STATISTICS FOR SPECIAL q,t-KOSTKA POLYNOMIALS

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ABSTRACT. Kirillov and Reshetikhin introduced rigged configurations as a new way to calculate the entries $K_{\lambda\mu}(t)$ of the Kostka matrix. Macdonald defined the two-parameter Kostka matrix whose entries $K_{\lambda\mu}(q,t)$ generalize $K_{\lambda\mu}(t)$. We use rigged configurations and a formula of Stembridge to provide a combinatorial interpretation of $K_{\lambda\mu}(q,t)$ in the case where μ is a partition with no more than two columns. In particular, we show that in this case, $K_{\lambda\mu}(q,t)$ has nonnegative coefficients.

Kirillov and Reshetikhin ont introduit le concept de "rigged configurations" fournissant un nouveau moyen de calculer les coefficients $K_{\lambda\mu}(t)$ de la matrice de Kostka. Macdonald définit la matrice de Kostka à deux paramètres dont les coefficients $K_{\lambda\mu}(q,t)$ généralisent les $K_{\lambda\mu}(t)$. Nous utilisous les "rigged configurations" et une formule de Stembridge pour fournir une interprétation combinatoire des $K_{\lambda\mu}(q,t)$ dans le cas où le diagramme de μ ne contient pas plus de deux colonnes. Nous montrons, en particulier, que dans ce cas, $K_{\lambda\mu}(q,t)$ a tous ses coefficients non-négatifs.

1. INTRODUCTION

In [Mac2], Macdonald defined a basis $P_\lambda(q,t)$ of the ring of symmetric functions. Hall–Littlewood symmetric functions, Jack polynomials, Schur functions, and zonal polynomials are all either limiting or special cases of the $P_\lambda(q,t)$s. He also defined a transition matrix, whose entries are denoted $K_{\lambda\mu}(q,t)$, between a renormalized version of the $P_\lambda(q,t)$s and another basis S_λ of the ring of symmetric functions. $(K_{\lambda\mu}(q,t))_{\lambda,\mu\vdash n}$ generalizes the Kostka matrix $(K_{\lambda\mu})_{\lambda,\mu\vdash n}$.

Macdonald conjectured that the entries in the two parameter Kostka matrix are polynomials in q and t with nonnegative integer coefficients. All that is known a priori is that the entries are rational functions of q and t. Garsia and Haiman have constructed, for each partition μ of n, a finite-dimensional bigraded S_n-module whose irreducible multiplicities they conjecture to be rescaled versions of the entries

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$K_{\lambda\mu}(q,t)$. In [GH], they give several constructions of S_n-modules conjectured to have this property, together with an announcement of the special cases for which they can prove their conjecture. The main special cases correspond to the entries $K_{\lambda\mu}(q,t)$ in which μ is either a hook, or has at most two rows or two columns. Their results do not provide any explicit combinatorial interpretation of the entries.

In the paper [Ste], Stembridge gave a direct proof of the hook case of Macdonald's conjecture, and gave a formula for the polynomial in the two-column case, which proves the entries are polynmials. His formula is

$$K_{\lambda 2^r 1^{n-2r}}(q,t) = \sum_{s=0}^{r} q^{r-s}(t^{n-r}q;t^{-1})_s \begin{bmatrix} r \\ s \end{bmatrix}_t K_{\lambda 2^s 1^{n-2s}}(t) \tag{1}$$

where $K_{\lambda 2^s 1^{n-2s}}(t)$ is the Kostka (charge) polynomial [Macl] and $|\lambda| = n$. We use (1) to show that there are statistics c_r and cut_r defined on the set $\mathcal{M}_0^0$ of Kirillov and Reshetikhin rigged configurations which correspond to standard Young tableaux of shape λ such that

Theorem 1.1.

$$K_{\lambda 2^r 1^{n-2r}}(q,t) = \sum_{(\alpha(0),L)\in\mathcal{M}_0^0} q^{cut_r(\alpha(0),L)} t^{c_r(\alpha(0),L)} \tag{2}$$

Theorem 1.1 proves that $K_{\lambda 2^r 1^{n-2r}}(q,t)$ has nonnegative coefficients.

This paper is divided into 5 sections. In Section 2, we explain the necessary Kirillov and Reshetikhin material, and introduce notation. In Section 3 we rewrite $K_{\lambda 2^r 1^{n-2r}}(q,t)$ as a sum of "difference" polynomials. In Section 4 we show the difference polynomials are nonnegative by showing they are generating functions for sets $\mathcal{M}_m^d$ of rigged configurations. In Section 5 we finish proving Theorem 1.1.

