Projective Invariants of Four Subspaces

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Abstract

We compute a set of generators of the ring of invariants for a set of 4 subspaces in a projective space. More precisely, let $K[Gr^4]$ be the ring of regular functions on the product of four Grassmannians. We present a set of generators for $K[Gr^4]^{SL(V)}$, where V is the underlining vector space. We also study the syzygies among the generators. In the case when the sum of the affine dimensions of the four subspaces is not equal to $2\dim(V)$, no syzygy exists and generators are algebraically independent. If the sum is $2\dim(V)$, there is one and only one syzygy, the precise form of which is given.

On calcule un système de générateurs de l'anneau des invariants d'un ensemble de 4 sous-espaces dans un espace projectif. Plus précisément, soit $K[Gr^4]$ l'anneau des fonctions réguliéres définies sur le produit de quatre variétés Grassmanniennes. On présente un ensemble de générateurs pour $K[Gr^4]^{SL(V)}$, où V est l'espace vectoriel sousjacent. On étude également les syzygies entre les générateurs. Si la somme des dimensions affines des quatre sous-espaces n'est pas égale à $2\dim(V)$, il n'existe aucune syzygie et les générateurs sont algébriquement indépendents. Si cette somme est égale à $2\dim(V)$, il existe une et une seule syzygie dont on précise la forme.

1. Introduction

The geometry of the configurations of linear subspaces of projective space is one of the most basic topics in projective geometry. One is interested in describing such configurations modulo collineations, i.e., modulo the action of the general linear group. For three or fewer subspaces, there are only finitely many possible configurations in a given dimension, which can be easily described. But for four subspaces, there are continuous families of configurations. Their classification is a so-called "tame" problem, in the sense of the theory of quivers [2], [8], while the classification of five or more subspaces is a "wild" problem and is extremely difficult. The configurations of four subspaces, which are "indecomposable" in the sense of the theory of quivers have been classified by Nazarova [9] and by Gelfand and Ponomarev [3].

In this paper, we study configurations of four subspaces from the viewpoint of invariant theory. A $(k-1)$-dimensional linear subspace of projective $(n-1)$-space can be identified with a k-dimensional linear subspace of an n-dimensional vector space, and thus may be thought of as a point in $G_{k,n}$, the Grassmann variety of k-planes in n-space. We can realize $G_{k,n}$ as the variety of non-zero decomposable vectors in the exterior power $\wedge^k(V)$,

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where V is an n-dimensional vector space over a field K of characteristic zero, modulo scalar multiples. The variety of all decomposable tensors together with zero is an affine variety, which we denote by $CG_{k,n}$. We call the algebra $K[CG_{k,n}]$ of restrictions to $CG_{k,n}$ of polynomial functions on $\wedge^k(V)$ the **coordinate ring** of $G_{k,n}$. The natural action of $GL(V)$ on $\wedge^k(V)$ preserves $CG_{k,n}$, and therefore $K[CG_{k,n}]$ carries the structure of $GL(V)$-module. The invariant-theoretic approach to the study of configurations of four subspaces asks for the $SL(V)$-invariants in $K[Gr^4] \stackrel{\text{def}}{=} K[CG_{k_1,n} \times CG_{k_2,n} \times CG_{k_3,n} \times CG_{k_4,n}] = K[CG_{k_1,n}] \otimes K[CG_{k_2,n}] \otimes K[CG_{k_3,n}] \otimes K[CG_{k_4,n}]$. We will give a complete answer to this question: for all choices of k_1, k_2, k_3 and k_4, we will describe generators and relations for the $SL(V)$-invariants in $K[Gr^4]$. The case of four medials, when n is even and $k_1 = k_2 = k_3 = k_4 = n/2$, which in some sense is the most interesting case, was considered by Turnbull [10] and described completely in [7]. Here we allow k_i to be arbitrary. We find that when $k_1 + k_2 + k_3 + k_4 = 2n$, there is a relative rich set of invariants, analogous to those of the case of four medials. We call these invariants of type II. When $k_1 + k_2 + k_3 + k_4 \neq 2n$, there are few invariants. Most of these arise when some subset of $\{k_1, k_2, k_3, k_4\}$ sum to n. We call these invariants of type I. However, there are other cases when invariants exist; these other invariants are more mysterious than those of type I or II, appearing only when $\delta = 2n - k_1 - k_2 - k_3 - k_4$ is a factor of k_i or $n - k_i$ for some i. Finally, we show that the $SL(V)$-invariants in $K[Gr^4]$ is a polynomial algebra generated by these generators.

