COMBINATORICS OF DIAGONALLY CONVEX DIRECTED POLYOMINOES

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Abstract. A new bijection between the diagonally convex directed (dcd-) polyominoes and ternary trees makes it possible to enumerate the dcd-polyominoes according to several parameters (sources, diagonals, horizontal and vertical edges, target cells). For a part of these results we also give another proof, which is based on the cycle lemma. Thanks to the fact that the diagonals of a dcd-polyomino can grow at most by one, the problem of q-enumeration of this object can be solved by an application of Gessel's q-analog of the Lagrange inversion formula.

Résumé. Une nouvelle bijection entre les polyominos dirigés diagonalement convexes (polyominos d.d.c.) et les arbres ternaires permet l'énumération des polyominos d.d.c. suivant plusieur paramètres (sources, diagonales, arêtes horisontales et verticales, cellules cibles). Pour une partie de ces résultats nous donnons une preuve supplémentaire, qui est basée sur le lemme généralisée de Raney. Grâce au fait que les diagonales d'un polyomino d.d.c. croissent au plus d'une unité, leur q-énumération peut être résolue en utilisant le q-analogue de la formule d'inversion de Lagrange dû à Gessel.

1. Definitions, conventions and notations

Binomial coefficients. Generally, we adopt the convention: if a binomial coefficient has a negative numerator or denominator, then the value of the coefficient is zero. Exceptionally, for those binomial coefficients which are indicated by an ↘ arrow we stipulate: $\binom{-1}{-1}_{\searrow}=1$.

The *Gaussian polynomials* are defined by

$$\begin{bmatrix} k \\ r \end{bmatrix} = \frac{\left(1-q^k\right)\left(1-q^{k-1}\right)\cdots\left(1-q^{k-r+1}\right)}{\left(1-q\right)\left(1-q^2\right)\cdots\left(1-q^r\right)}$$

If $k < 0$ or $r < 0$, we agree that $\begin{bmatrix} k \\ r \end{bmatrix} = 0$. Again, the only exception is $\begin{bmatrix} -1 \\ -1 \end{bmatrix}_{\searrow} = 1$.

Formal sums. Let $h(z)=h(z\,;q)$ be a formal power series in z, whose coefficients are formal Laurent series in q. For $n \geq 0$ we set

$\left\langle z^n \right\rangle h(z)$:= the coefficient of z^n in $h(z)$

$h^{[n]}(z)$:= $h(z)h(qz)\cdots h\left(q^{n-1}z\right)$

$\tilde{h}^{[n]}(z)$:= $h\left(z;q^{-1}\right)h\left(q^{-1}z;q^{-1}\right)\cdots h\left(q^{-(n-1)}z;q^{-1}\right)$

Segments. For $n \in N$, $\underline{n}$ denotes the set $\{i \in N : 1 \le i \le n\}$.

Lattice paths. We shall only work with those lattice paths whose step-set is $\{(1,0),(0,1)\}$. A path with vertices $v_0, v_1, \ldots, v_n$ is "1/2-good" if all the vertices $v_1, \ldots, v_n$ lie in the half plane $y < \frac{1}{2}x$.

Ternary trees. Given a ternary tree T, we first visit the root and then traverse its subtrees from left to right. Let *u* and *v* be two vertices of T. We put $u<v$ iff the first visit to *u* precedes the first visit to *v*. Thus we obtain the *prefix order* on T (Fig. 1a). Further, we say that *l* is an odd (resp. even) leaf of T if $|\{k \text{ leaf of T}: k \trianglelefteq l\}|$ is an odd (resp. even) number (Fig. 1b).

(a) (b)

Figure 1. A ternary tree T. (a) The vertices of T are labeled after the prefix order. (b) The odd and even leaves of T are shown.

A *cell* is a unit square $[i, i+1]\times[j, j+1]$, where $i, j \in Z$. A *polyomino* is a finite union of cells which is connected and has no finite cut set. Two polyominoes will be considered equivalent if there is a translation that transforms one into the other (reflections and rotations are not allowed).

A *diagonal* of a polyomino P is a nonempty intersection between P and a diagonal "line" $\bigcup_{i \in Z} [i, i+1]\times[j-i, j-i+1]$, where $j \in Z$. A polyomino whose diagonals consist of consecutive cells is said to be *diagonally convex*.

A polyomino P is *directed* if it has the following property:

if c is a cell of P not lying on the southwestern-most diagonal of P, then $c-(1,0) \subseteq P$ or $c-(0,1) \subseteq P$ (or both).