2. Kirillov and Reshetikhin's Rigged Configurations

Kirillov and Reshetikhin [KR1][KR2] introduced rigged configrations as a new way to calculate $K_{\lambda\mu}(t)$ for any pair of partitions λ and μ of n. Fix n, a positive integer and λ, a partition of n. Let $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^x)$ be a sequence of partitions such that $|\alpha^i| = \lambda_{i+1} + \lambda_{i+2} + \ldots$. For any such sequence, if μ is any partition of n, let $\alpha(\mu)$ be the sequence of partitions $(\mu' = \alpha^0, \alpha^1, \ldots, \alpha^x)$ and if m is a nonnegative integer, let $\alpha(m) = ((n-m, m), \alpha^1, \alpha^2, \ldots, \alpha^x)$. $\alpha(\mu)$ is called a configuration.

Define, for $k \geq 1$,

$$P_l^k(\alpha(\mu)) = \sum_{i=1}^{l}(\alpha_i^{k-1} - 2\alpha_i^k + \alpha_i^{k+1}) \tag{3}$$

A rigged configuration is a pair$(\alpha(\mu), L)$, where L labels the columns of the partitions in α. In particular,

(1) $0 \leq L_i^k \leq P_l^k(\alpha(\mu))$ for $k \geq 1$ and $1 \leq i \leq \alpha_1^k$ where the length of column i of α^k is l, .

(2) If column i of α^k has the same length as column $i+1$, then $L_i^k \leq L_{i+1}^k$.

Not all sequences $\alpha(\mu)$ will have labels. Kirillov and Reshetikhin call a sequence $\alpha(\mu)$ a μ-admissible λ configuration when there is at least one labelling function L, that is, when $P_i^k(\alpha(\mu)) \geq 0$. They have defined a bijection between μ-admissible λ rigged configurations and column strict tableaux of shape λ and content μ.

Further, Kirillov and Reshetikhin define the charge of a rigged configuration $c((\alpha(\mu), L))$. Let

$$c(\alpha(\mu)) = n(\mu) - \sum_{i\geq 1}\mu_i'\alpha_i^1 + \sum_{k,i\geq 1}\alpha_i^k(\alpha_i^k - \alpha_i^{k+1}),$$

where $n(\mu) = \sum(i-1)\mu_i$.

Then

$$c((\alpha(\mu), L)) = c(\alpha(\mu)) + \sum_{i,k\geq 1} L_i^k.$$

The Kirillov and Reshetikhin theorem is now

$$K_{\lambda\mu}(t) = \sum_{(\alpha(\mu),L)} t^{c((\alpha(\mu),L))}$$

where the sum is over all μ-admissible λ rigged configurations.

In this paper, $\mu = (2^m 1^{n-2m})$, so that $\alpha(\mu) = \alpha(m)$. We need several properties which are peculiar to this case.

(1) Let

$$c'(\alpha) = \sum_{k,i\geq 1}\alpha_i^k(\alpha_i^k - \alpha_i^{k+1}). \tag{4}$$

Then

$$c((\alpha(m))) = \binom{n-m}{2} + \binom{m}{2} - (n-m)\alpha_1^1 - m\alpha_2^1 + c'(\alpha) \tag{5}$$

(2) By the definition of $P_i^k(\alpha(m))$

$$P_i^k(\alpha(m+1)) = \begin{cases} P_i^k(\alpha(m)) - 1 & \text{if } i = k = 1 \\ P_i^k(\alpha(m)) & \text{otherwise} \end{cases} \tag{6}$$

Finally, let $\mathcal{M}_m^0 = \{(\alpha(m), L) | \alpha(m)$ is a $(n-m, m)$-admissable λ configuration and L is a label$\}$. Call $\mathcal{M}_0^0$ the set of standard rigged configurations.

3. Difference Polynomials

In this section, we rewrite Stembridge's formula (1), changing it to (8). (8) is crucial because we will show the polynomials $M_{r-k}^k(t)$ are generating functions for sets of rigged configurations.

