Intersection Numbers, Eigenvalues and Fusion in the Orbital Scheme $B_{3,2}(q)$

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§1. Introduction

Consider the transitive action of finite Chevalley group $G = X_l(q)$ on the set $\gamma_l^i = (G : P_i)$ of cosets of P_i in G, where l is the rank of G (i.e. the number of nodes in the Dynkin diagram X_l of G) and P_i is the maximal parabolic subgroup corresponding to the i^{th} node of the diagram. The orbitals of the permutation group (G, γ_l^i) are the relations of an association scheme, usually denoted $X_{l,i}(q)$, which is an important object in algebraic combinatorics (see [BI] and [BCN]). In the case of such schemes which are P–polynomial, formulas for the intersection numbers and eigenvalues are known (see [UZ-P] and [St], respectively), and a description of all fusion schemes can be found in [M], [IMU].

In this paper we determine the intersection numbers, eigenvalues, and fusion schemes in the simplest case of an orbital association scheme $X_{l,i}(q)$ which is *not* P–polynomial, namely the scheme $B_{3,2}(q)$ coming from the action of group $B_3(q)$ (equivalently, $PSp(6,q)$) on the set of 2–dimensional totally isotropic subspaces in a 6-dimensional symplectic space over $GF(q)$. Although the relations of $B3, 2(q)$ can be recovered as orbits of edge pairs of a generalized cubeoctahedron, it is our methodology that we wish to stress here as it has application to classical orbital schemes for which there is no apparent physical model.

Considérons l'action transitive du groupe de Chevalley fini $G = X_l(q)$ sur l'ensemble quotient $\gamma_l^i = (G : P_i)$, où l est le rang de G (c.a.d. le nombre de sommets du graphe

All authors were supported by NSF grant DMS–9115473.
The author A. Woldar was supported by NSF grant DMS–9304580.

de Dynkin X_l de G) et P_i est le sous-groupe parabolique maximal associé au $i^{ème}$ sommet du graphe. Les orbitales du groupe de permutations (G, γ_l^i) sont les relations d'un schéma d'association, noté d'habitude $X_{l,i}(q)$, qui est un objet important en combinatoire algébrique (voir [BI] et [BCN]). Dans le cas de tels schémas qui sont P-polynômiaux, on connait des formules pour les nombres d'intersection et pour les valeurs propres (voir [UZ-P] et [St], respectivement), et on trouve une description de tous les schémas de fusion dans [M], [IMU].

Dans cet article, on détermine les nombres d'intersection, les valeurs propres, et les schémas de fusion dans le cas le plus simple d'un schéma d'association orbital $X_{l,i}(q)$ qui *n'est pas* P-polynômial, c'est à dire le schéma $B_{3,2}(q)$ qui provient de l'action du groupe $B_3(q)$ (ou $PSp(6,q)$) sur l'ensemble des plans totalement isotropes d'un espace à 6-dimensions symplectique sur $GF(q)$. Quoique on puisse retrouver les relations de $B3,2(q)$ comme orbites de paires d'arêtes d'un cubeoctaèdre généralisé, c'est sur notre méthodologie que nous voulons mettre l'accent, puisqu'elle s'applique aux schémas orbitaux classiques pour lesquels il n'y a pas de modèle physique évident.

§2. Association Schemes

An *association scheme* $(X, \{R_i\}_{0\leq i\leq d})$ is a set X together with a family of binary relations $R_0, R_1, \ldots, R_d$ such that:

(i) the relations form a partition of $X \times X$, i.e. $X \times X = \cup_{0\leq i\leq d} R_i$ and $R_i \cap R_j = \emptyset$ for $i \neq j$;

(ii) R_0 is the diagonal relation on X, i.e. $R_0 = \{(x,x) | x \in X\}$;

(iii) for any relation R_i, its transpose relation ${}^tR_i = \{(y,x) | (x,y) \in R_i\}$ is again a relation of the scheme, i.e. ${}^tR_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$;

(iv) for any $(x,y) \in R_k$, the number p_{ij}^k of elements $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ depends only on i, j, k, i.e. p_{ij}^k is independent of the representative (x,y) from R_k.

