

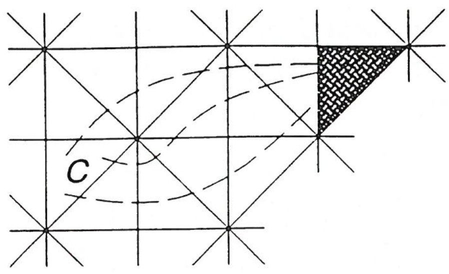

Figure 3: The interval $[C_0, C]$ with the three minimal galleries indicated.

a cone with the set of alcoves contained in it. This covering will have the following three properties :

$$(1)\ V_x \neq V_y \text{ if } x \neq y, \quad (2) \bigcap_{x \in X_n} V_x = C_0, \quad (3)\ C \in V \Rightarrow [C_0, C] \subset V \text{ for all } V \in \mathcal{V}$$

In words, there is one particular cone V_x for every element x in the finite group X_n, only the fundamental alcove C_0 lies in every cone, and all minimal galleries between C_0 and any given alcove C stay in the cone in which C lies. Note that (3) implies

$$(3')\ C \in \bigcap_{V \in \mathcal{V}'} V \Rightarrow [C_0, C] \subset \bigcap_{V \in \mathcal{V}'} V \text{ for all subsets } \mathcal{V}' \subset \mathcal{V},$$

that is, if the alcove C lies in the intersection of several cones, then every minimal gallery from C_0 to C must be also contained in this intersection.

The cones are constructed in the following way. Build a *thick wall* from a pair of successive parallel hyperplanes in $\tilde{\mathcal{H}}$ enclosing the fundamental alcove C_0. In other words, each of the hyperplanes in the finite arrangement $\mathcal{H}$ is thickened to a thick wall containing C_0. Now, what we have got is a thickened version of the hyperplane arrangement for X_n, so these thick walls bound a set of $|X_n|$ thick cones, and there is a natural way of assigning a thick cone V_v to every element v of X_n. (A thick cone contains its thick walls.)

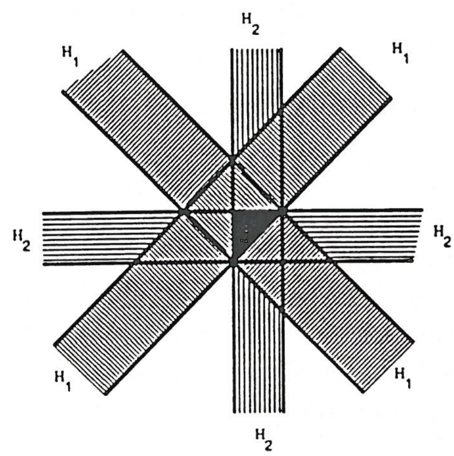

Figure 4: The thickened arrangement of B_2.

Lemma 1. *The set $\mathcal{V} = \{V_v : v \in X_n\}$ of thick cones has the properties (1), (2) and (3) above.*

PROOF (1) is clear from above. (2) follows immediately from the construction. To prove (3), let C be an alcove in V_v. Since a gallery from C to C_0 that leaves the cone V_v must also reenter the cone, it must cross some bounding hyperplane twice. Thus it cannot be minimal, so all minimal galleries from C to C_0 stay in the cone. □

Thanks to this lemma we know that when counting minimal galleries we can restrict our attention to one thick cone instead of the entire complex. We shall now break the thick cone V_v into smaller pieces. V_v is bounded by n thick walls, with a natural labeling $H_1, H_2, \ldots, H_n$ induced by the labeling of the corresponding thin walls. Make V_v into a lattice of *cells* by subdividing it by all hyperplanes of $\tilde{\mathcal{H}}$ that are parallel to any of the bounding hyperplanes of V_v. The figure should make the situation clear.

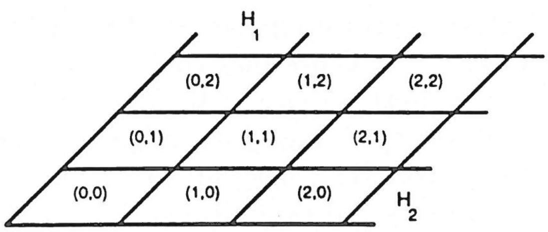

Figure 5: The cell decomposition of a thick cone.

Note that since the subdivision is caused by a subset of $\tilde{\mathcal{H}}$, the alcoves are finer objects than the cells. Every cell will be a union of alcoves of $\mathcal{C}$. The *apex* of a cell is the vertex closest to the origin.

