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Grothendieck polynomials and the Yang-Baxter equation

Abstract

A new development of the theory of Grothendieck polynomials based on an exponential solution of the Yang-Baxter equation in the degenerate Hecke algebra is given.

1. Introduction.

In this paper, we continue to study (cf. [FK1, FK2]) the connections between the Yang-Baxter equation and the theory of symmetric functions and Schubert and Grothendieck polynomials, with the emphasis on the latter ones.

It was shown in [FK1] that, for any exponential solution of the YBE, a theory of generalized Schubert polynomials and corresponding symmetric functions can be constructed. The solution related to the nilCoxeter algebra of the symmetric group gives the Schubert polynomials of A.Lascoux and M.-P.Schützenberger, as shown in [FS].

In this paper we study the solution (first mentioned in [FK1]) related to the degenerate Hecke algebra. We show that this solution leads to the Grothendieck polynomials of A.Lascoux and M.-P.Schützenberger [LS, L] who introduced them in their study of the Grothendieck ring of a flag manifold. These are non-homogeneous polynomials that can be defined inductively via isobaric divided differences π_i; the lowest-degree homogeneous component of a Grothendieck polynomial is the corresponding Schubert polynomial.

Thus, we give a new combinatorial definition of Grothendieck polynomials. As in the Schubert case, it can be formulated either in terms of "reduced decompositions and compatible sequences" (cf. [BJS, FS]) or in terms of "resolved braid configurations" [FK1].

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And even in the case of Schubert polynomials, our proof of the equivalence of the two definitions (cf. Theorem 2.3) is much simpler than the ones of [FS] and [BJS].

Furthermore, we define, for any number β, a polynomial which we call a β-polynomial. This polynomial reduces to Schubert and Grothendieck polynomials in the cases $\beta = 0$ and $\beta = -1$, respectively.

Stable β-polynomials are also defined. They are certain formal power series in β whose coefficients are symmetric functions. Again, the lowest-degree coefficient is the corresponding stable Schubert polynomial.

The theory of Schubert polynomials is a well-known tool in the enumerative combinatorics of reduced decompositions. Likewise, the β-polynomials allow to obtain enumerative results concerning so-called "sorting sequences".

A generalization is given for the formula of Macdonald [M] for the sum of the products of entries of reduced decompositions.

In Section 2 of this extended abstract, we give a full proof of the main result (Theorem 2.3) that justifies a combinatorial definition of β-polynomials (and thus of the Grothendieck and Schubert polynomials as their special cases). Section 3 contains statements (without proofs) of some results about these polynomials that we have been able to obtain using this approach.

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2. Generalized Schubert and Grothendieck polynomials

Let K be a field of zero characteristic, and let β, x_1, x_2, ... , be formal variables. Define a β-divided-difference operator $\pi_i^{(\beta)}$ acting in $K[x_1, x_2, \ldots]$ by

$$\pi_i^{(\beta)} f(x_1, x_2, \ldots) = \frac{(1+\beta x_{i+1}) f(x_1, x_2, \ldots) - (1+\beta x_i) f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}} \quad ;$$

in other words, $\pi_i^{(\beta)} = \partial_i \circ (1 + \beta X_{i+1})$ where ∂_i is the usual divided difference operator and X_{i+1} is the operator of multiplication by x_{i+1}.

2.1 Definition. *Generalized Schubert/Grothendieck polynomials.* Let S_n be the symmetric group of permutations of n elements; $s_i = (i\ i+1)$ is an adjacent transposition; $l(w)$ is the length of a permutation $w \in S_n$, i.e., the number of inversions; w_0 is the permutation of maximal length. For any $w \in S_n$, define the β-polynomial $\mathfrak{L}_w^{(\beta)}(x_1, \ldots, x_{n-1})$ recursively by

(i) $\mathfrak{L}_{w_0}^{(\beta)}(x_1, \ldots, x_{n-1}) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$;

(ii) $\mathfrak{L}_w^{(\beta)} = \pi_i^{(\beta)} \mathfrak{L}_{ws_i}^{(\beta)}$ whenever $l(ws_i) = l(w) + 1$.

This definition is self-consistent because operators $\pi_i^{(\beta)}$ satisfy the Coxeter relation $\pi_i^{(\beta)} \pi_{i+1}^{(\beta)} \pi_i^{(\beta)} = \pi_{i+1}^{(\beta)} \pi_i^{(\beta)} \pi_{i+1}^{(\beta)}$ (see [L]). In the case $\beta = 0$ the corresponding polynomials are, by definition, the Schubert polynomials of Lascoux and Schützenberger (see, e.g., [M]).