The numbers p_{ij}^k are called the *intersection numbers* of the association scheme $(X, \{R_i\}_{0\leq i\leq d})$ (see [BI] or [BCN]).

Let (G, X) be a transitive permutation group (so G acts faithfully and transitively on the set X). An *orbital* of (G, X) is, by definition, an orbit of $(G, X \times X)$, where the action of G on $X \times X$ is the natural induced action $(x,y)^g = (x^g, y^g)$. It is convenient to consider orbitals as a binary relations (or directed graphs) on X. When this is done, it is easy to see that set X together with the family of orbitals of (G, X) is an association scheme, called an *orbital scheme.*

Let $(X, \{R_i\}_{0\leq i\leq d})$ be an association scheme. The adjacency matrices $A_0 = I, A_1, \ldots, A_d$, which correspond, respectively, to the relations $R_0, R_1, \ldots, R_d$, generate a vector space $\mathcal{A}$ over the complex numbers, which is closed under matrix multiplication (and so is also an algebra!). $\mathcal{A}$ is called the *Bose-Mesner algebra* of the scheme $(X, \{R_i\}_{0\leq i\leq d})$. Moreover, the structure constants of $\mathcal{A}$ are precisely the intersection numbers of the scheme, i.e.

$$A_iA_j = \sum_{i=0}^{d} p_{ij}^k A_k.$$

In the case of an orbital scheme, the Bose-Mesner algebra is just the Hecke algebra (see [B]) or, equivalently, the Schur V–ring (see [FKM]).

As an abstract algebra, $\mathcal{A}$ is always semisimple. Thus, there exists a basis (unique up to order) consisting of the primitive orthogonal idempotents $E_0 = J, E_1, \ldots, E_d$ of $\mathcal{A}$. (Here J denotes the all–ones matrix.) The *eigenvalues* $p_i(j)$ are now defined to be the complex numbers determined by

$$A_i = \sum_{j=0}^{d} p_i(j)E_j.$$

We end this section with the definition of fusion scheme. Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ and $\mathcal{Y} = (X, \{S_j\}_{0\le j\le e})$ be association schemes on the common set X. We say $\mathcal{Y}$ is a *fusion scheme* of $\mathcal{X}$ if for every i, $0 \le i \le d$, relation R_i of $\mathcal{X}$ is a subset of relation S_j of $\mathcal{Y}$ for some j, $0 \le j \le e$. That is, fusion scheme $\mathcal{Y}$ is obtained by fusing together relations from scheme $\mathcal{X}$ in a very restricted way (so that $\mathcal{Y}$ is itself an association scheme).

§3. Coxeter and Lie Geometries

An *incidence system* over type set Δ is a triple (Γ, I, t), where Γ is a set (whose elements are called *objects*), I is a symmetric and reflexive binary relation on Γ (called the *incidence relation*) and t is a map from Γ into Δ (called the *type function*). The *rank* of the incidence system is defined to be $|\Delta|$. It is convenient to write Γ in place of (Γ, I, t) when doing so will not lead to confusion. Let Γ and Γ' be incidence systems defined over the same type set Δ. A *morphism* of Γ into Γ' is a map $\phi : \Gamma \to \Gamma'$ which preserves incidence. We say ϕ is *type–preserving* if, in addition, $t(A) = t(A^\phi)$ for all $A \in \Gamma$.

An important example of the above is the so–called *group incidence system* $\Gamma(G, G_s)_{s\in S}$. Here G is an abstract group and $\{G_s\}_{s\in S}$ is a family of distinct subgroups of G. The objects of $\Gamma(G, G_s)_{s\in S}$ are the cosets of G_s in G for all possible $s \in S$. Cosets α and β are incident precisely when $\alpha \cap \beta \neq \emptyset$. The type function is defined by $t(\alpha) = s$ where $\alpha = xG_s$ for some $x \in G$.