2. The umbral symbolic method for Grassmannians

We follow the notation in [4]. Let V be an n-dimensional vector space over a field K of characteristic 0. Identify $GL(V)$ with $GL_n(K)$ via a given basis $e_1, e_2, \cdots, e_n$ of V. The standard action of $GL_n(K)$ on V is denoted as $g \cdot v$ for $g \in GL_n(K), v \in V$. Let $\wedge^k(V)$ be the vector space consisting of all skew-symmetric tensors of step k. The group action of $GL_n(K)$ on $\wedge^k(V)$ is linearly extended by $g \cdot (v_1 \wedge v_2 \cdots \wedge v_k) = g \cdot v_1 \wedge g \cdot v_2 \wedge \cdots \wedge g \cdot v_k$. It acts on $K[\wedge^k(V)]$, the algebra of polynomial functions on $\wedge^k(V)$, by $g \cdot f(s) = f(g^{-1} \cdot s)$, for $f \in K[\wedge^k(V)], s \in \wedge^k(V)$. Denote by $s_{i_1 i_2 \cdots i_k}$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, the coordinate functions on $\wedge^k(V)$. We also set $s_{i_{\sigma_1} i_{\sigma_2} \cdots i_{\sigma_k}} = (-1)^\sigma s_{i_1 i_2 \cdots i_k}$ and $s_{i_1 i_2 \cdots i_k} = 0$ if $i_p = i_q$ for some $p \neq q$. Then $K[\wedge^k(V)]$ is the ring of polynomials on $s_{i_1 i_2 \cdots i_k}$. For the purpose of the present paper, consider the algebra of polynomial functions $K[W]$ on $W = \wedge^{k_1}(V) \times \wedge^{k_2}(V) \times \wedge^{k_3}(V) \times \wedge^{k_4}(V)$. The action of $GL_n(K)$ on $K[W]$ is given by $g \cdot f(s_1, s_2, s_3, s_4) = f(g^{-1} \cdot s_1, g^{-1} \cdot s_2, g^{-1} \cdot s_3, g^{-1} \cdot s_4), s_i \in \wedge^{k_i}(V)$. Applying the symbolic method [4] for $K[W]$, let $L = L^+ = L_1 \cup L_2 \cup L_3 \cup L_4$ be a set of linearly ordered positive letters, where each L_i is an infinite set **belonging** to $\wedge^{k_i}(V)$. Let $P = P^- = \{1, 2, \cdots, n\}$ be a set of n negative places. We shall simply write the letterplace algebra $Super[L|P] \otimes K$ as $Super[L|P]$. The group $GL_n(K)$ acts on $Super[L|P]$ by $g \cdot (a|i) = \sum_j x_{ij}(a|j)$, where

$g^{-1}\cdot e_j = \sum_i x_{ij} e_i, a \in L$.

We now quote two important results regarding the letterplace algebra $Super[L|P]$. Let M_1 and M_2 be two multisets on L and P, respectively, of the same size. Let $Super^{(M_1,M_2)}[L|P]$ be the linear subspace of $Super[L|P]$ spanned by monomials of the letterplace pairs $(a|i)$, $a \in L$ and $i \in P$, having the letter content (or degree) M_1 and place content M_2. We have

Theorem 1 *[ref.[4]] Standard bitableaux $(D|E)$ of the letter content M_1 and the place content M_2 form a linear basis of $Super^{(M_1,M_2)}[L|P]$.*

Theorem 2 *[ref. [1]] Standard symmetrized bitableaux $(D|\boxed{E})$ of the letter content M_1 and the place content M_2 form a linear basis of $Super^{(M_1,M_2)}[L|P]$. Furthermore, if D is a diagram of shape λ, then the linear subspace*

$$S_{\lambda,D} = \langle (D|\boxed{E}) : E \textit{ is standard on } P\rangle_K,$$

is a $GL_n(K)$-irreducible module (called Schur module), as long as it is not a zero subspace.