In the case $\beta = -1$ we obtain, after a change of variables $x_i \leftarrow 1 - x_i$, the Grothendieck polynomials of the same authors [LS, L]. We are going to give a direct combinatorial interpretation of these polynomials that extends the one(s) of [BJS, FS, FK1]; this will show, in particular, that $\mathfrak{L}_w^{(\beta)}$, as a polynomial in β, $x_1, \ldots, x_{n-1}$, has nonnegative integer coefficients.

2.2 Definition. Let $\mathcal{A}_n^{(\beta)}$ be the algebra with generators $u_1, \ldots, u_{n-1}$ satisfying commutation relations

$$u_i u_j = u_j u_i \ , |i-j| \geq 2 \ ;$$
$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \ ;$$
$$u_i^2 = \beta u_i \ .$$

In particular, $\mathcal{A}_n^{(0)}$ is the nilCoxeter algebra of the symmetric group [FS] and $\mathcal{A}_n^{(-1)}$ is the degenerate Hecke/Iwahori algebra $\mathcal{H}_n(0)$. Note that $\mathcal{A}_n^{(\beta)}$ has a natural linear basis formed by permutations of S_n; namely, each $w \in S_n$ is identified with a product $u_{a_1} \ldots u_{a_l}$ where $a_1 \ldots a_l$ is any reduced decomposition of w.

It was shown in [FK1] that the elements $h_i(t) = e^{tu_i}$ satisfy the Yang-Baxter equation

$$h_i(t) h_{i+1}(t+s) h_i(s) = h_{i+1}(s) h_i(t+s) h_{i+1}(t) \ ;$$

various consequences of this fact have been then obtained. Following [FS], let us define

$$A_i(t) = h_{n-1}(t) h_{n-2}(t) \cdots h_i(t)$$

and

$$\mathfrak{S}(t_1, \ldots, t_{n-1}) = A_1(t_1) A_2(t_2) \cdots A_{n-1}(t_{n-1}) \ ; \tag{2.1}$$

the latter is the generalized Schubert expression. It was shown in [FS] that in the case $\beta = 0$ the coefficients of $\mathfrak{S}$ in the basis of permutations are exactly the Schubert polynomials. Below we generalize this statement to the case of an arbitrary β.

Let us first note that $h_i(t) = e^{tu_i} = 1 + x u_i$ where $x = \frac{e^{\beta t} - 1}{\beta}$; we will write $x = [t]_\beta$. The map $t \to [t]_\beta$ converts the ordinary addition into the operation $\oplus$ defined by

$$x \oplus y = x + y + \beta x y \ ;$$

in other words, $[t]_\beta \oplus [s]_\beta = [t+s]_\beta$. (Also note that $(1 + x u_i)(1 + y u_i) = 1 + (x \oplus y) u_i$. Correspondingly, the β-subtraction $\ominus$ is defined by

$$z \ominus y = \frac{z - y}{1 + \beta y}$$

which is equivalent to $y \oplus (z \ominus y) = z$. Also, $(1 + x u_i)(1 + (\ominus x) u_i) = 1$ where $\ominus x = 0 \ominus x$.

The generalized Schubert expression (2.1) can be rewritten in terms of the variables $x_i = [t_i]_\beta$ by using the formula $h_i(t_j) = 1 + x_j u_i$; thus we get

$$\mathfrak{L}^{(\beta)}(x_1, \ldots, x_{n-1}) = \mathfrak{S}(x_1, \ldots, x_{n-1}) = \prod_{j=1}^{n-1} \prod_{i=n-1}^{j} (1 + x_j u_i) \tag{2.2}$$

where the interchanged bounds for i mean that the corresponding factors are multiplied in descending order, starting with $i = n-1$. Now, as in [FS, FK1], let us expand the last expression in the basis of permutations.

2.3 Theorem. $\mathfrak{L}^{(\beta)}(x_1,\ldots,x_{n-1}) = \sum_{w\in S_n} \mathfrak{L}_w^{(\beta)}\, w$.

In other words, to find a β-polynomial $\mathfrak{L}_w^{(\beta)}$, one needs to take a coefficient of w in the expression (2.2).

2.4 Example. $n = 3$.

$$\begin{aligned}\mathfrak{L}^{(\beta)}(x_1,x_2) &= (1+x_1u_2)(1+x_1u_1)(1+x_2u_2)\\ &= 1 + x_1u_1 + (x_1+x_2+\beta x_1x_2)u_2 + x_1x_2u_1u_2 + x_1^2u_2u_1 + x_1^2x_2u_2u_1u_2\end{aligned}$$

Therefore, e.g., $\mathfrak{L}_{u_2}^{(\beta)} = x_1 + x_2 + \beta x_1x_2$. Corresponding Schubert and Grothendieck polynomials are x_1+x_2 and $(1-x_1)+(1-x_2)-(1-x_1)(1-x_2) = -x_1x_2+1$, respectively.