The cells of the first (i. e. the southwestern-most) diagonal of a directed polyomino are called *sources*, those of the last diagonal are called *target cells*.

The polyomino in Fig. 2 is diagonally convex and directed. It has one source and two target cells.

To any 1- source diagonally convex directed (*dcd*-) polyomino P having k diagonals we associate a sequence $\langle p_1, \ldots, p_{2k}\rangle$ defined by:

$$p_1 = p_2 = 0, \qquad p_{2i-1} = X_{i-1} + 1 - X_i \quad (2 \le i \le k), \qquad p_{2i} = Y_{i-1} + 1 - Y_i \quad (2 \le i \le k)$$

where X_j (resp. Y_j) denotes the maximal abscissa (resp. ordinate) of the j^{th} diagonal of P. We call $\langle p_1, \ldots, p_{2k}\rangle$ the *sequence of losses* of P because p_{2i-1} (resp. p_{2i}) represents the number of unoccupied available places at the bottom (resp. top) of the polyomino's i^{th} diagonal. See Fig. 2 for an example.

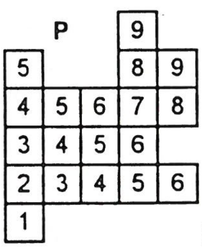

Figure 2. The sequence of losses of the polyomino P is: $p_1=p_2=0$, $p_3=1$, $p_4=\ldots=p_{11}=0$, $p_{12}=p_{13}=2$, $p_{14}=1$, $p_{15}=p_{16}=0$, $p_{17}=1$, $p_{18}=0$. For example, p_{11} is zero and p_{12} is two because the sixth diagonal occupies all the available places at the bottom and leaves two places free at the top.

2. Introduction

Polyominoes are used in physics and chemistry to model crystal growth, polymers etc. Despite strenuous efforts, counting the general polyominoes remains an unsolved problem. However, over the past 40 years considerable progress has been made in solving various simpler, but non-trivial models. For instance, nice results are known for the classes of parallelogram, column-convex, convex, directed and diagonally convex directed polyominoes. See [2] or [16] for a survey.

The dcd-polyominoes model was used for the first time by the physicists Privman and Švrakić [12; 13, p.99], who obtained the area generating function for the 1-source case. The enumeration by the perimeter was carried out later by Delest and Fédou [3] and Penaud [11].

In the present paper we enounce two easy propositions about dcd-polyominoes (Section 3), define a new bijection between the dcd-polyominoes and ternary trees (Sec. 4) and employ this new bijection in the dcd-polyominoes non-q-enumeration (Sec. 5). In Sec. 6 a part of the results of Sec. 5 is proved again by using Raney's generalized lemma. In Sec. 7 the dcd-polyominoes are q-enumerated with the aid of Gessel's q-analog of the Lagrange inversion formula.

3. Basic properties

It seems to be useful to state some simple facts about dcd-polyominoes, which can be proved easily by induction on k.

Proposition 1. Let P be a 1-source dcd-polyomino with k diagonals and let $<p_1, \ldots, p_{2k}>$ be its sequence of losses. Then

(a) For $j \in \underline{k}$, the j^{th} diagonal of P contains $j - \sum_{i=1}^{2j} p_i$ cells.

(b) P has $2 \cdot |\{j \in \underline{k} : p_{2j-1} = 0\}|$ horizontal edges and $2 \cdot |\{j \in \underline{k} : p_{2j} = 0\}|$ vertical edges.

Proposition 2. A sequence of nonnegative integers $<p_1, \ldots, p_{2k}>$ is the sequence of losses of some 1-source dcd-polyomino if and only if $\sum_{i=1}^{2j} p_i < j \quad \left(\forall j \in \underline{k}\right)$.

4. A new bijection between dcd-polyominoes with one source and ternary trees

Using Schützenberger's methodology [14], Delest and Fédou in [3] obtained the following interesting result: the number of 1-source dcd-polyominoes with k diagonals is equal to $\frac{1}{3k+1}\binom{3k+1}{k}$, which is also the number of ternary trees with k internal nodes. Although two different bijections between 1-source dcd-polyominoes and ternary trees were already given in [3] and [11], we believe that the following elementary one-to-one correspondence between those polyominoes and 1/2-good paths still deserves to be mentioned.

