On Reduced Expressions in Affine Weyl Groups

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Abstract

We investigate reduced expressions in the irreducible affine Weyl groups. A balance condition, similar to the condition defining the balanced tableaux of Edelman and Greene [4], is introduced to give a geometric characterization of the reduced expressions. Finite automata are constructed so that the words accepted by the automata correspond to the reduced expressions. This establishes the existence of a rational generating function with the values of $r(w)$ (the number of reduced expressions for a group element w), appearing as the coefficients. We also discuss the combinatorial properties of an arrangement of affine hyperplanes used in one of the constructions.

On étudie les décompositons réduites des éléments des groupes de Weyl affine irréductibles. Une condition dite d'équilibre, est introduite afin de donner un caractérisation géométrique des décompositions réduites. Cette condition est similare à celle qui définit les tableaux équilibrés d'Edelman et Greene [4]. On construit des automates finis, qui sont tels que les mots acceptés correspondent aux décompositions réduites. Cela entraine l'existence d'une fonction génératrice rationelle dont les coefficients sont les valeurs de $r(w)$ (le nombre de décompositions réduites d'un élément w du groupe). On discute aussi des propriétés combinatoires d'un arrangement affine d'hyperplans qui est employé dans une des constructions.

1 Introduction

The reduced expressions in finite Coxeter groups are a subject of recent interest. Stanley showed that there is a connection between the enumeration of reduced expressions in A_n and the enumeration of standard Young tableaux [11]. Edelman and Greene introduced balanced tableaux to find a bijective proof of this result [4]. Similar methods have been used to study the reduced expressions of B_n and D_n.

This paper is the result of investigations concerning reduced expressions in irreducible affine Weyl groups. The methods we use lead to some interesting geometric properties of the affine groups, which will also be discussed.

The results are divided into three parts. First, we show that the concept of balance found in [4] can be extended to affine Weyl groups. A reduced expression in an affine Weyl group corresponds to a finite sequence of hyperplanes crossed by a path in affine space. The balance condition describes the sequences that are possible.

Second, we explore the applications of finite automata to the enumeration of reduced expressions. We show that there is a rational generating function that counts the number of reduced expressions for each element of an affine Weyl group. The automaton used to establish this can be adapted to show that the language of reduced expressions is itself a regular language; i.e., there is an automaton that accepts precisely this set. We then introduce another construction of automata accepting the language of reduced expressions, requiring in general far fewer states. In fact this construction is optimal for $\tilde{A}_n$, in the sense that no automaton with fewer states can accept the language of reduced expressions.

This last construction has a particularly elegant description involving an arrangement of affine hyperplanes. In the third part we consider some combinatorial properties of this hyperplane arrangement. Shi encountered the same arrangement in the study of Kazhdan-Lusztig cells [10]. We apply his results to determine the number of states of the automata. We also build on Shi's work by applying the theory of intersection lattices to this arrangement.

2 Affine Weyl Groups

Let Φ be an irreducible crystallographic root system; i.e., a system of one of the types $A-G$. We will assume that Φ lies in a real vector space V such that $n = \dim V$ is the rank of Φ. For $\alpha \in \Phi$, $k \in \mathbf{Z}$, define $H_{\alpha,k}$ to be the affine hyperplane $\{\lambda \in V | (\lambda, \alpha) = k\}$, and define $s_{\alpha,k}$ to be the affine reflection through $H_{\alpha,k}$. Then the group generated by the $s_{\alpha,k}$ is an irreducible affine Weyl group. We will denote the affine group by placing a tilde over the name of the corresponding root system, and we will refer to an arbitrary irreducible affine Weyl group as $\tilde{W}$.

The connected components of $V - \bigcup H_{\alpha,k}$ are called *alcoves*. If Δ is a set of simple roots for Φ, and $\tilde{\alpha}$ the corresponding highest root, then the set $\{\lambda | (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Delta, (\lambda, \tilde{\alpha}) < 1\}$ is an alcove, known as the *fundamental alcove*, which will be designated A_0.

