INCOMPARABILITY GRAPHS OF $(3+1)$-FREE POSETS ARE s-POSITIVE

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March 16, 1994

SUMMARY In [5] Stanley associated to a (finite) graph G a symmetric function X_G generalizing the chromatic polynomial of G. Using an involution on a special type of arrays constructed by Gessel and Viennot [1], we show that if G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then X_G is a nonnegative linear combination of Schur functions. Since the elementary symmetric functions are nonnegative linear combinations of Schur functions, this result gives supportive evidence for a conjecture of Stanley and Stembridge ([5, Conjecture 5.1] or [6, Conjecture 5.5]).

Dans [5], Stanley associe à tout graphe (fini) G une fonction symétrique X_G qui généralise le polynôme chromatique de G. En utilisant une involution sur certains tableaux construits par Gessel et Viennot [1], nous démontrons que si G est le graphe de la relation d'incomparabilité d'un ensemble partiellement ordoné qui ne contient pas $(\mathbf{3}+\mathbf{1})$, alors X_G est une combinaison linéaire de fonctions de Schur dont les coefficients sont positifs. Puisque les fonctions symétriques élémentaires sont des combinaisons linéaires de fonctions de Schur dont les coefficients sont positifs, notre résultat confirme une conjecture de Stanley et Stembridge ([5, Conjecture 5.1] ou [6, Conjecture 5.5]).

1. Introduction

Let G be a (finite) graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_d\}$. A coloring of G is a function $\kappa : V \to \mathbf{P}$, where $\mathbf{P} = \{1, 2, \dots\}$ is the set of colors. A coloring κ is called proper, if $\kappa(u) \neq \kappa(v)$ whenever u and v are the vertices of an edge of G. The *chromatic polynomial* of G is the function $\chi_G : \mathbf{P} \to \mathbf{P}$ such that $\chi_G(n)$ is the number of proper colorings of G with n colors. (It is not difficult to see that χ_G is indeed a polynomial of degree d.) In [5] Stanley introduced and studied a symmetric function X_G, generalizing χ_G. It is defined as follows. Let $x_1, x_2, \dots$ be commuting indeterminates. Then

$$X_G = X_G(x) = X_G(x_1, x_2, \dots) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)} ,$$

where the sum ranges over all proper colorings of G. It is immediate from the definition that $X_G(1^n) = \chi_G(n)$, where $X_G(1^n)$ is the specialization of X_G obtained by letting $x_1 = x_2 = \cdots = x_n = 1$ and $x_{n+1} = x_{n+2} = \cdots = 0$. One very interesting question is to study the coefficients that arise in the expansion of X_G in terms of the "natural" bases for the vector space of symmetric functions Λ. (In [5] many results related to this question are proved.) In particular, we may ask whether these coefficients are nonnegative. Following

[5] we say that a symmetric function f is u-positive, where $\{u_\lambda\}$ is a basis for Λ, if the coefficients d_λ in the expansion $f = \sum_\lambda d_\lambda u_\lambda$ are all nonnegative. A graph G is said to be u-positive if X_G is u-positive. Let s and e stand for the Schur functions and the elementary symmetric functions, respectively. A poset P is called $(\mathbf{a}+\mathbf{b})$-free if P does not have an induced subposet isomorphic to the direct sum $\mathbf{a}+\mathbf{b}$ of an a-element chain and a b-element chain. The *incomparability graph* of a finite poset P, $\mathrm{inc}(P)$, is the graph with vertex set P and edge set $E = \{(u,v) \in P^2 \mid u \text{ and } v \text{ are incomparable in } P\}$. Stanley stated the following conjecture [5, Conjecture 5.1], which as he mentions is equivalent to [6, Conjecture 5.5].

Conjecture 1 (Stanley–Stembridge). *If P is $(\mathbf{3}+\mathbf{1})$-free, then inc(P) is e-positive.*

This conjecture has been verified for all posets with at most 7 elements by Stanley and Stembridge [6, pp. 277-278] and for all 8-element posets by Stembridge.

Since each e_λ is s-positive, Conjecture 1 implies that the incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets are s-positive. Further evidence in support of Conjecture 1 is Theorem 1 below, which as is mentioned in [5, p. 18] follows easily from a result of Haiman [2, Theorem 1.4]. An *indifference graph* (or *unit interval graph*) is an incomparability graph of a poset which is both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free.

Theorem 1. *Let G be an indifference graph. Then G is s-positive.* $\square$

To prove his result Haiman uses deep machinery from the theory of Hecke algebras and Kazhdan–Lusztig polynomials, in particular the Kazhdan–Lusztig conjectures on composition series of Verma modules (proved by Beilinson-Bernstein and Brylinski–Kashiwara).

Stanley remarked that there should be a proof of the innocent sounding Theorem 1 which does not use the Kazhdan–Lusztig conjectures.

In this paper we prove (a generalization of) the following theorem.