Let P be a 1-source dcd-polyomino with k diagonals and let $<p_1, \ldots, p_{2k}>$ be its sequence of losses. We assign to P a lattice path $B_1(P)$ starting at (0,0), ending at (2k+1,k), beginning with a horizontal step and making p_i verticals steps with abscissa i ($\forall i \in \underline{2k}$) (Fig. 3).

The north-most points of $B_1(P)$ with abscissas 2j-1 and 2j ($j \in \underline{k}$) are $Q_j = \left(2j-1, \sum_{i=1}^{2j-1} p_i\right)$ and $R_j = \left(2j, \sum_{i=1}^{2j} p_i\right)$, respectively. By Proposition 2, $\sum_{i=1}^{2j-1} p_i \le \sum_{i=1}^{2j} p_i < j$. Thus Q_j and R_j lie below the line $y = \frac{1}{2}x$ and $B_1(P)$ is a 1/2-good path.

Next, let W be a 1/2-good path from (0, 0) to (2k+1, k). It is well-known (see, for example, Dershowitz and Zaks [4]) that there is a unique ternary tree with k internal nodes $T=B_2(W)$ having the property:

the i^{th} ($i \in \underline{3k+1}$) vertex of T in prefix order is an internal node iff the i^{th} step *from the endpoint* of W is a vertical step (Fig. 4).

Theorem 1. The composition $B_2 \circ B_1$ is a bijection between the 1-source dcd-polyominoes with k diagonals and ternary trees with k internal nodes.

Remark. Notice that the dcd-polyominoes with r sources can be naturally embedded into those with one source. The embedding C_r consists in replacing the first diagonal of a given polyomino by a "triangle" $\triangle_r$ (Figs. 5 and 6).

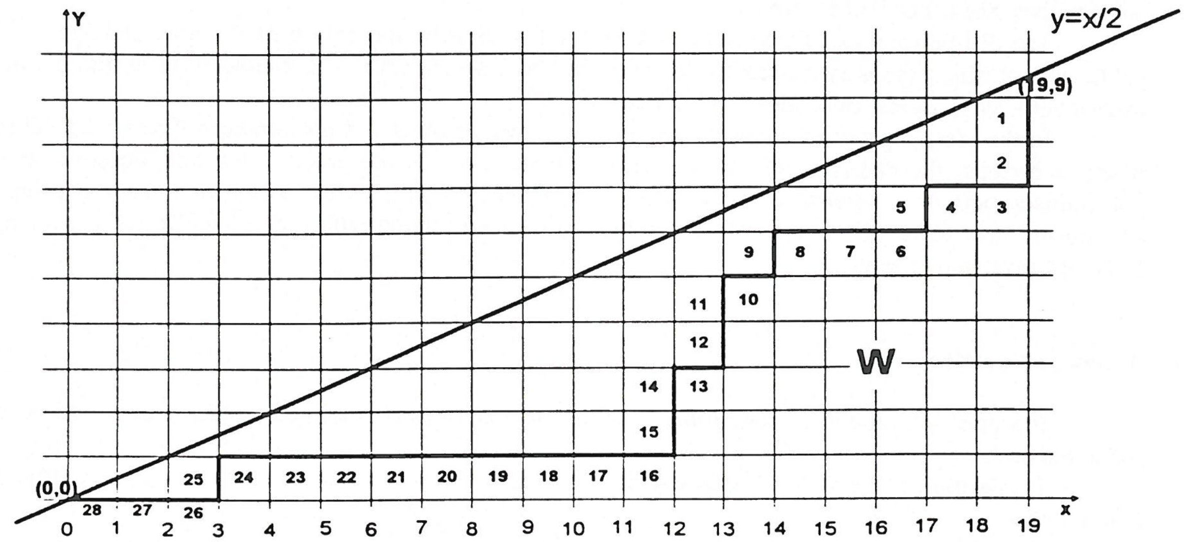

Figure 3. The path $W=B_1(P)$, where P is the polyomino of Fig. 2. The steps of W are numbered in the reverse order in view of Fig. 4.

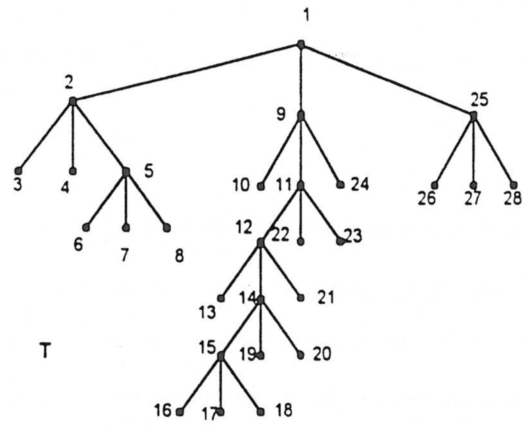

Figure 4. The ternary tree $T=B_2(W)$, where W is the path of Fig. 3.