Lemma 2. *Every cell of a cone V_v is composed of alcoves in the same way, and every possible orientation of alcoves occurs exactly once in every cell.*

PROOF With respect to the affine hyperplane arrangement, the apex of any cell is equivalent to the origin, since the n walls (of the cell) containing the apex are parallel to the n walls bounding a cone in the X_n-arrangement, so by reflections they generate an isomorphic hyperplane arrangement through the apex. Together with one of the other walls they generate the entire affine arrangement. Hence, from an apex the arrangement can only look in exactly one way, so in particular every cell must look the same. Also, from any alcove of the same orientation as the alcove at the apex of the cell, the arrangement can only look in one way. Since there are no additional hyperplanes parallel to cell walls, there can be just one alcove of this orientation in every cell, and by symmetry the same must hold for every orientation. □

Thus, if there are k possible orientations of alcoves, then an arbitrary cell D consists of k alcoves. Since one thick cone can be obtained from another by reflections and translations,

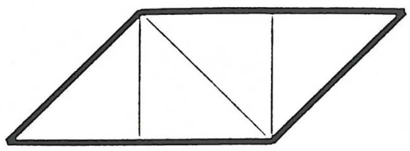

Figure 6: A $\tilde{B}_2$-cell and its decomposition into alcoves of four types.

the alcove decomposition of all thick cones are of course isomorphic. Therefore, we define the *type* of an alcove *in a specified cone* to be the orientation relative to the cone. If there are k different orientations of alcoves, then there are k types of alcoves.

Lemma 3. *Let $p(\tilde{X}_n)$ be the number of parabolic subgroups of $\tilde{X}_n$ that are isomorphic to X_n. Let k be the number of possible orientations of alcoves in $\tilde{X}_n$. Then*

$$k = \frac{|X_n|}{p(\tilde{X}_n)}$$

PROOF A maximal parabolic subgroup can be identified with a vertex of the fundamental alcove, since the n walls of the alcove that contain the vertex correspond to n generators. The $p(\tilde{X}_n)$ parabolic subgroups isomorphic to X_n can be identified with the vertices of the fundamental alcove that are equivalent to the origin with respect to the affine arrangement. Since hyperplanes of all orientations meet at the origin, clearly, all possible orientations of alcoves are represented among the $|X_n|$ alcoves that touch the origin. Of these there are $p(\tilde{X}_n)$ oriented as the fundamental alcove, one for each translation of the origin-equivalent vertices to the origin. By symmetry, there are $p(\tilde{X}_n)$ alcoves of each orientation among the $|X_n|$. □

For a cell D, let D_t be the alcove of type t in D, where t is one of the k possible types. We can index the cells of V_v by n nonnegative integer indices such that $D^v(0,0,\ldots,0)$ is the cell at the apex of the cone, and $D^v(m_1, m_2, \ldots, m_n)$ is the cell separated from $D^v(0,0,\ldots,0)$ by m_i hyperplanes parallel to H_i for all $i = 1,2,\ldots,n$.

Thus, every alcove of the cone V_v can be described uniquely as $D_t^v(m_1,\ldots,m_n)$ for some indices $m_1,\ldots,m_n \geq 0$ and some type t.

3 Counting minimal galleries

We are interested in the number of reduced decompositions for an element $w \in \tilde{X}_n$, which is equivalent to the number of minimal galleries between the corresponding alcove C_w and the fundamental alcove C_0. C_w belongs to some thick cone $V_v \in \mathcal{V}$ (possibly to several) and we know that the minimal galleries never leave this cone.

Let $r(C)$ denote the number of minimal galleries to the alcove C. A fundamental observation is that, using the covering relation of the weak order,

$$r(C) = \sum r(C') \quad \text{summed over all } C' \text{ covered by } C$$

By the analysis in the previous section, we know that any alcove C in cone V_v can be uniquely described as $D_t^v(m_1,\dots,m_n)$, for some type t and indices $m_1,\dots,m_n$. Thus, we may define $r_t^v(m_1,\dots,m_n) := r(C)$, the number of minimal galleries between the fundamental alcove C_0 and $D_t^v(m_1,\dots,m_n)$. We know that all covering relations occur within the cone V_v. For inner cells, when all $m_1,\dots,m_k \geq 1$, it is not hard to prove (but omitted here) that the form of the sum will depend on the type of the alcove only, that is, for all inner cells an alcove of a given type t has always isomorphic covering relations. Thus, if there are k types, the relation above implies a system of k recurrences for the r_t^v.

Example. In $\tilde{B}_2$ there are four types, which we may number 1 through 4 by decreasing distance from the apex of the cell. For brevity, we do not specify the cone v, since the recurrences do not differ, but only the boundary values. The figure below shows that the recurrences are:

$$\begin{cases} r_1(m_1,m_2) = r_2(m_1,m_2) \\ r_2(m_1,m_2) = r_3(m_1,m_2) \\ r_3(m_1,m_2) = r_4(m_1,m_2) + r_1(m_1,m_2-1) \\ r_4(m_1,m_2) = r_1(m_1-1,m_2) + r_2(m_1,m_2-1) \end{cases}$$

Elimination gives a single recurrence, which for any type t has the form:

$$r_t(m_1,m_2) = r_t(m_1-1,m_2) + 2r_t(m_1,m_2-1)$$

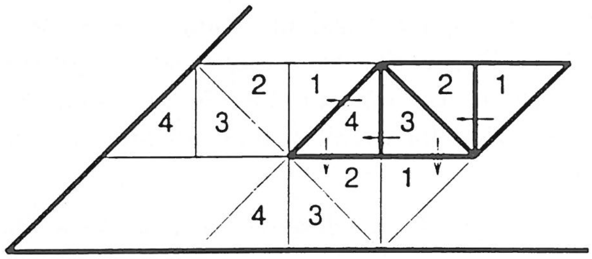

Figure 7: The arrows signify the covering relations between alcoves in $\tilde{B}_2$.