Theorem 2. *If G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then G is s-positive.* $\square$

This theorem provides new evidence in support of Conjecture 1. Its proof is relatively short and uses only standard facts from the theory of symmetric functions. In particular, it yields a simple proof of Theorem 1.

2. The main result

Definition. Let G be a graph with vertex set $V = \{v_1, v_2, \dots, v_d\}$. A *multicoloring* of G is a function $\kappa : V \to 2^{\mathbf{P}}$, where $2^{\mathbf{P}}$ is the set of all subsets of $\mathbf{P} = \{1, 2, \dots\}$, including the empty set. The multicoloring is *proper* if $\kappa(u) \cap \kappa(v) = \emptyset$ whenever u and v are the vertices of an edge of G. If $\mathbf{m} = (m_1, m_2, \dots, m_d)$ is a sequence of nonnegative integers, then an $\mathbf{m}$-multicoloring is a multicoloring κ such that $|\kappa(v_i)| = m_i$ for $i = 1, 2, \dots, d$. For a finite subset $S = \{s_1, s_2, \dots\}$ of $\mathbf{P}$ we define $x_S = x_{s_1} x_{s_2} \cdots$.

Definition. Let $\mathbf{m} = (m_1, m_2, \ldots, m_d)$ be as above. Define

$$\tilde{X}_G^{\mathbf{m}} = \tilde{X}_G^{\mathbf{m}}(x) = \tilde{X}_G^{\mathbf{m}}(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)} \ ,$$

where the sum ranges over all proper $\mathbf{m}$-multicolorings $\kappa : V \to 2^{\mathbf{P}}$.

It is clear that $\tilde{X}_G^{\mathbf{m}}$ generalizes X_G in the sense that $\tilde{X}_G^{(1,1,\ldots,1)} = X_G$. Then Theorem 2 is a special case of the following theorem.

Theorem 3. *If G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset with d elements, and $\mathbf{m} = (m_1, m_2, \ldots, m_d)$ is any sequence of nonnegative integers, then $\tilde{X}_G^{\mathbf{m}}$ is s-positive.*

Proof. Let $P = (P, \prec)$ be a partially ordered set. We define a P-array to be an array

$$\begin{array}{lll} a_{11} & a_{12} & \ldots \\ a_{21} & a_{22} & \ldots \\ \ldots & & \end{array}$$

of elements in P, arranged in left-justified rows, and satisfying the following condition:

(2.1) $$a_{ij} \prec a_{i,j+1}$$

A P-tableau is a P-array, satisfying the additional condition:

(2.2) If $a_{i+1,j}$ is defined, then a_{ij} is defined and $a_{i+1,j} \not\prec a_{ij}$.

Such arrays were first considered by Gessel and Viennot [1]. The weight of an array T with entries in P is the sequence $\mathrm{wt}(T) = (n_1, n_2, \ldots)$, where n_i is the number of occurences of v_i in T. Let G be the incomparability graph of P. To each proper multicoloring κ of G, we can associate a P-array T_κ in the following way.

For any $i \geq 1$, let $\{v_1^{(i)}, v_2^{(i)}, \ldots\} = \kappa^{-1}(i)$. Since κ is proper, it follows that $\kappa^{-1}(i)$ is a stable subset of V, i.e., no two vertices in $\kappa^{-1}(i)$ are connected by an edge in G. This implies that $\kappa^{-1}(i)$ is a chain in P, so we may assume that $v_1^{(i)} \prec v_2^{(i)} \prec \ldots$. Then T_κ is the array

$$\begin{array}{lll} v_1^{(1)} & v_2^{(1)} & \ldots \\ v_1^{(2)} & v_2^{(2)} & \ldots \\ \ldots & & \end{array}$$

It is clear that for any P-array T, there is a unique multicoloring κ of G such that $T = T_\kappa$. For a partition λ, let m_λ and h_λ denote respectively the monomial symmetric function and the complete symmetric function indexed by λ. It is well known (see [3, Chapter I, §4] for example) that the s_λ form an orthonormal basis for Λ with the inner product defined by $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$. Therefore, if $X_G^{\mathbf{m}} = \sum_\lambda c_\lambda s_\lambda$, then $c_\lambda = \langle X_G^{\mathbf{m}}, s_\lambda \rangle$. Let $l(\lambda)$

be the length of λ. Then the Jacobi–Trudi identity (see e.g. [3, Chapter I, (3.4)] or [4, Theorem 4.5.1]) is the following:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \le i,j \le l(\lambda)} \tag{2.3}$$