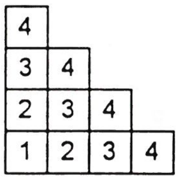

Figure 5. This is $\triangle_4$.

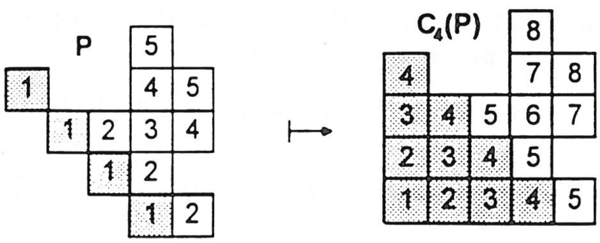

Figure 6. The polyominoes P and $C_4(P)$.

5. The non-q-enumeration with the aid of the new bijection

Definition 1. Let $\mathscr{P}(r, k, \ell, m, e)$ be the set of dcd-polyominoes having r sources, k diagonals, 2ℓ horizontal edges, 2m vertical edges and e target cells.

Let $\mathscr{T}(r, k, \ell, m, e)$ be the family of ternary trees T which have the following properties: i) T has r+k-1 internal nodes; ii) the event "the prefix order successor of an even (resp. odd) leaf of T is again a leaf" takes place ℓ (resp. m) times; iii) the left branch of T is of length e; iv) the prefix order list of vertices of T ends with at least 2r+1 leaves.

Proposition 3. The composition $B_2 \circ B_1 \circ C_\tau$ is a one-to-one correspodence between $\mathscr{P}(r, k, \ell, m, e)$ and $\mathscr{T}(r, k, \ell, m, e)$.

Proof. A closer look at the mappings B_2, B_1 and C_τ.

Thus it is of interest to study $f_{TL}(d, x, y, t, \ell)$, the generating function for non-trivial ternary trees in variables d=internal nodes, x=syllables (even leaf, leaf), y=syllables (odd leaf, leaf), t=length of the left branch, ℓ=final leaves.

We shall also need the functions $f_L(d, x, y, \ell) := f_{TL}|_{t=1}$, $f_T(d, x, y, t) := f_{TL}|_{\ell=1}$, $f(d, x, y) := f_{TL}|_{t=\ell=1}$, $g_{TL}(d, x, y, t, \ell) := f_{TL}(d, y, x, t, \ell)$ and similarly defined g_L, g_T and g.

Proposition 4. a) The coeff. of $d^k x^\ell y^m t^e$ in f_T (resp. g_T) is the number of 1-source dcd-polyominoes having k diagonals, 2ℓ (resp. 2m) horizontal edges, 2m (resp. 2ℓ) vertical edges and e target cells.

b) We have $f_T = g_T$.

Proof. Since every non-trivial ternary tree ends with at least three leaves, a) follows from Proposition 3 with r=1. Part b) follows from a) by reflecting the dcd-polyominoes in the line y=x.

Now we partition the non-trivial ternary trees into eight classes $\mathscr{T}_{000}, \mathscr{T}_{001}, \ldots, \mathscr{T}_{111}$: the trees belonging to the class $\mathscr{T}_{\alpha\beta\gamma}$ have a non-trivial left (resp. middle, right) subtree iff α (resp β, γ) is 1. Keeping in mind that every ternary tree has odd number of leaves, we find that the contributions to f_{TL} are:

from $\mathscr{T}_{000}$:	$dxyt\ell^3$	from $\mathscr{T}_{001}$:	$dytf_L$
from $\mathscr{T}_{010}$:	$dxt\ell g_L$	from $\mathscr{T}_{011}$:	$dtgf_L$
from $\mathscr{T}_{100}$:	$dxyt\ell^2 f_{TL}$	from $\mathscr{T}_{101}$:	$dytf_T f_L$
from $\mathscr{T}_{110}$:	$dxt\ell f_T g_L$	from $\mathscr{T}_{111}$:	$dtf_T g f_L$