We list some standard facts:

- The reflections $s_{\alpha,0}$, $\alpha \in \Delta$ and $s_{\tilde{\alpha},1}$ (i.e., the reflections through the $(n-1)$-dimensional faces of A_0) constitute a Coxeter system for $\tilde{W}$.
- The action of $\tilde{W}$ on the set of alcoves is transitive, and the map $w \to wA_0$ is a bijection from $\tilde{W}$ to the set of alcoves.
- The minimal length $l(w)$ of an expression for w as a product of elements of the Coxeter system is equal to the number of hyperplanes $H_{\alpha,k}$ separating A_0 from wA_0. An expression for w of length $l(w)$ is called a *reduced expression.*

Standard references for affine Weyl groups are [1] and [6].

3 A Balance Condition

While w can be identified by the hyperplanes separating A_0 from wA_0, the reduced expressions for w can be identified with certain linear orderings of these hyperplanes. To be precise, let $s_1 \ldots s_r$ be a reduced expression for w, and let H_i be the hyperplane fixed by s_i. The hyperplanes separating A_0 from wA_0 are

$$H_1, s_1H_2, s_1s_2H_3, \ldots, s_1 \ldots s_{r-1}H_r.$$

In fact there exists a continuous path from A_0 to wA_0, crossing the hyperplanes in the list in the order given. Such a path visits the alcoves $A_0, s_1A_0, s_1s_2 \ldots s_rA_0 = wA_0$, in that order. So different reduced expressions for w correspond to different orderings of the set of hyperplanes separating A_0 from wA_0.

Of course, not all orderings correspond to reduced expressions; for instance, the first hyperplane in the list must contain an $(n-1)$-dimensional face of A_0. The following theorem gives a characterization of the orderings that do correspond to reduced expressions. The conditions are related to the conditions in the definition of balanced staircase tableaux [4] and the root sequences of [7], both of which correspond to orderings of roots in finite Coxeter groups. There is an analogous result for general Coxeter groups, using the roots of the geometric representation [5].

Remark: The hyperplanes $H_{\alpha,-k}$ and $H_{-\alpha,k}$ are the same. For the statement of the following theorem, it will be helpful to give each hyperplane a unique labeling. A hyperplane will be called $H_{\alpha,k}$ so that $k > 0$, or $\alpha \in \Phi^-$ and $k = 0$.

Theorem 3.1 *Let $H_1, \ldots, H_n$ be a sequence of distinct hyperplanes of the form $H_{\alpha,k}$, labeled as in the remark above. There exists a reduced expression $s_1 \ldots s_n$ such that H_i separates $s_1 \ldots s_{i-1}A_0$ from $s_1 \ldots s_iA_0$ for all i if and only if the following condition is met for all triples of not necessarily distinct roots $\alpha, \beta, \gamma = c_\alpha\alpha + c_\beta\beta \in \Phi$, where $c_\alpha, c_\beta \in \mathbb{R}^+$: for all triples of hyperplanes H_{α,k_α}, H_{β,k_β}, H_{γ,k_γ}, with $c_\alpha k_\alpha + c_\beta k_\beta = c_\gamma k_\gamma$,*

- *If H_{γ,k_γ} appears in the sequence, then it is preceded by exactly one of H_{α,k_α}, H_{β,k_β}.*
- *If H_{α,k_α}, H_{β,k_β} both appear in the sequence, then exactly one of the two is preceded by H_{γ,k_γ}.*

4 Finite Automata

Let A be a finite set, which will be called an *alphabet.* Let A^* be the set of words over this alphabet (i.e., the set of finite sequences of elements of A). A (formal) *language* is defined to be a subset of A^*. A *finite automaton* determines a language. Several definitions of finite automata exist, all essentially equivalent. For our purposes a finite automaton consists of a finite set S of states, with one state S_0 designated the initial state and a subset designated as the final states, and a transition map $t : S \times A \to S \cup \{\emptyset\}$. A word $a_1 \ldots a_n$ is *accepted* by the automaton if there exist states $S_1, \ldots, S_n$ such that

- $t(S_i, a_{i+1}) = S_{i+1}$ for $0 \leq i \leq n-1$
- S_n is a final state.