Let (W, S) be a *Coxeter system*, i.e. W is a group with set of distinguished generators given by $S = \{s_1, s_2, \ldots, s_l\}$ and generic relations $(s_i, s_j)^{m_{ij}} = 1$. Here $M = (m_{ij})$ is a symmetric $l \times l$ matrix with $m_{ii} = 1$ and off–diagonal entries satisfying $m_{ij} \ge 2$ (allowing $m_{ij} = \infty$ as a possibility, in which case the relation $(s_i, s_j)^{m_{ij}} = 1$ is omitted). Letting $W_i = \langle S \setminus \{s_i\}\rangle$, $1 \le i \le l$, we obtain a group incidence system $\Gamma_W = \Gamma(W, W_i)_{1\le i\le l}$ called the *Coxeter geometry* of W. The W_i are referred to as the *maximal standard subgroups* of W (see [B]).

Let G be a group, B and N subgroups of G, and S a collection of cosets of $B \cap N$ in N. We call (G, B, N, S) a *Tits system* (or we say that G *has a* BN*-pair*) if

(i) $G = \langle B, N \rangle$ and $B \cap N$ is normal in N,

(ii) S is a set of involutions which generate $W = N/B \cap N$,

(iii) $sBw \subset BwB \cup BswB$ for any $s \in S$ and $w \in W$,

(iv) $sBs \neq B$ for all $s \in S$.

Properties (i)–(iv) imply that (W, S) is a Coxeter system (see [B]). Whenever (G, B, N, S) is a Tits system, we call group W the Weyl group of the system, or, more usually, the Weyl group of G. The subgroups P_i of G defined by $P_i = BW_iB$ are called the *standard maximal parabolic subgroups* of G. The group incidence system $\Gamma_G = \Gamma(G, P_i)_{1 \leq i \leq l}$ is commonly referred to as the *Lie geometry* of G (see [T]). Note that the Lie geometry of G and the Coxeter geometry of the corresponding Weyl group W have the same rank. In fact there is a type–preserving morphism from Γ_G onto Γ_W given by $gP_i \mapsto wW_i$, where w is determined from the equality $BgP_i = BwP_i$ of double cosets. This morphism is called *retraction* (see [T]).

For (G, Λ) a general permutation group with orbits $\Lambda_1, \ldots, \Lambda_r$ and corresponding one–point stabilizers $G_1, \ldots, G_r$, the orbitals of (G, Λ) are in one–one correspondence with the double cosets G_igG_j, $1 \leq i, j \leq r$, $g \in G$. In the case where G is a Chevalley group and $G_i = P_i$ are the maximal parabolic subgroups of G (so that Λ coincides with the set of objects of Γ_G), properties (i)–(iv) of a Tits system give a natural one–one correspondence between double cosets P_igP_j of G and double cosets W_iwW_j of corresponding Weyl group W. As a consequence we have a natural one–one correspondence between the respective orbitals of (G, Γ_G) and (W, Γ_W). Finally, this gives a bijection between the respective relations of the orbital schemes $X_{l,i}(q)$ and $X_{l,i}$, the latter coming from the action of $W = W(X_l)$ acting on the cosets of W_i in W.

More explicitly, we can identify Γ_W as a subgeometry of Γ_G in the following manner. For a fixed Borel subgroup B of G, define $T = \bigcap_{w \in W} B^w$. (We also have $T = B \bigcap N$ in the language of Tits systems. T is called a *maximal torus*.) Consider now the subset Ω of Γ_G consisting of all cosets which contain T. Then the incidence system $(\Omega, I_\Omega, t_\Omega)$—where I_Ω and t_Ω denote the respective restictions of incidence and type in Γ_G to Ω—is isomorphic to Γ_W. In fact, restriction to Ω of the retraction morphism defined above yields an isomorphism. (See [T] for a full discussion on this.) Thus, not only are the orbitals of (G, Γ_G) and (W, Γ_W) in one–one correspondence as mentioned above, but, more strongly, we can represent each orbital of (G, Γ_G) by an ordered pair of objects from Γ_W.