The example above is quite representative for the general situation. We have a system of k homogeneous, first-order linear recurrences:

$$r_t = \sum_{t'} \Delta_{t,t'} r_{t'}$$

where $\Delta_{t,t'}$ is a first-order difference operator, plus a term 1 whenever the alcove of type t covers the alcove of type t' in an inner cell. Give the types any ordering satisfying $t' > t$ when t covers t' in the cell. Thus, in this ordering the difference operator $\Delta_{t,t'}$ has no constant term when $t' \leq t$.

We can now express the system of recurrences as a matrix multiplication:

$$\begin{pmatrix} (1-\Delta_{1,1}) & -\Delta_{1,2} & \cdots & -\Delta_{1,k} \\ -\Delta_{2,1} & (1-\Delta_{2,2}) & \cdots & -\Delta_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\Delta_{k,1} & -\Delta_{k,2} & \cdots & (1-\Delta_{k,k}) \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

All elements below the diagonal lack constant terms, while all diagonal elements have constant term 1. It is easy to verify that this property is preserved during Gaussian elimination without pivoting. Thus, by completing the Gaussian elimination, we arrive at a diagonal matrix $\mathrm{diag}(1 - \Delta'_t)$, or equivalently, a system of k independent linear recurrences:

$$r_t = \Delta'_t r_\tau, \qquad t = 1, 2, \ldots, k,$$

for some constant-free difference operators Δ'_t. The total degree is bounded by 2^{k-1}, because in the initial matrix the degrees were bounded by one, and for every row that is eliminated the maximum degree can be at most doubled. Let us state this as a theorem.

Theorem 4. *For every alcove type t there is a constant-free difference operator Δ'_t of degree $\leq 2^{k-1}$ such that the numbers of minimal galleries $r_t(m_1, \ldots, m_n)$ satisfy the homogeneous n-dimensional linear recurrence*

$$r_t(m_1, \ldots, m_n) = \Delta'_t r_t(m_1, \ldots, m_n)$$

whenever all m_i are sufficiently large.

4 Remarks

Remark 1. For the affine group $\tilde{A}_n$, there is a more combinatorial way of stating the recurrences. Lemma 3 gives for $\tilde{A}_n$ that the number of types is $k = (n+1)!/(n+1) = n!$, so one should look for a natural bijection between orientations t and permutations $\pi = \pi_1\pi_2 \ldots \pi_n \in S_n = A_{n-1}$. The easy way is to change the shape of the cells a little, so that they coincide with cosets isomorphic to the parabolic subgroup A_{n-1}. After gaining a lot of insight (ask the author for details!) one can present the following permutational version of the recurrences:

Theorem 5. *Let $D(\pi)$ be the descent set of π, let s_i denote the transposition $(i, i+1)$ for $i = 1, 2, \ldots, n-1$ and let τ be the rotation operator defined by $\tau(\pi_1\pi_2 \ldots \pi_n) = \pi_n\pi_1 \ldots \pi_{n-1}$. The elements in a given cone of $\tilde{A}_n$ may be indexed by a permutation π and $m_1, \ldots, m_n$ such that r_π satisfies the recurrence*

$$r_\pi(m_1, \ldots, m_n) = \sum_{i \in D(\pi)} r_{s_i\pi}(m_1, \ldots, m_n) + r_{\tau\pi}(m_1, \ldots, m_{\pi_n} - 1, m_{\pi_n+1} + 1, \ldots, m_n)$$

Remark 2. The geometric construction also gives a canonical reduced expression for w. The cone C containing w is spanned by n directions, and a canonical way of getting to w is by first going in direction 1, then in direction 2, and so on, last going in direction n. Walking in direction i corresponds to an infinite periodic word v_i. Ask the author for details!

Theorem 6. *Every element of $\tilde{X}_n$ has a reduced expression of the form $u_1u_2 \ldots u_n$ where every u_i is a factor of the infinite periodic word v_i.*

For example, if s, t, u are the generators of $\tilde{A}_2$, then every $w \in \tilde{A}_2$ has a reduced word of the form $(\ldots stustu \ldots)(\ldots sutsut \ldots)$.

Remark 3. The obvious use of the recurrences would be to find asymptotics for r_t^v. No work in this area has been done as yet (at least not to the author's knowledge).

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