# Combinatorial applications of canonical modules of Cohen-Macaulay rings

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Abstract. We study combinatorial applications of algebraic technique on canonical modules of Cohen-Macaulay rings and obtain some linear inequalities for Ehrhart polynomials of convex polytopes and for face numbers of matroid complexes.

## Introduction.

Combinatorial applications of Cohen-Macaulay rings, which originated in Stanley [7] (see also [9]), have great influence on both algebraic combinatorics and commutative algebra. On the other hand, since the manuscript [2] appeared, the concept of canonical modules has been an indispensable tool in the study of Cohen-Macaulay rings. In the present paper, after a brief discussion about canonical modules of Cohen-Macaulay rings (Section 1), we study combinatorial applications of canonical modules to Ehrhart polynomials of convex polytopes (Section 2) and to face numbers of matroid complexes (Section 3).

## 1. Canonical modules of Cohen-Macaulay rings

(1.1) Let $k$ be a field and $A$ a commutative $k$-algebra. We say that $A$ is *semi-standard* if $A$ has a direct sum decomposition $A = \oplus_{n \geq 0} A_n$ such that (i) $A_0 = k$, (ii) $A$ is finitely generated as a module over the subalgebra $k[A_1]$ and (iii) $\dim_k A_1 < \infty$. The *Hilbert function* of $A$ is defined to be

$$H(A,n) := \dim_k A_n \qquad \text{for } n = 0, 1, \ldots .$$

while the *Hilbert series* of $A$ is given by

$$F(A,\lambda) := \sum_{n=0}^{\infty} H(A,n)\,\lambda^n .$$

It is known that

$$F(A,\lambda) = (h_0 + h_1\lambda + \ldots + h_s\lambda^s)/(1-\lambda)^d$$

for some integers $h_0, h_1, \ldots, h_s$ with $h_s \neq 0$. Here $d$ is the Krull-dimension of $A$. We say that the vector $h(A) := (h_0, h_1, \ldots, h_s)$ is the *h-vector* of $A$.

(1.2) Suppose that a semi-standard $k$-algebra $A$ is Cohen-Macaulay. Then $h(A) \geq 0$, i.e., each $h_i \geq 0$ ([7]). Let $K_A$ denote the canonical module (e.g., [1, Chapter 3]) of $A$. Then there exists a graded ideal $I$ of $A$ with $I \cong K_A$ (up to shift in grading) if and only if $A$ is "generically Gorenstein," i.e., the localization $A_q$ is Gorenstein for every minimal prime ideal $q$ of $A$.

(1.3) The fundamental technique in the present paper is the following result which first appeared in [11].

**LEMMA.** Let a Cohen-Macaulay semi-standard k-algebra $A = \bigoplus_{n \geq 0} A_n$ be generically Gorenstein, and let $I = \bigoplus_{n \geq a} (I \cap A_n)$ with $I \cap A_a \neq (0)$ denote a graded ideal of of $A$ with $I \cong K_A$. Suppose that there exists a non-zero divisor $\theta \in I \cap A_a$ on $A$. Then the h-vector $h(A) = (h_0, h_1, \ldots, h_s)$ of $A$ satisfies the linear inequality

$$h_0 + h_1 + \ldots + h_i \leq h_s + h_{s-1} + \ldots + h_{s-i} \qquad (*)$$

for every $0 \leq i \leq [s/2]$.

(1.4) We say that a Cohen-Macaulay semi-standard k-algebra $A = \bigoplus_{n \geq 0} A_n$ is *level* if the canonical module $K_A = \bigoplus_{n \geq a} (K_A)_n$ with $(K_A)_a \neq (0)$ of $A$ is generated by $(K_A)_a$ as an A-module.

**COROLLARY.** Suppose that a Cohen-Macaulay semi-standard k-algebra $A = \bigoplus_{n \geq 0} A_n$ is both generically Gorenstein and level. Then the h-vector $h(A) = (h_0, h_1, \ldots, h_s)$ of $A$ satisfies the linear inequality $(*)$ for every $0 \leq i \leq s$.