In the present context, a symmetrized bitableau $(D|\boxed{\mathrm{E}})$ is given by

$$(D|\boxed{\mathrm{E}}) = \sum_\sigma (D|\sigma E),$$

where σ ranges over all column stabilizers of the tableau E. For example,

$$\left(D\,\middle|\,\boxed{\begin{matrix}12\\13\end{matrix}}\right) = 2\left(D\,\middle|\,\begin{matrix}12\\13\end{matrix}\right) + 2\left(D\,\middle|\,\begin{matrix}13\\12\end{matrix}\right).$$

The umbral **linear** operator $\mathcal{U}$ from $Super[L|P]$ to $K[W]$ is defined by
(i) If $a \in L_i$ belonging to $\wedge^{k_i}(V)$, set

$$\mathcal{U}\left((a^{(k_i)}|j_1 j_2 \cdots j_{k_i})\right) = s_{j_1 j_2 \cdots j_{k_i}},$$

which is the corresponding coordinate function on $\wedge^{k_i}(V)$; set $\mathcal{U}\left((a^{(p)}|u)\right) = 0$ if $p \neq k_i$, where u is a word of length p on L.
(ii) For distinct letters $a < b < \cdots < c$ in L, set

$$\mathcal{U}\left((a^{(p)}|u)(b^{(p')}|u')\cdots(c^{(p'')}|u'')\right) = \mathcal{U}\left((a^{(p)}|u)\right)\mathcal{U}\left((b^{(p')}|u')\right)\cdots\mathcal{U}\left((c^{(p'')}|u'')\right).$$

where $u, u', ..., u''$ are words of lengths $p, p', ..., p''$ on L.

Proposition 3 *The umbral symbolic operator $\mathcal{U}$ is a well-defined $GL_n(K)$-equivariant surjective map.*

A k-dimensional subspace V' of V can be identified with (up to a scalar) a decomposable skew-symmetric tensor $s = v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \wedge^k(V)$, where $v_1, \cdots, v_k$ is a basis of V'. The set of all decomposable tensors in $\wedge^k(V)$, together with $\{0\}$, is called a Grassmannian and denoted as $G_{k,n}$ (or, to be rigorous, is called the cone of a Grassmannian and denoted as $CG_{k,n}$). The Grassmannian (necessary and sufficient) condition for a skew symmetric tensor $s \in \wedge^k(V)$ to be in $G_{k,n}$, when translated in terms of the symbolic method, is given by: for two distinct letters a, b belonging to $\wedge^k(V)$, the evaluation of the function

$$\mathcal{U}\left((a^{(k)}b|i_1 \cdots i_{k+1})(b^{(k-1)}|j_1 \cdots j_{k-1})\right) \tag{1}$$

vanishes at s, for all elements $i_1, \cdots, i_{k+1}, j_1, \cdots, j_{k-1}$ in P.

Let I be the ideal of $Super[L|P]$ generated by all elements of the form in (1), as (a, b) ranges over all pairs of distinct letters belonging to L_i (the number k in (1) is changed to k_i accordingly), $i = 1, 2, 3, 4$. Then $\mathcal{U}$ induces a $GL_n(K)$-equivalent surjective map

$$\bar{\mathcal{U}} : Super[L|P]/I \to K[Gr^4],$$

where $K[Gr^4]$ is the coordinate ring on four Grassmannians $Gr^4 \stackrel{\text{def}}{=} G_{k_1,n} \times G_{k_2,n} \times G_{k_3,n} \times G_{k_4,n}$.

Given a quadruple $\mathbf{m} = (m_1, m_2, m_3, m_4)$ of nonnegative integers, denote by $K^{\mathbf{m}}[Gr^4]$ the corresponding homogeneous component of $K[Gr^4]$. To focus on $K^{\mathbf{m}}[Gr^4]$ symbolically, choose letters $a_1, \cdots, a_{m_1} \in L_1, b_1, \cdots, b_{m_2} \in L_2, c_1, \cdots, c_{m_2} \in L_3, d_1, \cdots, d_{m_4} \in L_4$. Let $M^{\mathbf{m}}$ be the $GL_n(K)$-submodule of $Super[L|P]/I$ spanned by the bitableaux $(D|E) + I$, where the content of D is

$$(a_1 \cdots a_{m_1})^{k_1}(b_1 \cdots b_{m_2})^{k_2}(c_1 \cdots c_{m_3})^{k_3}(d_1 \cdots d_{m_4})^{k_4}.$$