To prove Theorem 2.3, we will need the following lemmas.

2.5 Lemma. *Let f be a polynomial in x_1, x_2, ... ; denote*

$$\tilde f(\ldots,x_i,x_{i+1},\ldots) = f(\ldots,x_{i+1},x_i,\ldots)\ .$$

Then

$$(\pi_i^{(\beta)}+\beta)f = \frac{\tilde f - f}{x_{i+1}\ominus x_i}\ .$$

Proof.

$$(\pi_i^{(\beta)}+\beta)f = \frac{(1+\beta x_{i+1})f-(1+\beta x_i)\tilde f}{x_i - x_{i+1}} + \beta f = \frac{1+\beta x_i}{x_{i+1}-x_i}(\tilde f - f) = \frac{\tilde f - f}{x_{i+1}\ominus x_i}\ . \qquad \square$$

2.6 Lemma. [FK1] *Let $\alpha_i(x) = A_i(t)$ where $x = [t]_\beta$; in other words,*

$$\alpha_i(x) = (1+xu_{n-1})(1+xu_{n-2})\cdots(1+xu_i)\ .$$

Then, for any variables x and y, the expressions $\alpha_i(x)$ and $\alpha_i(y)$ commute. $\square$

Proof of Theorem 2.3. In the notation of Lemma 2.6,

$$\begin{aligned}&\mathfrak{L}^{(\beta)}(x_1,\ldots,x_{n-1})\\ &= \alpha_1(x_1)\cdots\alpha_{n-1}(x_{n-1})\\ &= \alpha_1(x_1)\cdots\alpha_i(x_i)\alpha_i(x_{i+1})\,(1+(\ominus x_{i+1})u_i)\,\alpha_{i+2}(x_{i+2})\cdots\alpha_{n-1}(x_{n-1})\ .\end{aligned}$$

Lemma 2.6 implies that in the last product the expressions to the left and to the right of $(1+(\ominus x_{i+1})u_i)$ are symmetric in x_i and x_{i+1}. Therefore they behave as constants with respect to divided differences. Since, according to Lemma 2.5,

$$\begin{aligned}(\pi_i^{(\beta)}+\beta)(1+(\ominus x_{i+1})u_i) &= \frac{(1+(\ominus x_i)u_i)-(1+(\ominus x_{i+1})u_i)}{x_{i+1}\ominus x_i}\\ &= \frac{(1+(x_{i+1}\ominus x_i)u_i)-1}{x_{i+1}\ominus x_i}(1+(\ominus x_{i+1})u_i) = (1+(\ominus x_{i+1})u_i)u_i\end{aligned}$$

and u_i commutes with $\alpha_{i+2}(x_{i+2})\cdots\alpha_{n-1}(x_{n-1})$, we conclude that

$$(2.3) \qquad (\pi_i^{(\beta)}+\beta)\mathfrak{L}^{(\beta)}(x_1,\ldots,x_{n-1}) = \mathfrak{L}^{(\beta)}(x_1,\ldots,x_{n-1})u_i \, .$$

This identity contains the recurrence (ii) of Definition 2.1. Indeed, let $\bar{\mathfrak{L}}_w^{(\beta)}(x_1,\ldots,x_{n-1})$ be the coefficient of a permutation w in $\mathfrak{L}^{(\beta)}(x_1,\ldots,x_{n-1})$. Assume $l(ws_i)=l(w)+1$. Then the coefficient of ws_i in the right-hand side of (2.3) is $\bar{\mathfrak{L}}_w^{(\beta)}+\beta\mathfrak{L}_{ws_i}^{(\beta)}$ whereas the coefficient of ws_i in the left-hand side is $\pi_i^{(\beta)}\mathfrak{L}_{ws_i}^{(\beta)}+\beta\mathfrak{L}_{ws_i}^{(\beta)}$; this gives the desired recurrence. It only remains to check that the coefficient of w_0 in $\mathfrak{L}^{(\beta)}(x_1,\ldots,x_{n-1})$ is $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$; this follows from the fact that the only way to obtain w_0 from (2.2) is to take x_ju_i from each other. $\square$

3. Further results

Stable β-polynomials.

One can define stable β-polynomials $\mathfrak{M}_w^{(\beta)}(x_1,x_2,\ldots)$ similarly to the stable Schubert polymomials, or Stanley's symmetric functions [S], as done in [FK1]. Namely, let

$$\alpha_1(x_1)\alpha_1(x_2)\cdots = \sum_{w\in S_n}\mathfrak{M}_w^{(\beta)}w \, .$$

Then $\mathfrak{M}_w^{(\beta)}$ are some *power series* in β whose coefficients $(\mathfrak{M}_w^{(\beta)})_j$ are symmetric functions in $x_1,x_2\ldots$ (cf. Lemma 2.6). The constant term $(\mathfrak{M}_w^{(\beta)})_0$ is the corresponding stable Schubert polynomial. In general, $(\mathfrak{M}_w^{(\beta)})_j$ is a homogeneous symmetric function of degree $l(w)+j$.