Lemma 3.1. *The coefficient of q^k in $K_{\lambda 2^r 1^{n-2r}}(q,t)$ is*

$$\sum_{s=0}^{r} (-1)^{k-(r-s)} t^{(k-r+s)(n-r-s+1)+\binom{k-(r-s)}{2}} \begin{bmatrix} s \\ r-k \end{bmatrix}_t \begin{bmatrix} r \\ s \end{bmatrix}_t K_{\lambda 2^s 1^{n-2s}}(t) \tag{7}$$

Proof. Lemma 3.1 is a consequence of the q-binomial theorem [And, 3.3.6]. □

Definition 3.1. *Define the polynomials $M_m^d(t)$ recursively by $M_m^0(t) = K_{\lambda 2^m 1^{n-2m}}(t)$ and $M_m^{d+1}(t) = M_m^d(t) - t^{n-2m-(d+1)} M_{m+1}^d(t)$.*

Lemma 3.2. *The coefficient of q^k in $K_{\lambda 2^r 1^{n-2r}}(q,t)$ is $\left[\begin{smallmatrix} r \\ k \end{smallmatrix}\right]_t M_{r-k}^k(t)$, so that*

$$K_{\lambda 2^r 1^{n-2r}}(q,t) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} M_{r-k}^k(t) q^k \tag{8}$$

Proof: We use induction on $r+k$ and (7) to show that $M_{r-(k+1)}^{k+1}(t) = \left[\begin{smallmatrix} r \\ k+1 \end{smallmatrix}\right]_t^{-1}$(coeff. of q^{k+1} in $K_{\lambda 2^r 1^{n-2r}}(q,t)$). The lemma is true if $r = 0$ or $k = 0$. Assume $M_{b-a}^a = \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]_t^{-1}$(coeff. of q^a in $K_{\lambda 2^b 1^{n-2b}}(q,t)$) if a and b are nonnegative integers such that $a + b \leq r + k$.

$$
\begin{aligned}
M^{k+1}_{r-(k+1)}(t) &= M^{k+1}_{(r-1)-k}(t) \\
&= M^{k}_{(r-1)-k}(t) - t^{n-2(r-1-k)-(k+1)} M^{k}_{(r-1)-k+1}(t) \\
&= \begin{bmatrix} r-1 \\ k \end{bmatrix}_t^{-1} (\text{coeff. of } q^k \text{ in } K_{\lambda 2^{(r-1)}1^{n-2(r-1)}}(q,t)) - \\
&\quad t^{n-2(r-1-k)-(k+1)} \begin{bmatrix} r \\ k \end{bmatrix}_t^{-1} (\text{coeff. of } q^k \text{ in } K_{\lambda 2^r 1^{n-2r}}(q,t)) \\
&= \begin{bmatrix} r-1 \\ k \end{bmatrix}_t^{-1} \sum_{s=0}^{r-1} (-1)^{k-(r-1-s)} t^{(k-(r-1)+s)(n-(r-1)-s+1)+\binom{k-(r-1-s)}{2}} \times \\
&\quad \begin{bmatrix} s \\ r-1-k \end{bmatrix}_t \begin{bmatrix} r-1 \\ s \end{bmatrix}_t K_{\lambda 2^s 1^{n-2s}}(t) - \\
&\quad t^{n-2(r-1-k)-(k+1)} \begin{bmatrix} r \\ k \end{bmatrix}_t^{-1} \sum_{s=0}^{r} (-1)^{k-(r-s)} t^{(k-r+s)(n-r-s+1)+\binom{k-(r-s)}{2}} \times \\
&\quad \begin{bmatrix} s \\ r-k \end{bmatrix}_t \begin{bmatrix} r \\ s \end{bmatrix}_t K_{\lambda 2^s 1^{n-2s}}(t)
\end{aligned}
$$

We will finish proving the lemma assuming $0 \leq s \leq r-1$. The case $s = r$ is a degenerate special case of what follows. Now we need to show the coefficient of $K_{\lambda 2^s 1^{n-2s}}(t)$, $0 \leq s \leq r-1$, in the last expression in this string of equalities is equal to

$$
(-1)^{k+1-r+s} t^{(k+1-r+s)(n-r-s+1)+\binom{k+1-(r-s)}{2}} \begin{bmatrix} r \\ k+1 \end{bmatrix}^{-1} \begin{bmatrix} s \\ r-1-k \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix},
$$

which is the coefficient of $K_{\lambda 2^s 1^{n-2s}}(t)$ in $\left[\begin{smallmatrix} r \\ k+1 \end{smallmatrix}\right]^{-1}$(coefficient of q^{k+1} in $K_{\lambda 2^r 1^{n-2r}}(q,t)$).