To study the coordinate ring of two Grassmannians, let $\mathbf{m} = (m_1, m_2, 0, 0)$. Then $K^{\mathbf{m}}[Gr^4]$ becomes $K^{(m_1,m_2)}[G_{k_1,n} \times G_{k_2,n}]$ and $M^{(m_1,m_2)}$ is spanned by the bitableaux $(D|E) + I$ where the content of D is $(a_1 \cdots a_{m_1})^{k_1}(b_1 \cdots b_{m_1})^{k_2}$. Assume $k_1 \geq k_2$ and impose the linear order $a_1 > \cdots > a_{m_1} > b_1 > \cdots > b_{m_2}$ for the letters. Form a Young diagram D by first juxtaposing horizontally the $m_1 \times k_1$ rectangular diagram $(a_1^k, a_2^{k_1}, \cdots, a_{m_1}^{k_1})$ with a $m_2 \times k_2$ rectangular diagram $(b_1^{k_2}, b_2^{k_2}, \cdots, b_{m_2}^{k_2})$, then, out of the lower right corner of the thinner rectangular diagram, cut a diagram such that the remaining part of the $m_2 \times k_2$ rectangular diagram has at most m_1 rows and at most $n - k_1$ columns. Turn it around by 180° and place it below the fatter rectangular diagram. The following is an example of such a diagram with

$k_1 = 5, k_2 = 4$:

$$\begin{array}{l} a_1a_1a_1a_1a_1b_1b_1b_1b_1 \\ a_2a_2a_2a_2a_2b_2b_2b_2 \\ a_3a_3a_3a_3a_3b_3 \\ a_4a_4a_4a_4a_4b_4 \\ a_5a_5a_5a_5a_5 \\ b_4b_4b_4 \\ b_3b_3b_3 \\ b_2 \quad . \end{array}$$

Let $\mu = (\mu_1, \mu_2, \cdots, \mu_{m_1})$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m_1} \geq 0$; set $\mu_i = 0$ if $i > m_1$. Then the shape of D is $\lambda = (k_1 + \mu_1, \cdots, k_1 + \mu_{m_1}, k_2 - \mu_{m_2}, \cdots, k_2 - \mu_1)$, where $k_1 + \mu_1 \leq n$. Call the collection of all such shapes as Λ. For a fixed pair (m_1, m_2), the diagram D only depends on λ. So we write D as D_λ. To be explicit,

$$D_\lambda = (a_1^{k_1} b_1^{\mu_1}, \cdots, a_1^{k_1} b_{m_1}^{\mu_{m_1}}, b_{m_2}^{k_2 - \mu_{m_2}}, \cdots, b_1^{k_2 - \mu_1})$$

Proposition 4 *The restriction of the umbral operator on the direct sum of Schur modules*

$$\oplus_{\lambda \in \Lambda} S_{\lambda, D_\lambda} \xrightarrow{\mathcal{U}} K^{(m_1, m_2)}[G_{k_1,n} \times G_{k_2,n}]$$

gives an isomorphism between the two $GL_n(K)$-modules. In particular, the set

$$\cup_{\lambda \in \Lambda} \{\mathcal{U}((D_\lambda \| \boxed{E})) : E \text{ is standard on } P \text{ of shape } \lambda\}$$

is a linear basis of $K^{(m_1,m_2)}[G_{k_1,n} \times G_{k_2,n}]$.

3. Invariants of four subspaces

Denote by $K[Gr^4]^{SL_n(K)}$ the set of elements X in $K[Gr^4]$ such that $g \cdot X = X$ for every $g \in SL_n(K)$. We call $K[Gr^4]^{SL_n(K)}$ the ring of invariants of 4 subspaces.