If a language L is the set of accepted words for some finite-state automaton, then L is said to be a *regular language.* If we set $\Psi(a) = t_a$ for $a \in A$, then

$$\sum_{a_1 \ldots a_n \in L} \Psi(a_1) \ldots \Psi(a_n)$$

is a rational generating function in $\mathbb{C}[[T]]$, where T is the set of indeterminates $\{t_a\}_{a\in A}$. See Chapter 4 of [12] (where different terminology is used) for details.

Theorem 4.1 *Let $\tilde{W}$ be an irreducible affine Weyl group. For $w \in \tilde{W}$, let $R(w)$ be the set of reduced expressions for w, and let $r(w) = |R(w)|$. Let $f(w) = \prod_{\alpha\in\Phi^+} x_\alpha^{n_\alpha(w)}$, where $n_\alpha(w)$ is the integer such that $n_\alpha(w) \leq (\lambda, \alpha) \leq n_\alpha(w) + 1$ for $\lambda \in wA_0$. Then*

$$\sum_{w\in\tilde{W}} \sum_{r\in R(w)} f(w) = \sum_{w\in\tilde{W}} r(w) f(w)$$

is a rational generating function.

Proof. We construct a finite automaton on the alphabet $A = \{x_\alpha, x_\alpha^{-1} | \alpha \in \Phi^+\}$. Let T be the (finite) group of translations that preserve the alcoves of W, and let $S_0, \ldots, S_{m-1}$ be the orbits of the T-action on the alcoves, with $A_0 \in S_0$. The set of states of the automaton will consist of ordered pairs (S_i, B), with B a subset of A containing at least one of x_α, x_α^{-1} for all $\alpha \in \Phi^+$. The initial state is (S_0, A), and all states are final. Assume the only hyperplane separating an alcove of S_i from one of S_j is a hyperplane perpendicular to α. Then we will write $S_j = S_i^\alpha$ if the alcove of S_j is on the positive side (with respect to α) of the hyperplane, and $S_j = S_i^{-\alpha}$ if on the negative side. The transition function is defined by

$$t((S_i, B), x_\alpha) = \begin{cases} (S_i^\alpha, B - x_\alpha^{-1}) & \text{if } S_i^\alpha \text{ exists and } x_\alpha \in B \\ \emptyset & \text{otherwise} \end{cases}$$

$$t((S_i, B), x_\alpha^{-1}) = \begin{cases} (S_i^{-\alpha}, B - x_\alpha) & \text{if } S_i^{-\alpha} \text{ exists and } x_\alpha^{-1} \in B \\ \emptyset & \text{otherwise} \end{cases}$$

Words accepted by this automaton correspond to the sequence of hyperplanes crossed by a path from alcove to alcove through affine space beginning at A_0. The words that are accepted correspond to the paths that never cross a hyperplane in both directions, which are the paths that never cross a hyperplane twice. Thus, these paths correspond to the reduced expressions. The image of a word corresponding to path from A_0 to wA_0 in the commutative formal power series ring over A is clearly $f(w)$, and the result follows. □

It is possible to construct a similar automaton on the alphabet of Coxeter generators so that the language accepted by the automaton is the language of reduced expressions. If T is replaced by $T \cap \tilde{W}$ in the proof above, then each transition corresponds to paths from wA_0 to ws_iA_0 for some unique s_i. Using this correspondence, we can change the alphabet of the automaton to the set of Coxeter generators. Such automata on alphabets of group generators have been the subject of much recent research [3]. A proof that the language of reduced expressions in an arbitrary Coxeter group is regular appears in [2]; a somewhat different proof will be presented in [5].