The coefficient of $K_{\lambda 2^s 1^{n-2s}}(t)$ in the last expression in the string of equalities is equal to

$$
\begin{gathered}
(-1)^{k+1-r+s} t^{(k+1-r+s)(n-r-s+1)+\binom{k+1-(r-s)}{2}} \times \\
\left(\begin{bmatrix} r-1 \\ k \end{bmatrix}^{-1} t^{k+1-r+s} \begin{bmatrix} s \\ r-1-k \end{bmatrix} \begin{bmatrix} r-1 \\ s \end{bmatrix} + \right. \\
\left. t^{(n-2(r-1-k)-(k+1))-(n-r-s+1)-(k-r+s)} \begin{bmatrix} r \\ k \end{bmatrix}^{-1} \begin{bmatrix} s \\ r-k \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \right)
\end{gathered}
$$

Using the q-factorial definition of the q-binomial coefficient, this turns into

$$
\begin{gathered}
(-1)^{k+1-r+s} t^{(k+1-r+s)(n-r-s+1)+\binom{k+1-(r-s)}{2}} \begin{bmatrix} r \\ k+1 \end{bmatrix}^{-1} \begin{bmatrix} s \\ r-1-k \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \times \\
\left(\frac{r_t}{(k+1)_t} t^{k+1-r+s} \times \frac{(r-s)_t}{r_t} + \frac{(r-k)_t}{(k+1)_t} \times \frac{(s-r+k+1)_t}{(r-k)_t} \right),
\end{gathered}
$$

where $n_t = 1 - t^n$. The quantity in the brackets boils down to one, so we are done.
□

4. Difference polynomials are nonnegative

In this section, we show that the polynomials $M_m^k(t)$, defined in the last section, are the generating functions for sets of rigged configurations, thus showing that $K_{\lambda 2^r 1^{n-2r}}(q,t)$ has nonnegative coefficients.

Definition 4.1. *Let* $\mathcal{M}_m^k = \{(\alpha(m), L) \in \mathcal{M}_m^0 | L^1_{\alpha_2^1+1} = L^1_{\alpha_2^1+2} = \cdots = L^1_{\alpha_2^1+k} = 0\}$. *Note that if* $\alpha_1^1 - \alpha_2^1 < k$, *then* $(\alpha(m), L) \notin \mathcal{M}_m^k$ *for any* L.

Lemma 4.1. *The generating function for the set* $\mathcal{M}_m^k$ *is* $M_m^k(t)$*; that is,*

$$\sum_{(\alpha(m),L)\in\mathcal{M}_m^k} t^{c((\alpha(m),L))} = M_m^k(t)$$

Proof: The proof is by induction. The lemma is true if $k = 0$, by the orginal Kirillov and Reshetikhin result. The definition of $M_m^{k+1}(t)$ is $M_m^k(t) - t^{n-2m-(k+1)} M_{m+1}^k(t)$. In order to prove the lemma we need an injection $\phi_k : \mathcal{M}_{m+1}^k \longrightarrow \mathcal{M}_m^k$ such that

(1) $c(\phi_k((\alpha(m+1), L))) = c((\alpha(m+1), L)) + n - 2m - (k+1)$ and
(2) $(\alpha(m), L) \in \mathcal{M}_m^k$ is not in the image of ϕ_k if and only if $(\alpha(m), L) \in \mathcal{M}_m^{k+1}$.

Let $\phi_k(\alpha(m+1), L) = (\alpha(m), \hat{L})$, where

$$\hat{L_j^i} = \begin{cases} L_j^i + 1 & \text{if } i = 1 \text{ and } j \geq \alpha_2^1 + k + 1 \\ L_j^i & \text{otherwise} \end{cases}$$

Please note $(\alpha(m), \hat{L}) \in \mathcal{M}_m^k$ by (6) and also that 2. above is satisfied.