It is known that $K[Gr^4]^{SL_n(K)}$ is the sum of all one dimensional invariant subspaces of the $GL_n(K)$-module $K[Gr^4]$. Starting from the decompositions

$$\begin{aligned} K^{(m_1,m_2)}[G_{k_1,n} \times G_{k_2,n}] &\cong \oplus_{\lambda \in \Lambda} S_{\lambda, D_\lambda}, \quad k_1 \geq k_2, \\ K^{(m_3,m_4)}[G_{k_3,n} \times G_{k_4,n}] &\cong \oplus_{\lambda' \in \Lambda'} S_{\lambda', D_{\lambda'}}, \quad k_3 \geq k_4 \end{aligned}$$

we have

$$\begin{aligned} K^{\mathbf{m}}[Gr^4] &\cong K^{(m_1,m_2)}[G_{k_1,n} \times G_{k_2,n}] \otimes K^{(m_3,m_4)}[G_{k_3,n} \times G_{k_4,n}] \\ &\cong \oplus_{\lambda \in \Lambda, \lambda' \in \Lambda'} S_{\lambda, D_\lambda} \otimes S_{\lambda', D'_{\lambda'}}. \end{aligned} \tag{2}$$

It is also known that the tensor product $S_{\lambda,D_\lambda} \otimes S_{\lambda',D_{\lambda'}}$ has a 1-dimensional invariant subspace, and at most one such subspace, if and only if one can fit the diagrams D_λ and $D^*_{\lambda'}$, which is the 180° rotation of $D_{\lambda'}$, into a rectangular diagram D of width n. In this case, the 1-dimensional invariant subspace in $S_{\lambda,D_\lambda} \otimes S_{\lambda',D_{\lambda'}}$ is spanned by the single element $(D|E)$, where each row of E is $12\cdots n$. Its image $\mathcal{U}((D|E))$ is an invariant in $K[Gr^4]^{SL_n(K)}$. The pair of diagrams $(D_\lambda, D^*_{\lambda'})$ is called a **complementary pair**. Since (2) is an isomorphism, we have

Proposition 5 *All invariants in $K[Gr^4]^{SL_n(K)}$ are given by linear combinations of the images $\mathcal{U}((D|\boxed{E}))$, where each row of E is $12\cdots n$ and D comes from a complementary pair $(D_\lambda, D^*_{\lambda'})$.*

We say a diagram D is a juxtaposition, or sum, of diagrams D' and D'' if D' consists of some rows of D and D'' has the rest. Write $D = D' + D''$. A complementary pair $(D_\lambda, D^*_{\lambda'})$ is said to be a **juxtaposition**, or **sum**, of complementary pairs $(D_\mu, D^*_{\mu'})$ and (D_ν, D^*_ν) if $D_\lambda = D_\mu + D_\nu$ and $D^*_{\lambda'} = D^*_{\mu'} + D^*_{\nu'}$, such that D_μ contains a certain row of D_λ if and only if $D^*_{\mu'}$ contains the corresponding row of $D^*_{\lambda'}$. It will suffice to describe complementary pairs which are not juxtapositions. They correspond to generators of the ring of invariants $K[Gr^4]^{SL_n(K)}$. Such a complementary pair $(D_\lambda, D^*_{\lambda'})$ and the corresponding rectangular diagram D is called **primitive**.

We will describe 3 types of primitive complementary pairs, and hence 3 types of primitive diagrams. Without loss of generality, we may assume $k_1 + k_2 + k_3 + k_4 \le 2n$ (by taking duals if necessary) and $k_1 \ge k_2 \ge k_3 \ge k_4$.

Type I. Suppose some pair, say (k_1, k_3), or some triple, say (k_1, k_2, k_3), or the quadruple (k_1, k_2, k_3, k_4), sum to n. Then the pairs

$$(a^{k_1}, c^{k_3}), (a^{k_1}b^{k_2}, c^{k_3}), (a^{k_1}b^{k_2}, d^{k_4}c^{k_3})$$

define primitive complementary pairs.

Type II. Suppose $k_1 + k_2 + k_3 + k_4 = 2n$. Then a pair of diagrams of the form

$$\begin{pmatrix} a^{k_1}b^s & d^{k_4-t} \\ b^{k_2-s} , & d^t c^{k_3} \end{pmatrix}$$

where $0 \le s \le \min\{n - k_1, k_2\} = n - k_1$, $0 \le t \le \min\{n - k_3, k_4\} = k_4$, $s - t = n - k_1 - k_4$, form primitive complementary pairs.