We now give another construction of an automaton that accepts the language of reduced expressions, a construction leading to some interesting geometry. If $s_2s_3 \ldots s_k$ is reduced, and A_0 and $s_2s_3 \ldots s_kA_0$ lie on the same side of the hyperplane fixed by s_1, then $s_1 \ldots s_k$ is reduced. If A_0 and $s_2s_3 \ldots s_kA_0$ lie on opposite sides, then $s_1s_2 \ldots s_k$ is not reduced. This suggests a way to generate reduced expressions from right to left, but since $s_1 \ldots s_k$ is reduced if and only if $s_k \ldots s_1$ is reduced, it provides a method for generating reduced expressions from left to right as well. To employ this in an automaton, we use the following result

Lemma 4.2 *Let $\mathcal{H} = \{H_{\alpha,k} | \alpha \in \Phi^+, k = 0, 1\}$. Let $\mathcal{R}$ be the set of connected components of $V - \bigcup_{H\in\mathcal{H}} H$. If $R_1 \in \mathcal{R}$, and R_1 and A_0 lie on the same side of the hyperplane fixed by s, a generator in the Coxeter system, then sR_1 is contained in a unique $R_2 \in \mathcal{R}$.*

This allows us to construct an automaton with states corresponding to the members of $\mathcal{R}$ so that the automaton accepts the language of reduced expressions. It is a fairly efficient construction; in fact, it has the least possible number of states for an automaton accepting the reduced words of $\tilde{A}_n$.

5 The Combinatorics of the Hyperplane Arrangement

In this section $\mathcal{H}$ and $\mathcal{R}$ will be defined as in last section's lemma. We begin with Shi's result about $|\mathcal{R}|$, the number of states in the automaton described at the end of the last section.

Theorem 5.1 (Shi) *The cardinality of $\mathcal{R}$ is $(h+1)^n$, where n is the rank of Φ and h is the Coxeter number of the finite Coxeter group corresponding to Φ.*

Shi's method of proof is very different from the usual methods of investigating hyperplane arrangements and their complements [10]. One such method is the use of the intersection lattice (or semi-lattice, in the case of arrangements with empty intersection) [9] [13]. Let L

be the set of nonempty intersections of members of $\mathcal{H}$, partially ordered by reverse inclusion. Define the Poincaré polynomial of $\mathcal{H}$ to be

$$\pi(\mathcal{H},t) = \sum_{X \in L} \mu(V,X)(-t)^{codim(X)},$$

where μ is the Möbius function of L. Then $|\mathcal{R}| = \pi(\mathcal{H},1)$. In fact, we can combine Shi's result with information from the tables in [8] to show

$$\pi(\mathcal{H},t) = (h+t)^n.$$

An additional fact that can be learned from the Poincaré polynomial is the number of members of $\mathcal{R}$ that are bounded, which is $\pi(\mathcal{H},-1) = (h-1)^n$.

We have found intersection-lattice proofs of Shi's result for the arrangements corresponding to $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$. We will sketch the proof for $\tilde{A}_n$. It is easiest to think of $\tilde{A}_n$ as lying in a real vector space V of dimension $n+1$ generated by e_i, $1 \leq i \leq n+1$. The roots are the vectors of the form $e_i - e_j$, $i \neq j$, and $\mathcal{H} = \{x_i - x_j = k : i < j, k = 0 \text{ or } 1\}$. Instead of finding all intersections of members of $\mathcal{H}$, we examine the situation in each connected component of $V - \bigcup_{i \neq j}\{x_i - x_j = 0\}$. These are the interiors of the Weyl chambers of the finite Coxeter group A_n, and they are in a bijection with the permutations of $\{1,\ldots,n+1\}$. These components are open and convex, and this allows us to apply the intersection-lattice theory to each component. All of the intervals in the resulting posets are Boolean. Letting $\mathcal{C}$ be the set of Weyl chamber interiors, we have

$$|\mathcal{R}| = \sum_{C \in \mathcal{C}} |\{X \in L : C \cap X \neq \emptyset\}|.$$