*Proof.* A routine technique enables us to assume that $k$ is an infinite field. Let $I = \bigoplus_{n \geq a} (I \cap A_n)$ with $I \cap A_a \neq (0)$ denote a graded ideal of $A$ with $I \cong K_A$. Thanks to Lemma (1.3), what we must show is the existence of a non-zero divisor $\theta \in I \cap A_a$ on $A$. Let $\mathfrak{N}_A$ be the set of prime ideals of $A$ which belong to the ideal $(0)$. Since $A$ is Cohen-Macaulay, we know that the Krull-dimension of $A/q$ equals that of $A$ for each $q \in \mathfrak{N}_A$. We write $\mathfrak{U}$ for the (set-theoretic) union of all prime ideals $q \in \mathfrak{N}_A$. Recall that the set $\mathfrak{U}$ coincides with the set of zero-divisors on $A$. If $I \cap A_a \subset \mathfrak{U}$, then $I \cap A_a \subset q$ for some $q \in \mathfrak{N}_A$ since $k$ is infinite. Now, $A$ is level, thus $I$ is generated by $I \cap A_a$ as an A-module. Hence, if $I \cap A_a \subset q$ then $I \subset q$, which contradicts [1, Proposition (3.3.18)]. Q. E. D.

## 2. Ehrhart polynomials of convex polytopes

(2.1) A *polyhedral complex* $\Gamma$ in $\mathbb{R}^N$ is a finite set of convex polytopes in $\mathbb{R}^N$ such that (i) if $\mathcal{P} \in \Gamma$ and $\mathcal{F}$ is a face of $\mathcal{P}$ then $\mathcal{F} \in \Gamma$ and (ii) if $\mathcal{P}, \mathcal{Q} \in \Gamma$ then $\mathcal{P} \cap \mathcal{Q}$ is a face of $\mathcal{P}$ and of $\mathcal{Q}$. We are concerned with a polyhedral complex $\Gamma$ in $\mathbb{R}^N$ which satisfies the following conditions: (i) every vertex $\alpha$ of $\mathcal{P} \in \Gamma$ has integer coordinates, i.e., $\alpha \in \mathbb{Z}^N$, and (ii) the underlying space $X := \cup_{\mathcal{P}\in\Gamma} \mathcal{P}$ $( \subset \mathbb{R}^N )$ of $\Gamma$ is homeomorphic to the d-ball. Let $\partial X$ denote the boundary of $X$, thus $\partial X$ is homeomorphic to the (d-1)-sphere.

(2.2) Given an integer $n > 0$, write $nX$ for $\{ n\alpha \,;\, \alpha \in X \}$ and define $i(X,n)$ to be $\#(nX \cap \mathbb{Z}^N)$, the cardinality of $nX \cap \mathbb{Z}^N$. In other words, $i(X,n)$ is equal to the number of rational points $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in X$ with each $n\alpha_i \in \mathbb{Z}$. It is known that (i) $i(X,n)$ is a polynomial in $n$ of degree $d$, called the *Ehrhart polynomial* of $X$, (ii) $i(X,0) = 1$ and (iii) $(-1)^d i(X,-n) = \#[n(X-\partial X) \cap \mathbb{Z}^N]$ for every $n \geq 1$.

(2.3) Define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$( 1 - \lambda )^{d+1} \left[ 1 + \sum_{n=1}^{\infty} i(X,n) \lambda^n \right] = \sum_{i=0}^{\infty} \delta_i \lambda^i .$$

Then (i) $\delta_0 = 1$ and $\delta_1 = \#(X \cap \mathbb{Z}^N) - (d+1)$, (ii) $\delta_i = 0$ for each $i > d$, and (iii) $\delta_d = \#[(X-\partial X) \cap \mathbb{Z}^N]$. We say that $\delta(X) = (\delta_0, \delta_1, \ldots, \delta_d)$ is the *$\delta$-vector* of $X$. See, e.g., [3, Chapter IX] for geometric proofs of the above results due to Ehrhart.