Type III. Suppose $k_1 + k_2 + k_3 + k_4 < 2n$. Write

$$\delta = 2n - (k_1 + k_2 + k_3 + k_4).$$

Let p be the largest nonnegative integer such that $p\delta < k_4$ and q be the largest nonnegative integer such that $q\delta < n - k_1$. Consider the following possibilities:

(1) $1 \le q \le p$ and $(q+1)\delta = n - k_1$;
(2) $1 \le q \le p$ and $(q+1)\delta = n - k_2$;
(3) $0 \le p < q$ and $(p+1)\delta = k_4$;
(4) $0 \le p < q$ and $(p+1)\delta = k_3$.

Remark. Alternatives (1) and (2) exclude alternatives (3) and (4), and vice versa. Also only one of (1) or (2) can hold unless $k_1 = k_2$; and similarly for (3) and (4).

Corresponding to these 4 possibilities are the following 4 different primitive complementary pairs of diagrams:

(1) The numerical condition implies that $(2q+1)n = (q+1)(k_1+k_2+k_3+k_4) - k_1$, which suggests that the pair of diagrams have $(2q+1)$ rows, the diagram D_λ has content $(a_1 \cdots a_q)^{k_1}(b_1 \cdots b_{q+1})^{k_2}$ and the diagram $D_{\lambda'}$ has content $(c_1 \cdots c_{q+1})^{k_3}(d_1 \cdots d_{q+1})^{k_4}$. The complementary pair is given by

$$\begin{pmatrix} a_1^{k_1} b_1^{q\delta} & d_2^{\delta} \\ \vdots & \vdots \\ a_q^{k_1} b_q^{\delta} & d_{q+1}^{q\delta} \\ b_{q+1}^{k_2} & d_{q+1}^{k_4-q\delta} c_{q+1}^{k_3} \\ b_q^{k_2-\delta} & d_q^{k_4-(q-1)\delta} c_q^{k_3} \\ \vdots & \vdots \\ b_2^{k_2-(q-1)\delta} & d_2^{k_4-\delta} c_2^{k_3} \\ b_1^{k_2-q\delta} \quad , & d_1^{k_4} c_1^{k_3} \end{pmatrix}$$

(2) We have $(2q+1)n = (q+1)(k_1+k_2+k_3+k_4) - k_2$. So we use letters $a_1, ..., a_{q+1}$, $b_1, ..., b_q$, $c_1, ..., c_{q+1}$, $d_1, ..., d_{q+1}$ to make a complementary pair having $(2q+1)$ rows:

$$\begin{pmatrix} a_1^{k_1} b_1^{n-k_1-\delta} & d_2^{\delta} \\ \vdots & \vdots \\ a_q^{k_1} b_q^{n-k_1-q\delta} & d_{q+1}^{q\delta} \\ a_{q+1}^{k_1} & d_{q+1}^{k_4-q\delta} c_{q+1}^{k_3} \\ b_q^{k_1-\delta} & d_q^{k_4-(q-1)\delta} c_q^{k_3} \\ \vdots & \vdots \\ b_2^{k_1-(q-1)\delta} & d_2^{k_4-\delta} c_2^{k_3} \\ b_1^{k_1-q\delta} \quad , & d_1^{k_4} c_1^{k_3} \end{pmatrix}$$

(3) We get $2(p+1)n = (p+1)(k_1+k_2+k_3+k_4) + k_4$. So we use letters $a_1, ..., a_{p+1}$,

$b_1, \ldots, b_{p+1}$, $c_1, \ldots, c_{p+1}$, $d_1, \ldots, d_{p+2}$ to make a complementary pair having $(2p+2)$ rows:

$$\begin{pmatrix} a_1^{k_1} b_1^{n-k_1-\delta} & d_2^{\delta} \\ \vdots & \vdots \\ a_p^{k_1} b_p^{n-k_1-p\delta} & d_{p+1}^{p\delta} \\ a_{p+1}^{k_1} b_{p+1}^{n-k_1-k_4} & d_{p+2}^{k_4} \\ b_{p+1}^{n-k_3-\delta} & d_{p+1}^{k_4-p\delta} c_{p+1}^{k_3} \\ \vdots & \vdots \\ b_2^{n-k_3-p\delta} & d_2^{k_4-\delta} c_2^{k_3} \\ b_1^{n-k_3-k_4} & d_1^{k_4} c_1^{k_3} \end{pmatrix},$$