On account f=g, for ℓ=1 these contributions add up to

$$f_T = dt\,(f_T+1)\,(f+x)\,(f+y) \tag{1}$$

For the function $f_1 := f\,(1+f)^{-1}$ we have $f = f_1(1-f_1)^{-1}$. By letting t=1 in (1) we obtain the following equation for f_1 (in the form appropriate for Lagrange inversion):

$$f_1 = d\,[f_1(1-f_1)^{-1}+x\,]\,[f_1(1-f_1)^{-1}+y]\ . \tag{2}$$

Further, by solving (1) with respect to f_T we get

$$f_T = \frac{dt(f+x)(f+y)}{1-dt(f+x)(f+y)} = \frac{tf_1}{1-tf_1} = \sum_{e\geq 1} f_1^e t^e . \tag{3}$$

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Theorem 2. The number of 1-source dcd-polyominoes having k diagonals, 2ℓ horizontal edges, 2m vertical edges and e target cells is equal to

$$\frac{e}{k}\binom{k-e-1}{2k-\ell-m-1}\binom{k}{\ell}\binom{k}{m}\ .$$

Proof. By Proposition 4.a) and (3), the number of polyominoes in question is $< d^k x^l y^m t^e > f_T =< d^k x^l y^m > f_1^e$. Now the theorem follows by an application of the Lagrange inversion formula [8] to (2).

Theorem 2 generalizes the results of Delest, Fédou and Penaud, who obtained the coefficients of f_T in three cases: x=y=t=1; d=t=1 & x=y ; x=y=1.

Let us make an agreement: the equation obtained by swapping x and y in a given equation (n) will be denoted by (n').

In the case t=1 & $l \neq 1$ the eight contributions to f_{TL} sum to

$$[1-dxyl^2 - d\,(f+1)\,(f+y)]\,f_L - dxl\,(f+1)\,g_L = dxyl^3 \quad . \tag{4}$$

Using (1) with t=1, we can write the equation (5):=$x^{-1}(f+x)\cdot(4)$ - $l\cdot(4')$ in a way that there are no f's in the coefficients of f_L and g_L. Then the system (5) & (5') gives us f_L and g_L as linear functions of f.

Next we substitute these expressions for f_L and g_L into what the sum (contribution from $\mathcal{J}_{000}$)+...+(contribution from $\mathcal{J}_{111}$) is for $t \neq 1$ and $l \neq 1$. By a rather long algebra including one more application of (1) we obtain

$$f_{TL} = dl^3 \frac{\left[(xl+y)A + dxyl^2\right]A_T f_T + xyt\left[\left(1-l^2\right)A + dxl^2(1-l)\right]A}{\left[\left(A+dxl^2\right)\left(A+dyl^2\right) - l^2 A^2\right]A_T} \quad , \tag{6}$$

where $A_T= 1\text{-}dxytl^2$ and $A= 1\text{-}dxyl^2$.

Then we define f_{TL}^{+} (resp. f_{TL}^{-}) to be the sum of terms of f_{TL} containing even (resp. odd) powers of l,

$f_{ST}^{\bullet}$ to be $f_{TL}^{+} + l f_{TL}^{-}$ with l substituted by $s^{1/2}$ and

f_{ST} to be $d(1-s)^{-1}\cdot\left(sf_T - f_{ST}^{\bullet}\right)$ with s substituted by sd^{-1}.

Theorem 3. a) f_{ST} is the (s=sources, d=diagonals, x=1/2 horizontal perimeter, y=1/2 vertical perimeter, t=target cells) generating function for dcd-polyominoes.

b) We have $f_{ST} = s\dfrac{f_T - sxyt(1-sxyt)^{-1}}{\left[1+sx(1-sxy)^{-1}\right]\left[1+sy(1-sxy)^{-1}\right]-sd^{-1}}$.

c) The number of dcd-polyominoes with r sources, k diagonals, 2ℓ horizontal edges, 2m vertical edges and e target cells is equal to

$$\sum_{a,b,c\geq 0} \frac{(-1)^{b+c}e}{a+k}\binom{a+b}{a}\binom{a+c}{a}\binom{r-a-2}{b+c-1}\binom{a+k-e-1}{2r+2k-\ell-m-b-c-3}\binom{a+k}{a+b-r+l+1}\binom{a+k}{a+c-r+m+1}$$

Proof. a) It follows from the definitions of f_{ST} and $\mathcal{J}$(r, k, l, m, e) that $< s^r d^k x^l y^m t^e > f_{ST} = |\mathcal{J}(r, k, l, m, e)|$ By Proposition 3, $|\mathcal{J}(r, k, l, m, e)|=|\mathcal{P}(r, k, l, m, e)|$. Thus the assertion is proved.

b) follows easily from (6).

c) First we expand the rhs of the formula in b) as a geometric series. Then the formula follows by using Proposition 4.a) and Theorem 2.