(2.4) Fix a field $k$ and let $\xi_1, \ldots, \xi_N, t$ be indeterminates over $k$. If $\alpha = (\alpha_1, \ldots, \alpha_N) \in nX \cap \mathbb{Z}^N$, then we set $\xi^{\alpha} t^n = \xi_1^{\alpha_1} \ldots \xi_N^{\alpha_N} t^n$. We write $[A_k(\Gamma)]_n$ for the vector space spanned by all monomials $\xi^{\alpha} t^n$ with $\alpha \in nX \cap \mathbb{Z}^N$. Thus, $\dim_k [A_k(\Gamma)]_n = i(X,n)$. Let $A_k(\Gamma)$ denote $\oplus_{n \geq 0} [A_k(\Gamma)]_n$ with $[A_k(\Gamma)]_0 = k$ and define multiplication $(\xi^{\alpha} t^n)(\xi^{\beta} t^m)$ of monomials $\xi^{\alpha} t^n$ and $\xi^{\beta} t^m$ in $A_k(\Gamma)$ as follows: $(\xi^{\alpha} t^n)(\xi^{\beta} t^m) = \xi^{\alpha+\beta} t^{n+m}$ if there exists $\mathcal{P} \in \Gamma$ with $\alpha \in n\mathcal{P}$ and $\beta \in m\mathcal{P}$; $(\xi^{\alpha} t^n)(\xi^{\beta} t^m) = 0$ otherwise. Then $A_k(\Gamma)$ is a Cohen-Macaulay semi-standard $k$-algebra with $h(A_k(\Gamma)) = \delta(X)$ (see [10]). Let $\Omega(A_k(\Gamma))$ be the graded ideal $\oplus_{n \geq 1} [\Omega(A_k(\Gamma))]_n$ of $A_k(\Gamma)$ generated by those monomials $\xi^{\alpha} t^n$ with $n \geq 1$ and $\alpha \in n(X - \partial X) \cap \mathbb{Z}^N$. Then $\Omega(A_k(\Gamma))$ is the canonical module of $A_k(\Gamma)$.

(2.5) We say that $X$ is "star-shaped" with respect to a point $\alpha \in X - \partial X$ if $t\alpha + (1-t)\beta \in X - \partial X$ for every point $\beta \in X$ and for each real number $0 < t < 1$.

**THEOREM** ([6]). With the same notation as above, suppose that the set $(X - \partial X) \cap \mathbb{Z}^N$ is non-empty and that the underlying space $X$ is star-shaped with respect to some $v_1 \in (X - \partial X) \cap \mathbb{Z}^N$. Then the $\delta$-vector $\delta(X) = (\delta_0, \delta_1, \ldots, \delta_d)$ of $X$ satisfies the linear inequalities as follows:

$$\delta_0 + \delta_1 + \ldots + \delta_i \leq \delta_d + \delta_{d-1} + \ldots + \delta_{d-i}, \quad 0 \leq i \leq [d/2].$$

*Sketch of proof.* Fix an arbitrary polyhedral complex $\Gamma'(0)$ in $\mathbb{R}^N$ with the vertex set $\partial X \cap \mathbb{Z}^N$ whose underlying space is the boundary $\partial X$ of $X$. Since $X$ is star-shaped with respect to $v_1 \in (X - \partial X) \cap \mathbb{Z}^N$, we can define the cone $\Gamma'(1)$ over $\Gamma'(0)$ with apex $v_1$. Hence the vertex set of $\Gamma'(1)$ is $(\partial X \cap \mathbb{Z}^N) \cup \{v_1\}$

and the underlying space of $\Gamma'(1)$ is $X$. Let $(X-\partial X)\cap\mathbb{Z}^N = \{v_1,v_2,\ldots,v_\ell\}$ and, for each $2 \le j \le \ell$, construct a polyhedral complex $\Gamma'(j)$ with the vertex set $(\partial X\cap\mathbb{Z}^N)\cup\{v_1,v_2,\ldots,v_j\}$ and with the underlying space $X$ by the same way as in [5]. We write $\Gamma'$ for $\Gamma'(\ell)$. Then the element $\theta = \xi^{v_1}t + \xi^{v_2}t + \ldots + \xi^{v_\ell}t$ of $[\Omega(A_k(\Gamma'))]_1$ is a non-zero divisor on $A_k(\Gamma')$. Thus, Lemma (1.3) enables us to obtain the required inequalities.

Q. E. D.

EXAMPLE. Let $N = d = 3$ and $X = \mathcal{P} \cup \mathcal{Q}$, where $\mathcal{P}$ (resp. $\mathcal{Q}$) is the tetrahedron in $\mathbb{R}^3$ with the vertices (1,0,0), (0,1,0), (0,0,1), (-1,-1,-1) (resp. (1,0,0), (0,1,0), (0,0,1), (1,1,0) ). Then $(X-\partial X)\cap\mathbb{Z}^3 = \{ (0,0,0) \}$ and $\delta(X) = (1,2,1,1)$. Even though $X$ is star-shaped with respect to, e.g., (1/3,1/3,1/3), $X$ is not star-shaped with respect to (0,0,0).