§4. Embeddings in the Lie Algebra

Throughout this section we assume (G, B, N, S) is a Tits system which arises in connection with Chevalley group G, although we point out that the results of this section remain valid in a far more general setting (see [U1],[U2],[U3]). We write $G = X_l(K)$ to signify that G is a Chevalley group over the field K, with associated Dynkin diagram X_l. We are most interested in the case when K is finite, and we shall write $X_l(q)$ instead of $X_l(GF(q))$ in that case.

So, fix Chevalley group $G = X_l(K)$ with corresponding Weyl group W. As in the previous section, denote by Γ_W and Γ_G their associated Coxeter and Lie geometries. Let $\mathcal{L} = \mathcal{H} \oplus \mathcal{L}^+ \oplus \mathcal{L}^-$ be the Lie algebra corresponding to G. Following convention, we refer to $\mathcal{H}$, $\mathcal{L}^+$, $\mathcal{L}^-$ and $\mathcal{H} \oplus \mathcal{L}^+$ as, respectively, the *Cartan subalgebra*, *positive root space*, *negative root space* and (*positive*) *Borel subalgebra* with respect to the given decomposition of $\mathcal{L}$. We also use the familiar bracket notation $[\ ,\ ]$ to indicate Lie product.

Below, we turn our attention to a method for embedding Γ_W and Γ_G in $\mathcal{L}$. As the reader shall see, this method actually embeds Γ_W in the Cartan subalgebra $\mathcal{H}$ and Γ_G in the Borel subalgebra $\mathcal{U} = \mathcal{H} \oplus \mathcal{L}^+$.

It is well known (see [FV], for example) that the Coxeter geometry Γ_W of W can be embedded in l-dimensional Euclidean space, which, in the case when K is the real number field, can be identified with the Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$. Let's consider this embedding more precisely.

Let $A = (a_{ij})$ be the Cartan matrix corresponding to root system Φ of W. We consider the lattice $\mathcal{R}$ which is generated by the simple roots $\alpha_1, \ldots, \alpha_l$, and the reflections $r_1, \ldots, r_l$ of $\mathcal{R}$ defined by the equality

$$(\alpha_i)^{r_j} = \alpha_i - a_{ij}\alpha_j.$$

The set $S = \{r_1, \ldots, r_l\}$ is a set of Coxeter generators of Weyl group W. Let $\{\alpha_1^*, \ldots, \alpha_l^*\}$ be a dual basis of $\{\alpha_1, \ldots, \alpha_l\}$, i.e. α_i^* is the linear functional on $\mathcal{R}$ which satisfies $\alpha_i^*(\alpha_j) = \delta_{ij}$. We define the action of W on the dual lattice $\mathcal{R}^*$ by $l(x)^s = l(x^s)$, where $l(x) \in \mathcal{R}^*$ and $s \in S$. Consider the orbit $H_i = \{(\alpha_i^*)^w | w \in W\}$ of permutation group $(W, \mathcal{R}^*)$ which contains α_i^*. We give the set $H = \bigcup_i H_i$ the structure of an incidence system as follows. Linear functionals $l_1(x)$ and $l_2(x)$ are incident if and only if $l_1(\alpha)l_2(\alpha) \geq 0$ for all $\alpha \in \Phi$. The type function is defined by $t(l(x)) = i$ where $l(x) \in H_i$. It can be shown that (H, I, t) is isomorphic to the Coxeter geometry Γ_W. (In fact, there is a unique isomorphism of Γ_W with (H, I, t) which sends W_i to α_i^*, $1 \leq i \leq l$.) This gives the desired embedding since $H \subset \mathcal{R}^* \subset \mathcal{H}$. Moreover, this embedding obtains for K a field of sufficiently large characteristic since, in that case, $H \subset \mathcal{R}^* \otimes K = \mathcal{H}$. This latter fact is crucial to what follows.