The following are some of the examples of stable β-polynomials we computed:

$$\begin{aligned}
\mathfrak{M}_1^{(\beta)} &= 1\\
\mathfrak{M}_{s_1}^{(\beta)} = \mathfrak{M}_{s_2}^{(\beta)} &= \sum e_{k+1}\beta^k\\
\mathfrak{M}_{s_1s_2}^{(\beta)} &= \sum (k+1)e_{k+2}\beta^k\\
\mathfrak{M}_{s_2s_1}^{(\beta)} &= \sum (e_1e_{k+1}-e_{k+2})\beta^k\\
\mathfrak{M}_{s_1s_2s_1}^{(\beta)} &= \sum\Big(\sum_{i=0}^{k} e_{i+1}e_{k-i+2}-(k+1)e_{k+3}\Big)\beta^k \, .
\end{aligned}$$

Sorting sequences.

An elementary sorting operation u_i compares the ith and $(i+1)$st elements of a permutation in S_n and switches them if they form an inversion. A sequence of such operations is called a "sorting sequence" if it sorts out any permutation. For example, 21323212 is a sorting sequence for S_4. In general, a sequence is sorting if and only if it contains a reduced decomposition of w_0.

Let $\mathcal{R}_{nL}$ be the set of sorting sequences of a given length L. The cardinality N_{nL} of this set is the coefficient of a square-free monomial in the symmetric function $(\mathfrak{M}^{(\beta)}_{w_0})_{L-\binom{n}{2}}$. Thus one can obtain a formula for N_{nL} by computing these symmetric functions (we have a conjectured determinantal formula for them).

For example, for S_2, S_3, and S_4 this approach gives the following number of sorting sequences of length L:

$$
\begin{aligned}
S_2 &: \quad N_{2,L} = 1 \;; \\
S_3 &: \quad N_{3,L} = 2^L - 2L \;; \\
S_4 &: \quad N_{4,L} = 3^L - 2^{L-1}(L-1)(L-2) - 2L^2 - 1 \;.
\end{aligned}
$$

Stability and limits.

Both β-polynomials and the stable ones are independent of the parameter n of the symmetric group S_n; that is, they are well-defined for the elements $w \in S_\infty$. There is a formula that explicitly expresses stable β-polynomials in terms of the "unstable" ones. Also the $\mathfrak{M}^{(\beta)}_w$'s can be obtained from $\mathfrak{L}^{(\beta)}_w$'s by a certain limiting procedure. These results are analogous to their counterparts in [FK1].

Generalized Macdonald formula.

The following formula for the specialization $x_1 = x_2 = \cdots = 1$ generalizes the one of Macdonald [M]:

$$\sum_L \sum_{(a_1,\ldots,a_L)\in\mathcal{R}_{n,L}} \frac{a_1 a_2 \cdots a_L}{L!} z^L = (e^z - 1)^{\binom{n}{2}}$$

or, equivalently,

$$\sum_{(a_1,\ldots,a_L)\in\mathcal{R}_{n,L}} a_1 a_2 \cdots a_L = l_0!\; S[L, l_0] \quad , \quad l_0 = \binom{n}{2}$$

where $S[\ldots,\ldots]$ are the Stirling numbers of the second kind. This reduces to Macdonald's formula when $\beta = 0$.

There is an analogous formula for any dominant (cf. [M]) permutation.

Other identities for $x_1 = x_2 = \cdots = 1$.

For $\beta = -1$ (the case of Grothendieck polynomials), $\mathfrak{L}^{(\beta)}_w(1,1,\ldots,1) = 1$ for *any* permutation w.

For an arbitrary β,

$$\mathfrak{L}^{(\beta)}_w(1,1,\ldots,1) = \mathfrak{L}^{(\beta)}_{w^{-1}}(1,1,\ldots,1)\;.$$

Double β-polynomials and the Cauchy identity.

The notion of the double Schubert/Grothendieck polynomials can be straightforwardly generalized along the lines of [FS,FK1] to obtain the double β-polynomials and corresponding super-symmetric functions. The Cauchy identity in this case becomes

$$\mathfrak{L}^{(\beta)}_{w_0}(y,x) = \prod_{i+j\le n} (x_i + y_j + \beta x_i y_j)$$

where $\mathfrak{L}_w^{(\beta)}(y,x)$ is the double β-polynomial.

The B_n case.

It is possible to combine the main constructions of [FK3] and the present paper to obtain the B_n-analogues of the β-polynomials and Grothendieck polynomials.

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