COROLLARY (Stanley [11]). Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension $d$ and suppose that $(\mathcal{P}-\partial\mathcal{P})\cap\mathbb{Z}^N$ is non-empty. Then the $\delta$-vector $\delta(\mathcal{P}) = (\delta_0,\delta_1,\ldots,\delta_d)$ of $\mathcal{P}$ satisfies the following linear inequalities:

$$\delta_0 + \delta_1 + \ldots + \delta_i \le \delta_d + \delta_{d-1} + \ldots + \delta_{d-i}, \quad 0 \le i \le [d/2].$$

## 3. Face numbers of matroid complexes

(3.1) Let $V$ be a finite set, called the vertex set, and $\Delta$ a simplicial complex on $V$. Thus $\Delta$ is a collection of subsets of $V$ such that (i) $\{ x \} \in \Delta$ for every $x \in V$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma$ imply $\tau \in \Delta$. Each element of $\Delta$ is called a *face* of $\Delta$. Set $d := \max\{ \#(\sigma) ; \sigma \in \Delta \}$. Here $\#(\sigma)$ is the cardinality of $\sigma$ as a finite set. Then the dimension of $\Delta$ is defined to be $\dim \Delta := d - 1$. We say that $\Delta$ is *pure* if every maximal face has the same cardinality. We write $f_i = f_i(\Delta)$, $0 \le i < d$, for the number of faces $\sigma$ of $\Delta$ with $\#(\sigma) = i + 1$. Thus, $f_0 = \#(V)$.

We say that $f(\Delta) := (f_0, f_1, \ldots, f_{d-1})$ is the *f-vector* of $\Delta$. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of $\Delta$ by the formula

$$\sum_{i=0}^{d} f_{i-1} ( \lambda - 1 )^{d-i} = \sum_{i=0}^{d} h_i \lambda^{d-i}$$

with $f_{-1} = 1$.

(3.2) A simplicial complex $\Delta$ on the vertex set $V$ is called a *matroid complex* if the following conditions are satisfied :

(i) If $\sigma, \tau \in \Delta$ and $\#(\sigma) < \#(\tau)$, then there exists $x \in \tau$ such that $x \notin \sigma$ and $\sigma \cup \{ x \} \in \Delta$.

(ii) $\dim ( \Delta - x ) = \dim \Delta$ for every $x \in V$. Here $\Delta - x$ is the subcomplex $\{ \sigma \in \Delta ; x \notin \sigma \}$ of $\Delta$ on $V - \{ x \}$.

We remark that the above condition (ii) is required only to avoid the inessential case ; if $\dim ( \Delta - x ) < \dim \Delta$ then $\Delta$ is a cone over $\Delta - x$ with apex $x$, thus we should study $\Delta - x$ rather than $\Delta$.

For example, let $V$ be a finite set of non-zero vectors of a vector space over a field and suppose that the subspace spanned by $V$ is equal to the subspace spanned by $V - \{ x \}$ for every $x \in V$. Then the set $\Delta$ of linearly independent subsets of $V$ is a matroid complex.

(3.3) Now, what can be said about the h-vector of an arbitrary matroid complex ?

**THEOREM** ([4]). Suppose that $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is the h-vector of a matroid complex $\Delta$ of dimension $d - 1$. Then we have the linear inequality

$$h_0 + h_1 + \ldots + h_i \leq h_d + h_{d-1} + \ldots + h_{d-i}$$

for every $0 \leq i \leq [d/2]$.

*Sketch of Proof.* Let $V = \{x_1, x_2, \ldots, x_t\}$ be the vertex set of $\Delta$ and $k[\Delta] = k[x_1, x_2, \ldots, x_t]/I_\Delta$ the Stanley-Reisner ring ([9]) of $\Delta$ over a field $k$ with the standard grading, i.e., each $\deg x_i = 1$. Then the Krull-dimension of $k[\Delta]$ is $d$, and the Hilbert series of $k[\Delta]$ is $(h_0 + h_1\lambda + \ldots + h_d\lambda^d)/(1-\lambda)^d$. It is known [8] that $k[\Delta]$ is a level ring with $h_d = 0$. Moreover, $k[\Delta]$ is generically Gorenstein. Hence, thanks to Corollary (1.4), we obtain the inequalities as desired. Q. E. D.

**CONJECTURE.** (i) $h_i \le h_{d-i}$ for every $0 \le i \le [d/2]$;
(ii) $h_0 \le h_1 \le \ldots \le h_{[d/2]}$.

The above Conjecture is true if $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is a *pure* O-sequence (defined in, e.g., [8]).

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