We now consider an analogous embedding of the Lie geometry Γ_G of G into the Borel subalgebra $\mathcal{U} = \mathcal{H} \oplus \mathcal{L}^+$ of $\mathcal{L}$. Let $d = \sum_{i=1}^{l} \alpha_i^*$. Then we can take

$$\Phi^+ = \{\alpha \in \Phi | d(\alpha) \geq 0\}$$

to be our set of positive roots in Φ. For any $l(x) \in \mathcal{R}^*$, define

$$\eta^-(l) = \{\alpha \in \Phi^+ | l(\alpha) < 0\}.$$

Let $\mathcal{L}_\alpha$ be the root space corresponding to positive root α, so that $\mathcal{L}^+ = \sum_{\alpha \in \Phi^+} \mathcal{L}_\alpha$. For each $h \in H$ we define the subalgebra $\mathcal{L}_h = \sum_{\alpha \in \eta^-(h)} \mathcal{L}_\alpha$. Let $U_i = \{h + v | h \in H_i, v \in \mathcal{L}_h\}$ and $U = \bigcup_i U_i$. We give U the structure of an incidence system as follows. Elements $h_1 + v_1$ and $h_1 + v_1$ are incident if and only if each of the following hold:

(i) $h_1(\alpha)h_2(\alpha) \geq 0$ for all $\alpha \in \Phi$, i.e. h_1 and h_2 are incident in (H, I, t),

(ii) $[h_1 + v_1, h_2 + v_2] = 0$.

Element $h+v$ has type i if $h+v \in U_i$. In [U2] it is shown that this newly defined incidence system (U, I, t) is isomorphic to the Lie geometry Γ_G, provided the characteristic of K is sufficiently large to ensure isomorphism at the level of the subgeometries (H, I, t) and Γ_W. So let's make the assumption that the characteristic is sufficiently large. Then, analogous to the Weyl case, there exists a unique isomorphism of Γ_G onto (U, I, t) which sends P_i to α_i^*, $1 \leq i \leq l$. This gives an obvious embedding of Γ_G in $\mathcal{U}$, in which the image of subset Ω is H. ¿From our discussion in Section 3, it is clear that each orbital of (G, Γ_G) can be represented by an ordered pair (h, h') of objects from H. Moreover, by transitivity on objects of fixed type, we can further choose $h = \alpha_i^*$ where $t(h) = i$.

Finally, observe that the map $h + v \mapsto h$ (that is, the *canonical projection* of $\mathcal{U}$ onto $\mathcal{H}$) is a type-preserving morphism of incidence systems from (U, I, t) onto (H, I, t); in fact, it is essentially the retraction of Γ_G onto Γ_W introduced in Section 3.

§5. Characterizing Relations in $B_{3,2}(q)$

In this section we restrict our attention to the association scheme $B_{3,2}(q)$, q a prime power. That is, we consider the orbital scheme which corresponds to the action of Chevalley group $B_3(q)$ on cosets of the maximal parabolic subgroup which corresponds to the middle node of the diagram B_3. In classical terms, this is the action of the symplectic group $PSp(6, q)$ on the 2-dimensional totally isotropic subspaces of a 6-dimensional space equipped with a nondegenerate symplectic form.

A standard model for the Coxeter geometry Γ_W, $W = W(B_3)$, is obtained by considering as objects the vertices, edges and facets of the ordinary cube, with usual incidence. The relations of the scheme $B_{3,2}$ then become the orbits of edge pairs of the cube under its symmetry group. Letting e_0, e_1, e_2, e_3, e_4 denote the edges $\{(0,0,0),(1,0,0)\}$, $\{(0,0,0),(0,1,0)\}$, $\{(0,1,0),(1,1,0)\}$, $\{(0,1,0),(0,1,1)\}$, $\{(0,1,1),(1,1,1)\}$, respectively, the relations are characterized below.

Relation:	R_0	R_1	R_2	R_3	R_4
Edge pair:	(e_0, e_0)	(e_0, e_1)	(e_0, e_2)	(e_0, e_3)	(e_0, e_4)

Although this model provides a nice characterization of the relations of $B_{3,2}$, there is no easy way to extend it to a characterization of the relations of $B_{3,2}(q)$ that suits our objective. Thus it is preferable to return to our earlier model of geometry Γ_W as a subset H of the Cartan subalgebra $\mathcal{H}$.

Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ be a fundamental basis of root system Φ of W, so that we obtain, as the set of positive roots,

$$\Phi^+ = \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_2, \alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3, \alpha_3, 2\alpha_2+\alpha_3, 2\alpha_1+2\alpha_2+\alpha_3\}.$$

For each $(h, h') \in H^2$, define

$$\rho(h, h') = |\{\alpha \in \Phi^+ : h(\alpha)h'(\alpha) < 0\}|.$$

Then $\rho(h, h')$ uniquely determines the relation of $B_{3,2}$ to which (h, h') belongs:

Relation:	R_0	R_1	R_2	R_3	R_4
Value of ρ:	0	1	3	5	7

¿From this characterization one can readily determine all $h' \in H_2$ for which (α_2^*, h') belongs to relation R_i of $B_{3,2}(q)$ (see Table 1). This suffices to characterize the relations of $B_{3,2}(q)$ since, for any $v \in L_{h'}$, $(\alpha_2^*, h' + v)$ and (α_2^*, h') belong to the same relation.

Relation	Functionals $h' \in H_2$ such that (α_2^*, h') belongs to the relation
R_0	α_2^*
R_1	$-\alpha_1^*+2\alpha_2^*-2\alpha_3^*$, $-\alpha_1^*+3\alpha_3^*$, $\alpha_1^*-\alpha_2^*+2\alpha_3^*$, $\alpha_1^*+\alpha_2^*-2\alpha_3^*$
R_2	$2\alpha_1^*-\alpha_2^*$, $-2\alpha_1^*+\alpha_2^*$
R_3	$\alpha_1^*-2\alpha_2^*+2\alpha_3^*$, $\alpha_1^*-2\alpha_3^*$, $-\alpha_1^*-\alpha_2^*+2\alpha_3^*$, $-\alpha_1^*+\alpha_2^*-2\alpha_3^*$
R_4	$-\alpha_2^*$

TABLE 1. Relations of $B_{3,2}(q)$ characterized by representative pairs.

In what follows, it will be convenient to represent the root $a\alpha_1 + b\alpha_2 + c\alpha_3$ by its coordinate vector (a, b, c) with respect to the basis Π.

Recall that incidence in (U, I, t) (and so in Γ_G) is defined in terms of Lie product. For completeness, we list below those properties of Lie product which suffice in determining incidence.

(a) $[\ ,\ ]$ is bilinear,

(b) $[\ ,\ ]$ is skew-symmetric, i.e. $[x, y] = -[y, x]$ for all $x, y \in \mathcal{U}$,

(c) $[h, h'] = 0$ for all $h, h' \in \mathcal{H}$,

(d) $[h, e_\alpha] = h(\alpha)e_\alpha$ for all $h \in \mathcal{H}$, $\alpha \in \Phi^+$,

(e) For $\alpha, \beta \in \Phi^+$,

$$[e_\alpha, e_\beta] = \begin{cases} k_{\alpha\beta}e_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Phi^+; \\ 0, & \text{otherwise} \end{cases}$$

where $k_{\alpha\beta}$ is the (α, β)–entry in the array of Table 2.

In the above, e_α, $\alpha \in \Phi^+$, are appropriately chosen root vectors (see discussion on Chevalley basis in [C]).

	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(0,1,1)	(1,1,1)	(0,2,1)	(1,2,1)	(2,2,1)
(1,0,0)	0	1	0	0	1	0	1	2	0
(0,1,0)	−1	0	1	0	2	1	0	0	0
(0,0,1)	0	−1	0	1	0	0	0	0	0
(1,1,0)	0	0	−1	0	1	2	0	0	0
(0,1,1)	−1	−2	0	−1	0	0	0	0	0
(1,1,1)	0	−1	0	−2	0	0	0	0	0
(0,2,1)	−1	0	0	0	0	0	0	0	0
(1,2,1)	−2	0	0	0	0	0	0	0	0
(2,2,1)	0	0	0	0	0	0	0	0	0

TABLE 2. Chevalley structure constants $k_{\alpha\beta}$ $(\alpha, \beta \in \Phi^+)$

§6. Intersection Numbers, Eigenvalues and Fusion

Let us say that incidence chain $x_1 I x_2 I \ldots I x_m$ has type t_1–t_2–$\cdots$–t_m if $t_i = t(x_i)$ is the type of object x_i, $1 \le i \le m$. We begin with a proposition which provides a criterion for membership in relation R_j of $B_{3,2}(q)$.