(4) The condition implies $2(p+1)n = (p+1)(k_1+k_2+k_3+k_4)+k_3$. Hence we use letters $a_1, \ldots, a_{p+1}$, $b_1, \ldots, b_{p+1}$, $c_1, \ldots, c_{p+2}$, $d_1, \ldots, d_{p+1}$ to make a complementary pair having $(2p+2)$ rows:

$$\begin{pmatrix} a_1^{k_1} b_1^{n-k_1-\delta} & d_2^{\delta} \\ \vdots & \vdots \\ a_p^{k_1} b_p^{n-k_1-p\delta} & d_{p+1}^{p\delta} \\ a_{p+1}^{k_1} b_{p+1}^{n-k_1-k_3} & c_{p+2}^{k_3} \\ b_{p+1}^{n-k_4-\delta} & d_{p+1}^{k_4-p\delta} c_{p+1}^{k_3} \\ \vdots & \vdots \\ b_2^{n-k_4-p\delta} & d_2^{k_4-\delta} c_2^{k_3} \\ b_1^{n-k_4-k_3} & d_1^{k_4} c_1^{k_3} \end{pmatrix},$$

If $k_1 + k_2 < n$, then in addition to the above four possibilities is a collection of 4 similar alternatives with first row $(a_1^{k_1} b_1^{k_2}\ d_1^{n-k_1-k_2})$ and last row $(b_2^{\delta}\ d_1^{n-k_3-\delta} c_1^{k_3})$. If p' is the largest integer such that $p'\delta < k_2$, q' is the largest integer such that $q'\delta < n - k_3$, the corresponding numerical conditions for these 4 alternatives are

(1′) $1 \le q' \le p'$ and $(q'+1)\delta = n - k_3$;
(2′) $1 \le q' \le p'$ and $(q'+1)\delta = n - k_4$;
(3′) $0 \le p' < q'$ and $(p'+1)\delta = k_2$;
(4′) $0 \le p' < q'$ and $(p'+1)\delta = k_1$.

Following is the main proposition.

Proposition 6 *The three types of complementary pairs listed above are the only primitive complementary pairs.*

As a consequence , we have

Theorem 7 *A set of algebraic generators of the ring of invariants* $K[Gr^4]^{SL_n(K)}$ *of four subspaces is given by* $\mathcal{U}((D|E))$, *where* D *is formed by a complementary pair* $(D_\lambda, D^*_{\lambda'})$ *of one of the types (I) – (III), and* E *is the rectangular Young diagram having the same number of rows as* D *and having* $12\cdots n$ *for each of its rows.*

For a given set of four Grassmannians, the number of generators is quite few. In fact, the number of type I invariants is at most 6; this number is reached when n is even and $k_1 = k_2 = k_3 = k_4 = \frac{n}{2}$. Type II invariants exist only when $k_1 + k_2 + k_3 + k_4 = 2n$, with the number of such invariants being $1+min\{n-k_1, k_2, n-k_3, k_4\} = 1+min\{n-k_1, k_4\} \leq 1+\frac{n}{2}$. Type III invariants occur when $k_1+k_2+k_3+k_4 < 2n$ and δ is a factor of some of the integers in the list $\{k_1, k_2, k_3, k_4, n-k_1, n-k_2, n-k_3, n-k_4\}$. The number of different type III invariants is $\leq \min(4, i)$, where i is the number of integers in the list that has δ as a factor.

Remark. We can have the situation where δ is a factor of every integer in the list, whereas no type III invariants exists. For example, if $n = 6$, $k_1 = 4$, $k_2 = k_3 = k_4 = 2$, then $\delta = 2$, $p = 0$, $q = 0$, so none of the conditions (1)-(4) exists. Since $k_1 + k_2 = n$, no diagrams (1′)-(4′) exists either.

The following theorem gives the structure of the invariant ring $K[Gr^4]^{SL(V)}$.

Theorem 8 *The ring of invariants of four subspaces is a polynomial ring generated by the generators described in the previous theorem. To be more precise, the generators are algebraically independent as long as the quadruple* (k_1, k_2, k_3, k_4) *is not of the form* (k, k, l, l) *where* $k + l = n$*; otherwise, the invariant* $\mathcal{U}((D|E))$ *where* $D = (a^k c^l, b^k d^l)$ *and* $E = (12\cdots n, 12\cdots n)$ *can be expressed in terms of the rest of invariants, which are algebraically independent.*

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