Let S_l denote the group of permutations of $\{1, 2, \ldots, l\}$. If $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of length l, and $\pi \in S_l$, then we denote by $\pi(\lambda)$ the sequence $\{\lambda_{\pi(j)} - \pi(j) + j\}_{j=1}^{l}$. Expanding the determinant on the right side of (2.3) we get that

$$s_\lambda = \sum_{\pi \in S_l} \operatorname{sgn}(\pi) h_{\pi(\lambda)},$$

where, for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_l)$, $h_\alpha = h_{\alpha_1} \ldots h_{\alpha_l}$. (We set $h_r = 0$ if $r < 0$.) Thus $c_\lambda = \sum_{\pi \in S_l} \operatorname{sgn}(\pi) \langle X_G^{\mathbf{m}}, h_{\pi(\lambda)} \rangle$, where $\langle X_G^{\mathbf{m}}, h_{\pi(\lambda)} \rangle$ is, by the definition of the inner product $\langle -, - \rangle$, the coefficient of $m_{\pi(\lambda)}$ in $X_G^{\mathbf{m}}$, i.e., the coefficient of $x^{\pi(\lambda)}$ in $X_G^{\mathbf{m}}$, which in turn is the number of proper $\mathbf{m}$-multicolorings of G with weight $\pi(\lambda)$. (The weight $\mathrm{wt}(\kappa)$ of a multicoloring κ is the sequence $(|\kappa^{-1}(1)|, |\kappa^{-1}(2)|, \ldots)$.) Since the shape of T_κ is $\mathrm{wt}(\kappa)$, we get that $\langle X_G^{\mathbf{m}}, h_{\pi(\lambda)} \rangle$ is the number of P-arrays of shape $\pi(\lambda)$ and weight $\mathbf{m}$. Let

$$A = \{(\pi, T) \mid \pi \in S_l \text{ and } T \text{ is a } P\text{-array of shape } \pi(\lambda) \text{ and weight } \mathbf{m}\}.$$

Then $c_\lambda = \sum_{(\pi,T) \in A} \operatorname{sgn}(\pi)$. Let

$$B = \{(\pi, T) \in A \mid T \text{ is } \textit{not} \text{ a tableau}\}$$

and note that if T is a tableau, then $\pi(\lambda)_1 \ge \pi(\lambda)_2 \ge \ldots$, hence $\pi = id$. Thus to prove that $c_\lambda \ge 0$ it will be enough to find an involution $\varphi : B \longrightarrow B$ such that if $(\sigma, T') = \varphi(\pi, T)$, then $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\pi)$. One such involution is constructed in [1, Proof of Theorem 11]. We describe a slight modification of it below. Let

$$\begin{array}{l} \phantom{T = {}} a_{11}\ a_{12} \ldots \\ T = a_{21}\ a_{22} \ldots \\ \phantom{T = {}} \ldots \end{array}$$

and let $c = c(T)$ be the smallest positive integer such that (2.2) fails for some i and $j = c$. Let $r = r(T)$ be the largest i with this property. Then define $\sigma = \pi \circ (r, r+1)$, where $(r, r+1)$ is the permutation that interchanges r and $r+1$. Define

$$\begin{array}{l} \phantom{T' = {}} b_{11}\ b_{12} \ldots \\ T' = b_{21}\ b_{22} \ldots \\ \phantom{T' = {}} \ldots \end{array}$$

by letting

$$\begin{aligned} b_{ij} = a_{ij} \text{ if } i \ne r, r+1 \text{ or } i = r \text{ and } j \le c-1 \\ \text{or } i = r+1 \text{ and } j \le c\,; \end{aligned}$$

$$b_{rj} = a_{r+1,j+1} \text{ if } j \geq c \text{ and } a_{r+1,j+1} \text{ is defined ;}$$
$$b_{r+1,j} = a_{r,j-1} \text{ if } j \geq c+1 \text{ and } a_{r,j-1} \text{ is defined.}$$

Since by assumption a_{rc} is not defined or $a_{r+1,c} \prec a_{rc}$, it follows that T' satisfies (2.1) for $i = r+1$. So to show that T' is a P-array, it suffices to show that T' satisfies (2.1) for $i = r$, i.e., $a_{r,c-1} \prec a_{r+1,c+1}$. But $a_{r+1,c-1} \prec a_{r+1,c} \prec a_{r+1,c+1}$ is a 3-element chain in P and $a_{r+1,c-1} \not\prec a_{r,c-1}$, so by the assumption that P is $(\mathbf{3}+\mathbf{1})$-free, it follows that $a_{r,c-1} \prec a_{r+1,c+1}$. Thus T' is a P-array which is not a tableau ($b_{r+1,c} = a_{r+1,c} \prec b_{rc} = a_{r+1,c+1}$ if b_{rc} is defined) and clearly $c(T') = c(T)$ and $r(T') = r(T)$. This shows that $\varphi(\sigma, T') = (\pi, T)$, so φ is an involution. Moreover, $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\pi)$. □

From the proof of Theorem 3 we get the following combinatorial interpretation of the coefficients c_λ.

Theorem 4. *If* $X_G^{\mathbf{m}} = \sum_\lambda c_\lambda s_\lambda$, *then* c_λ *is the number of* P*-tableaux of shape* λ *and weight* $\mathbf{m}$. □

Acknowledgment. The author is grateful to Ira Gessel and Richard Stanley for helpful discussions.

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