6. The non-q-enumeration with the aid of Raney's generalized lemma

Let P be an element of $\mathcal{P}(1)$:=$\mathcal{P}$(1, k, l, m, e), the family of 1-source dcd-polyominoes having k diagonals, 2 ℓ horizontal edges, 2m vertical edges and e target cells. Let $<p_1, \ldots, p_{2k}>$ be the sequence of losses of P. Now we define a kind of "Raney mapping" by

$$R(P) := < 1, -p_1, -p_2, 1, -p_3, -p_4, \ldots, 1, -p_{2k-1}, -p_{2k}> \quad . \tag{7}$$

Using Propositions 1 and 2, it is easy to characterize the image $R(\mathcal{S}(1))$ of the (injective) correspodence R. It consists of those integer sequences $< 1, a_1, a_2, 1, a_3, a_4, \ldots, 1, a_{2k-1}, a_{2k}>$ which satisfy:

i) $a_i \le 0 \quad (\forall i \in \underline{2k})$;

ii) $|\{i \in \underline{k} : a_{2i-1}\}| = \ell$;

iii) $|\{i \in \underline{k} : a_{2i} = 0\}| = m$;

iv) $\sum_{i=1}^{j}(1 + a_{2i-1} + a_{2i}) > 0 \quad (\forall j \in \underline{k})$;

v) $\sum_{i=1}^{k}(1 + a_{2i-1} + a_{2i}) = e$

Let $\mathcal{A}$ be the set of integer sequences having all the above properties, except the property *iv*). To define an element of $\mathcal{A}$ we first choose any l odd positions (indices) and any m even positions (indices) where we want $a_i=0$. In the remaining $2k-l-m$ positions we can put any composition of the number $-(k-e)$ with parts $a_i<0$. Thus the number of ways to define an element of $\mathcal{A}$ is

$$|\mathcal{A}| = \binom{k}{l}\binom{k}{m}\binom{k-e-1}{2k-l-m-1}. \tag{8}$$

Notice that for $s_1 = < 1, a_1, a_2, 1, a_3, a_4, \ldots, 1, a_{2k-1}, a_{2k} > \in \mathcal{A}$ its cyclic shifts s_1, $s_2 = < 1, a_3, a_4, \ldots, 1, a_{2k-1}, a_{2k}, 1, a_1, a_2 >, \ldots, s_k = < 1, a_{2k-1}, a_{2k}, 1, a_1, a_2, \ldots, 1, a_{2k-3}, a_{2k-2} >$ belong to $\mathcal{A}$ too. Raney's generalized lemma [9, p. 348] tells us that exactly e of the sequences $s_1, s_2, \ldots, s_k$ have all partial sums positive. This is equivalent to say that exactly e of the sequences $s_1, s_2, \ldots, s_k$ belong to $R(\mathcal{S}(1))$.

Imagine all $|\mathcal{A}|$ elements of $\mathcal{A}$ together with all k of their cyclic shifts being listed in an array. Since the columns of the array are permutations of $\mathcal{A} \supseteq R(\mathcal{S}(1))$, the elements of $R(\mathcal{S}(1))$ occur $|R(\mathcal{S}(1))|$ times in each column and $k|R(\mathcal{S}(1))|$ times in the whole array. Since the elements of $R(\mathcal{S}(1))$ occur e times in each row, they occur $e|\mathcal{A}|$ times in the whole array. Therefore $k|R(\mathcal{S}(1))| = e|\mathcal{A}|$ and

$$|\mathcal{S}(1)| = |R(\mathcal{S}(1))| = \frac{e}{k}|\mathcal{A}| = \frac{e}{k}\binom{k-e-1}{2k-l-m-1}\binom{k}{l}\binom{k}{m}.$$

Thus we have got a new proof of Theorem 2. Let us mention that Raney's lemma can also be applied in the enumeration of column-convex directed polyominoes [6].

7. The q-enumeration

In this section the generating functions for dcd-polyominoes have four variables: d=diagonals, x=1/2 horizontal perimeter, y=1/2 vertical perimeter, q=area. Instead of $\varphi(d,x,y,q)$ we usually write φ or $\varphi(d)$.