PROPOSITION. *Let $y, z \in U_2$, with $(z, y) \in R_j$. Then j is uniquely determined by the number of chains from z to y of type 2–1–3–2. The correspondence is given in Table 3.*

Proof. It is clear that the number of such chains is an invariant of the relation R_j. To complete the proof, one merely needs to compute the number of chains for a representative pair from each relation. For example, one can do this for the pairs (α_2^*, h') which appear in Table 1. (Of course for R_1, R_2 and R_3, one has a choice of h'.) The number of chains corresponding to each relation is recorded in Table 3 below. Details on the underlying computations, as well as complete tables of intersection numbers, can be found in [HUW].

Relation of (z, y)	Number of Chains
R_0	$(q+1)^2$
R_1	$2q+1$
R_2	$q+1$
R_3	1
R_4	0

TABLE 3. The number of chains for each relation.

The eigenvalues of $B_{3,2}(q)$ are presented in Table 4 as entries of the first eigenmatrix $P = (p_j(i))$. They were obtained by standard methods: each row of P corresponds to a common left eigenvector of the intersection matrices $B_i = (p_{ij}^k)$. The corresponding multiplicities are given by $m_0 = 1$, $m_1 = q(q+1)(q^3+1)/2$, $m_2 = q^3(q^4+q^2+1)$, $m_3 = q^2(q^4+q^2+1)$ and $m_4 = q(q^2+1)(q^2+q+1)$, as can be computed directly from the table (see [BI]).

$$\begin{pmatrix} 1 & q(q+1)^2 & q^3(q+1) & q^4(q+1)^2 & q^7 \\ 1 & -(q+1) & q(q^2+1) & -q^3(q+1) & q^4 \\ 1 & -(q+1) & 0 & q(q+1) & -q^2 \\ 1 & (q+1)(q-1) & -q(q+1) & -q(q+1)(q-1) & q^3 \\ 1 & (2q-1)(q+1) & q(q+1)(q-1) & q^2(q-2)(q+1) & -q^4 \end{pmatrix}$$

TABLE 4. The first eigenmatrix $P = (p_j(i))$ of $B_{3,2}(q)$.

We now investigate the possible existence of fusion schemes for $B_{3,2}(q)$. In [V] all nontrivial fusion schemes of $B_{l,2}$ are classified. We see from that article that there are three such fusion schemes for $l = 3$: the first results from fusing relations 1 and 3; the second from simultaneously fusing relations 1 and 3 and relations 2 and 4; the third from fusing relations 1, 2 and 3. Each of these can easily be checked in the Lie case, and none works. For example, if the first fusion pattern were to work for general q, we would need

$$p_{12}^1 + p_{32}^1 = p_{12}^3 + p_{32}^3,$$

which yields (see Tables 7 and 9 of [HUW])

$$q^2 + q^4 = 2q^3.$$

But the only solution to the above is $q = 1$, bringing us back to the $B_{3,2}$ case. Thus there do not exist fusion schemes of this type in the Lie case. Similarly, one can check that neither of the remaining two fusion patterns give rise to fusion schemes in the Lie case.

Finally, it is possible for a fusion scheme to exist for some $q \neq 1$ which does not correspond to a fusion scheme of $B_{3,2}$. Of course, since the intersection numbers are polynomials, such a fusion scheme could exist for only a finite number of values of q. We checked all such possibilities and no fusion schemes were found. We conclude that $B_{3,2}(q)$ has no nontrivial fusion schemes.

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