To see that 1. above is satisfied,

$$c((\alpha(m+1), L)) = \binom{n-(m+1)}{2} + \binom{m+1}{2} - (n-(m+1))\alpha_1^1 - (m+1)\alpha_2^1 + c'(\alpha) + \sum_{i,j\geq 1} L_j^i$$

and

$$\begin{aligned} c(\phi_k(\alpha(m+1),L)) &= c(\alpha(m),\hat{L}) \\ &= \binom{n-m}{2}+\binom{m}{2}-(n-m)\alpha_1^1-m\alpha_2^1+c'(\alpha)+\sum_{i,j\geq 1}\hat{L_j^i} \\ &= \binom{n-m}{2}+\binom{m}{2}-(n-m)\alpha_1^1-m\alpha_2^1+c'(\alpha)+\sum_{i,j\geq 1}L_j^i+(\alpha_1^1-(\alpha_2^1+k)) \\ &= \binom{n-(m+1)}{2}+n-m+\binom{m+1}{2}-(m+1)-(n-(m+1))\alpha_1^1- \\ &\quad (m+1)\alpha_2^1+\sum_{i,j\geq 1}L_j^i-k \\ &= c(\alpha(m+1),L)+n-2m-k-1 \end{aligned}$$

□

A consequence of Lemma 3.1 and Lemma 3.2 is the following formula:

$$M_m^k(t)=t^{\binom{n-m}{2}+\binom{m}{2}}\sum_{j=0}^{k}(-1)^j t^{-\binom{j}{2}-\binom{n-(m+j)}{2}-\binom{m+j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_{t^{-1}}K_{\lambda 2^{m+j}1^{n-2(m+j)}}(t).$$

Lemma 4.1 therefore has the following corollary.

Corollary 4.1. *The polynomial*

$$t^{\binom{n-m}{2}+\binom{m}{2}}\sum_{j=0}^{k}(-1)^j t^{-\binom{j}{2}-\binom{n-(m+j)}{2}-\binom{m+j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_{t^{-1}}K_{\lambda 2^{m+j}1^{n-2(m+j)}}(t)$$

has nonnegative coefficients.

Let $\tilde{K}_{\lambda\mu}(t)$ be the cocharge polynomial; that is, $\tilde{K}_{\lambda\mu}(t)=t^{n(\mu)}K_{\lambda\mu}(t^{-1})$, where $n(\mu)=\sum(i-1)\mu_i$. Then we rewrite the sum in Corollary 4.1 in terms of cocharge polynomials and Lemma 4.1 has a second corollary.

Corollary 4.2. *The polynomial*

$$\sum_{j=0}^{k}(-1)^j t^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_{t}\tilde{K}_{\lambda 2^{m+j}1^{n-2(m+j)}}(t)$$

has nonnegative coefficients.

5. Proof of Theorem 1.1

In this section we finish proving the main theorem of this paper. We define the statistics c_r and cut_r on $\mathcal{M}_0^0$, the standard rigged configurations. Then we construct a surjection from $\mathcal{M}_0^0$ onto $\cup_{k=0}^r \mathcal{M}_{r-k}^k$ which respects the statistics.

Definition 5.1. *Let $(\alpha(0), L) \in \mathcal{M}_0^0$ and let $a_i = L^1_{\alpha_2^1+i}$, that is, a_i is the label of the ith column of length 1 in $(\alpha(0), L)$. Let $a_{\alpha_1^1-\alpha_2^1+1} = \infty$ and let $a_0 = 0$. We define $cut_r(\alpha(0), L)$ to be the least j, $0 \leq j \leq r$, such that $j + 1 + a_{j+1} > r$. If there is no such j, we let $cut_r(\alpha(0), L) = r$. Note that if $\alpha_1^1 - \alpha_2^1 \leq r$, then $cut_r(\alpha(0), L) \leq \alpha_1^1 - \alpha_2^1$.*

Definition 5.2. *Let $(\alpha(0), L) \in \mathcal{M}_0^0$. Let $k = cut_r(\alpha(0), L)$. Then we define $c_r((\alpha(0), L)) = c((\alpha(0), L)) - (n - r)(r - k)$.*

Definition 5.3. *Let $k = cut_r(\alpha(0), L)$. Define $\Psi((\alpha(0), L)) = (\alpha(r-k), \tilde{L})$, where*

$$\tilde{L_j^i} = \begin{cases} 0 & i = 1 \text{ and } \alpha_2^1 + 1 \leq j \leq \alpha_2^1 + k \\ L_j^i - (r - k) & i = 1 \text{ and } \alpha_2^1 + k + 1 \leq j \leq \alpha_1^1 \\ L_j^i & \text{otherwise} \end{cases}$$

Since $P_1^1(\alpha(m+1)) = P_1^1(\alpha(m)) - 1$, if $L_j^1 \leq P_1^1(\alpha(0))$, then $\tilde{L_j^1} \leq P_1^1(\alpha(r-k))$, so that $\Psi(\alpha(0), L) \in \mathcal{M}_{r-k}^0$. Since $\tilde{L_j^1} = 0$ for $\alpha_2^1 + 1 \leq j \leq \alpha_2^1 + k$, $\Psi((\alpha(0), L)) \in \mathcal{M}_{r-k}^k$. Also note that the image of Ψ is $\cup_{j=0}^r \mathcal{M}_{r-j}^j$.