Definition 2. As before, Δ_1 denotes the one-cell polyomino. Let $\mathcal{D}_\beta$ be the set of one-source dcd-polyominoes with β target cells. Let $\mathcal{D}_{\alpha\beta}$ be the subset of $\mathcal{D}_\beta$ containg those polyominoes whose next to last diagonal is of length α. Further, let ${}_\alpha\mathcal{D}_\beta$ stand for the set of dcd-polyominoes with α sources and β target cells (thus $\mathcal{D}_\beta = {}_1\mathcal{D}_\beta$).

The generating functions for the sets $\mathcal{D}_\beta$, $\mathcal{D}_{\alpha\beta}$ and ${}_\alpha\mathcal{D}_\beta$ will be denoted by f_β, $f_{\alpha\beta}$ and ${}_\alpha f_\beta$, respectively.

Definition 3. The number of diagonals, horizontal perimeter, vertical perimeter and area of a given dcd-polyomino P will be denoted by D(P), H(P), V(P) and Area(P), respectively.

Let P be an element of $\mathcal{D}_e$. As the diagonals of P grow at most by one, for every $i \in \underline{e}$ there is a number z(i) such that the $z(i)^{th}$ diagonal is the last diagonal of length i in P. For convenience, we put z(0)=0.

For $i \in \underline{e}$, the $z(i-1)+1^{th}, z(i-1)+2^{th}, \ldots, z(i)^{th}$ diagonal of P form a polyomino belonging to ${}_i\mathcal{D}_i$. Let us denote that polyomino by $\Pi_i(P)$. Let $\pi_i(P)$ be that what remains of $\Pi_i(P)$ after we cut off the i-1 top cells from each of its diagonals. It is easy to see that $\pi_i(P) \in \mathcal{D}_1$. Thus we have associated to $P \in \mathcal{D}_e$ the e-tuple $\pi(P) = (\pi_1(P), \ldots, \pi_e(P)) \in \mathcal{D}^e_1$. See Fig.7.

Clearly, $D(P)=\sum_{i=1}^{e} D(\pi_i(P))$. The sequence of losses of P can be obtained from those of $\pi_i(P)$'s by concatenation. Hence by Proposition 1.b) $H(P)=\sum_{i=1}^{e} H(\pi_i(P))$ and $V(P)=\sum_{i=1}^{e} V(\pi_i(P))$. But with the area the things are different: $\text{Area}(P)=\sum_{i=1}^{e}\left[\text{Area}(\pi_i(P))+(i-1)D(\pi_i(P))\right]$.

The above properties of the decomposition $\pi:\mathcal{A}_e\to\mathcal{A}^e$, lead us to the conclusion:

$$f_e(d)=f_1(d)f_1(qd)\cdots f_1(q^{e-1}d)=f_1^{[e]}(d) \qquad (\forall e\in N). \tag{9}$$

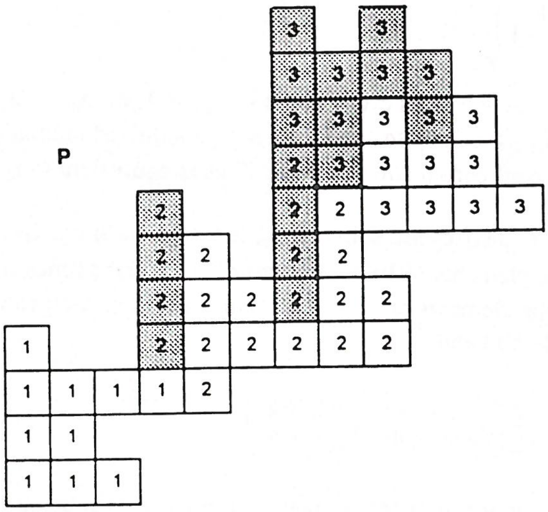

Figure 7. The decomposition π. The cells of $\Pi_i(P)$ $(i=1,2,3)$ are labeled i. The shaded cells are those being cancelled from Π_i's to obtain π_i's.

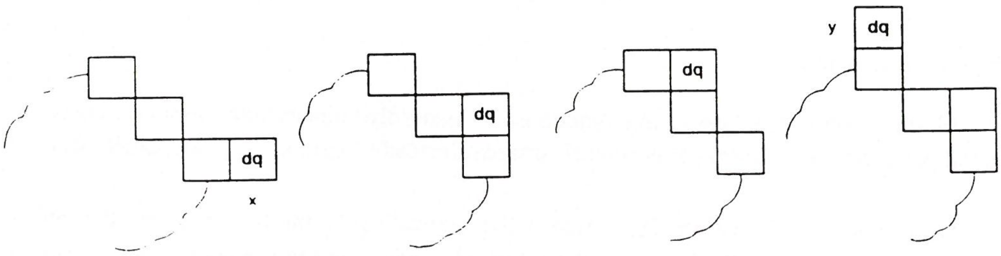

Figure 8. The four types of elements of $\mathcal{A}_{3,1}$. Their contributions to $f_{3,1}$ are, from left to right, $dqxf_3$, dqf_3, dqf_3 and $dqyf_3$. Thus $f_{3,1}=dq(x+y+2)f_3$

We see that the function f_1 is standing out among the f_e's. So let us take a closer look at f_1.