Reversing the sum, we have

$$|\mathcal{R}| = \sum_{X \in L} |\{C \in \mathcal{C} : C \cap X \neq \emptyset\}|.$$

So, given $X \in L$, we now find the number of $C \in \mathcal{C}$ such that $C \cap X$ is nonempty. We only need concern ourselves with those X that are not contained in any of the hyperplanes $x_i - x_j = 0$. These intersections are in a bijection with the partitions of the set $\{1,\ldots,n+1\}$. Let $B_1,\ldots,B_k$ be such a partition. Label the elements of B_i as $b_{i,1}, b_{i,2},\ldots$ so that $b_{i,j}$ is the j-th smallest element of B_i. Then the corresponding intersection is $\bigcap\{x_i - x_j = 1 : i = b_{l,m}, j = b_{l,m+1} \text{ for some } l,m\}$.

Theorem 5.2 *If X corresponds to the partition $B_1,\ldots,B_k$, then* $|\{C \in \mathcal{C} : C \cap X \neq \emptyset\}| = (n+1)!/(n+2-k)!$.

Sketch of Proof: One way of describing a chamber C with $C \cap X \neq \emptyset$ is with an ordered pair consisting of

1. a permutation τ of $\{1,\ldots,k\}$ such that $x_{b_{\tau(1),1}} > x_{b_{\tau(2),1}} > \ldots > x_{b_{\tau(k),1}}$ in C, and

2. a subset P of $\{1,\ldots,n+1\}$ of size $k-1$, with the property that $p \in P$ implies x_p immediately precedes $x_{b_{i,1}}$ for some i in the linear ordering of the coordinates corresponding to C.

There are $k! \cdot \binom{n+1}{k-1} = k \cdot (n+1)!/(n+2-k)!$ such ordered pairs. However, not all of them correspond to chambers. We also need the condition

$$\sum_{i=1}^{l} |P \cap B_{\tau(i)}| \geq l,\ 1 \leq l \leq k-1, \tag{1}$$

because the immediate predecessor of $x_{b_{\tau(i),1}}$ must correspond to a member of one of the blocks $B_{\tau(1)},\ldots,B_{\tau(i-1)}$.

Let ρ be the $n+1$-cycle $(12\ldots k)$. We claim that for any choice of τ and P, exactly one of the pairs $(\tau,P),(\tau\rho,P),\ldots,(\tau\rho^{k-1},P)$ satisfies (1). Choose j such that

$$\left(\sum_{i=1}^{j} |P \cap B_{\tau(i)}|\right) - j \cdot (k-1)/k$$

is minimal for $1 \leq j \leq k$. We can then check that $\tau\rho^j$ is the proper choice. Thus, $|\{C \in \mathcal{C} : C \cap X \neq \emptyset\}| = (1/k) \cdot k \cdot (n+1)!/(n+2-k)! = (n+1)!/(n+2-k)!$.□

In terms of Stirling numbers, we see that

$$|\mathcal{R}| = \sum_{k} S(n+1,k)(n+1)!/(n+2-k)!.$$

This sum can be interpreted combinatorially as the number of ways to partition $\{1,\ldots,n+1\}$ into blocks and then assign each block a distinct element of $\{1,\ldots,n+2\}$, with the condition that the block containing 1 is assigned 1. Thus, the sum is equal to the number of functions f from $\{1,\ldots,n+1\}$ to $\{1,\ldots,n+2\}$ such that $f(1)=1$. This number is clearly $(n+2)^n$, which is the desired result since the Coxeter number of A_n is $n+1$.

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