Lemma 5.1. *Let $(\alpha(r-k), \tilde{L}) \in \mathcal{M}_{r-k}^k$. Then*

$$\sum_{(\alpha(0),L)\in\Psi^{-1}((\alpha(r-k),\tilde{L}))} t^{c_r((\alpha(0),L))} = \begin{bmatrix} r \\ k \end{bmatrix}_t t^{c((\alpha(r-k),\tilde{L}))}$$

Proof: First we show that

$$c(\Psi((\alpha(0), L))) = c(\alpha(0), L) - \Big(\sum_{i=\alpha_2^1+1}^{\alpha_2^1+k} L_i^1\Big) - (n - r)(r - k)$$

so that

$$c_r(\alpha(0), L) = \sum_{i=\alpha_2^1+1}^{\alpha_2^1+k} L_i^1 + c(\Psi(\alpha(0), L))$$

(9)
$$= \sum_{i=\alpha_2^1+1}^{\alpha_2^1+k} L_i^1 + c(\alpha(r-k), \tilde{L})$$

$$\begin{aligned}
c(\Psi((\alpha(0),L))) &= \sum_{i=k+1+\alpha_2^1}^{\alpha_1^1} L_i^1 - (\alpha_1^1 - \alpha_2^1 - k)(r-k) + \sum_{i>1, j\geq 1} L_j^i + c(\alpha(r-k)) \\
&= \sum_{i\geq 1, j\geq 1} L_j^i - \sum_{j=\alpha_2^1+1}^{\alpha_2^1+k} L_j^1 - (\alpha_1^1 - \alpha_2^1 - k)(r-k) + \\
&\quad [\binom{n-(r-k)}{2} + \binom{r-k}{2} - (n-(r-k))\alpha_1^1 - (r-k)\alpha_2^1 + c'(\alpha)] \text{ by (5)} \\
&= \sum_{j,i\geq 1} L_j^i - \sum_{j=\alpha_2^1+1}^{\alpha_2^1+k} L_j^1 + \binom{n-(r-k)}{2} + \binom{r-k}{2} + n\alpha_1^1 + c'(\alpha) + k(r-k) \\
&= c(\alpha(0),L) - \binom{n}{2} - \sum_{j=\alpha_2^1+1}^{\alpha_2^1+k} L_j^1 + \binom{n-(r-k)}{2} + \binom{r-k}{2} + k(r-k) \\
&= c(\alpha(0),L) - \sum_{j=\alpha_2^1+1}^{\alpha_2^1+k} L_j^1 - (n-r)(r-k).
\end{aligned}$$

Also note that

$$\begin{aligned}
&\Psi^{-1}((\alpha(r-k),\tilde{L})) = \\
&\qquad \{(\alpha(0),L) | L_j^1 \leq r-k \text{ for } \alpha_2^1 + 1 \leq j \leq \alpha_2^1 + k, \\
&\qquad L_j^1 = \tilde{L}_j^1 + (r-k) \text{ for } j \geq \alpha_2^1 + k + 1, \\
&\qquad \text{and } L_j^i = \tilde{L}_j^i \text{ otherwise}\}
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{(\alpha(0),L)\in\Psi^{-1}(\alpha(r-k),\tilde{L})} t^{c_r(\alpha(0),L)} \\
&= \sum_{0\leq L^1_{\alpha_2^1+1}\leq\cdots\leq L^1_{\alpha_2^1+k}\leq r-k} t^{\sum_{j=1}^k L^1_{\alpha_2^1+j} + c(\alpha(r-k),\tilde{L})} \text{ by (9)} \\
&= \begin{bmatrix} r \\ k \end{bmatrix}_t t^{c(\alpha(r-k),\tilde{L})}
\end{aligned}$$

This last lemma, the fact that the image of Ψ is $\cup_{j=0}^r \mathcal{M}_{r-j}^j$, and Lemma 3.2 and Lemma 4.1, finish the proof of Theorem 1.1. □

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