Since the sets $\{\square_1\}$ and $\mathcal{A}_{e1}(e\in N)$ form a partition of $\mathcal{A}_1$, we have $f_1(d)=dqxy+\sum_{e\geq 1} f_{e1}(d)$. Then, Figure 8 should suffice to convince the reader that $f_{e1}(d)=dq(x+y+e-1)f_e(d)$. These considerations together with (9) imply

$$f_1(d)=dq\left\{xyf_1^{[0]}(d)+\sum_{e\geq 1}(x+y+e-1)f_1^{[e]}(d)\right\}. \tag{10}$$

Fortunately, we need not bother about how to solve (10), because Gessel's q-analog of the Lagrange inversion formula [7] comes to our aid. Indeed, the q-analog has following obvious consequence:

Corollary 2. Let $f_1(d)=f_1(d,q)$ satisfy

$$f_1(d)=dq\sum_{e\geq 0} g_e f_1^{[e]}(d) \quad . \tag{11}$$

where the g_e are indeterminates. Let $g(t)=\sum_{e\geq 0} g_e t^e$. Then for $e\in N$,

$$f_1^{[e]}(d)=\frac{\sum_{k\geq 1} d^k q^{\frac{(k+1)k}{2}} \left\langle t^{k-e}\right\rangle \tilde{g}^{[k]}(q^{-1}t)}{1+\sum_{k\geq 1} d^k q^{\frac{(k+1)k}{2}} \left\langle t^{k}\right\rangle \tilde{g}^{[k]}(q^{-1}t)} \tag{12}$$

In our problem

$$g(t)=xyt^0+\sum_{e\geq 1}(x+y+e-1)t^e=xy\left[1+(1-x)x^{-1}t\right]\left[1+(1-y)y^{-1}t\right](1-t)^{-2} \tag{13}$$

and ($\forall i\in Z,\ k\in N_0$)

$$\left\langle t^i\right\rangle \tilde{g}^{[k]}(q^{-1}t)=\sum_{a,b,c\geq 0}\begin{bmatrix}a+k-1\\ k-1\end{bmatrix}\begin{bmatrix}i+k-a-b-c-1\\ k-1\end{bmatrix}\begin{bmatrix}k\\ b\end{bmatrix}\begin{bmatrix}k\\ c\end{bmatrix}(1-x)^b x^{k-b}(1-y)^c y^{k-c} q^{\frac{b(b-1)+c(c-1)}{2}-ik} \quad . \tag{14a}$$

The computation of (14a) includes the use of two identities for Gaussian polynomials [10, p.18, ex.3]. In the case x=y=1 (14a) simplifies to

$$\left\langle t^i\right\rangle \tilde{g}^{[k]}(q^{-1}t)=x^k y^k q^{-ik}\sum_{a\geq 0}\begin{bmatrix}a+k-1\\ k-1\end{bmatrix}\begin{bmatrix}i+k-a-1\\ k-1\end{bmatrix} \tag{14b}$$

Thus the generating function for 1-source dcd-polyominoes with e target cells f_e is given by (9), (12) and (14a,b). This results improves that obtained by Privman and Švrakić [12; 13, p. 99]. Observe that our formula for f_e is what Bousquet-Mélou and Fédou [1] would call a formula perfectly developed in d.

In the case of r>1 sources a literal application of the q-analog is not possible. However, considerations similar to those in Gessel's proof lead us to the following

Theorem 4. The generating function for r-source dcd-polyominoes with e target cells ($r,e\in N$) is

$${}_rf_e(d)=d^{-(r-1)}q^{-\frac{r(r-1)}{2}}(xy)^r\sum_{j\geq r} d^j q^{\frac{(j+1)j}{2}}\left[\left\langle t^{j-e}\right\rangle-f_e(d)\left\langle t^j\right\rangle\right]\tilde{g}^{[j-r]}(q^{-1}t). \tag{15}$$

Proof will be given in a future paper of ours.

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