Algebraic and analytic approaches for the genus series for 2-cell embeddings on orientable and nonorientable surfaces

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February 9, 1994

1 The genus series and some background

This talk is concerned with properties of the generating series that counts the number of rooted graphs embedded on orientable surfaces with respect to the number of vertices, edges, faces, and genus, and with combinatorial aspects of the symmetric group.

Of course, the genus is determined by the Euler-Poincaré formula once the numbers of vertices, edges and faces has been specified so, in this sense, the preservation of genus is redundant. Informally, it is to be imagined that a graph has been drawn on a surface in such a way that the continuous line segments representing edges do not intersect each other, except at the vertices. On deletion of the graph, the surface separates into regions homeomorphic to open discs, to be called the faces of the map. Finally, a map is rooted by selecting a mutually incident vertex, edge and face. Equivalently, this can be done by selecting an edge and a direction to the edge. The generating series constructed in this way is called the *genus series* for orientable surfaces. Throughout, it is to be assumed that all maps are rooted.

There is a long history of work on maps by combinatorialists, particularly on the sphere, driven partly by the Four Colour Problem. Tutte's enumerative work in this connexion has inspired subsequent investigations extending his original constructions. Work on the genus series may lead to information about classes of maps and their interrelation.

Interest in maps has been heightened by its connexion to questions arising in mathematical physics. For example, random surfaces occur in the study of matter coupled to 2-dimensional gravity, and serve as a toy model for string theory. The genus series can be regarded as a discretisation of a functional integral of a particular action, the Einstein action, over all metrics. Vertices of degree one and two are not allowed in this context. In the double scaling limit, when certain combinatorial parameters are mutually scaled and allowed to tend to infinity, a finite limit is obtained. It is then supposed that information of physical importance survives the discretisation and the passage to the limit. The resulting series is called the *partition function* for the physical model.

Of particular interest in this context is the discretisation where only vertices of degree four occur. This corresponds to the situation where the fourth power of the potential function ϕ occurs, and the model is therefore called the *ϕ^4-model.* To the combinatorialist, the maps are therefore *vertex 4-regular.* In addition, there is interest in the case when there is no condition imposed on vertex degrees. This corresponds to the situation where $\log(1-\phi)^{-1}$ occurs. The model is called the *Penner model.* Combinatorially, this corresponds to the set of all maps. These observations account for an interest in the set of all maps, and the set of vertex k-regular maps which will be evident in this talk.

A great deal of work has been done in mathematical physics from the seventies' onwards on the partition function, Feynman integrals, and the discretisation of the partition function. In this talk, I will give a combinatorialisation of the general approach that they have used for the vertex 4-regular case. This is the *analytic approach* referred to in the title. I shall also give an entirely different approach that rests on the group algebra of the symmetric group, and for this reason I have termed it the *algebraic approach*, to distinguish it from the former. It is interesting that there are combinatorial relations that are obtainable from one approach and not the other, and by both.

It is not my intention to give an encylopaedic account of these topics. Instead, I have focussed upon the genus series itself, which is of sufficient interest in its own right, for the reasons given above. This is reflected in the material I have cited in the bibliography. However, I trust that these references will serve an interested reader as good sources for backtracking to important material of further interest. In the interests of brevity, I have not included bibliographic references in this extended abstract: the reader is directed to the separate bibliography whose annotations take care of these points.

2 The algebraic approach

Let G be a graph embedded in an orientable surface. Assign directions to each edge, and then assign a label from 1 to n to each of the n edges. For an edge labelled e, let e^- denote its tail and e^+ its head. These are called edge-end labels. ¿From this we construct two permutations $\nu, \varepsilon \in \mathfrak{S}_{2n}$, as follows. For each vertex, list the edge-end labels encountered in describing a small circle around it in a direction consistent with the orientation of the surface. The collection of these lists form the cycle decomposition of a permutation in $\mathfrak{S}_{2n}$, since these lists are pairwise disjoint. The permutation is denoted by ν. Now list the pair of edge-end labels associated with each edge. Each pair can be regarded as a 2-cycle, and together they form the cycle decomposition of a fixed-point free involution ε in $\mathfrak{S}_{2n}$ since the labels are pairwise disjoint.

Let $\phi = \nu\varepsilon$. The cycle structure of ϕ can be determined combinatorially by moving along an edge under the action of ν, rotating under the action of ν to find the next edge, and repeating this until all edge-end labels have been used, or such a label has been encountered for a second time. In this case a cycle has been constructed, and this corresponds to a circumperipheral tour of a face. The process is repeated until each symbol has been encountered at least once.

By the embedding theorem, an embedding is uniquely specified by a permutation ν. Moreover, there are $2^{n-1}(n-1)!$ distinct ways of assigning edge-end labels to a rooted map. The permutation ν is called a *rotation system*. However, not every permutation in $\mathfrak{S}_{2n}$ is a rotation system for a map, since the resulting structure may not be connected. Instead, an arbitrary permutation $\nu \in \mathfrak{S}_{2n}$ corresponds to an unordered set of embeddings of maps on surfaces, and we call such a collection a *premap*. The permutation ν is then the rotation system for a premap.

Let $\mathbf{x} = (x_1, x_2, \ldots)$ where x_j is an indeterminate marking a vertex of degree j. Let $\mathbf{y} = (y_1, y_2, \ldots)$ where y_j is an indeterminate marking a face of degree j, so such a face is bounded by j edges. Let z be an indeterminate marking the number of edges. Then the generating series for the number of rooted maps counted with respect to the degree sequences for vertices and faces and the number of edges is denoted by $M(\mathbf{x}, \mathbf{y}, z)$. If u is an indeterminate marking the genus then

$$\Gamma(u^2, \mathbf{x}, \mathbf{y}, z) = u^2 M(u^{-1}\mathbf{x}, u^{-1}\mathbf{y}, uz),$$

by the Euler-Poincaré formula, is the *genus series* for maps on orientable surfaces. However, it suffices to set $u = 1$, and we therefore regard $M(\mathbf{x}, \mathbf{y}, z)$ as the genus series instead. Note that $\Gamma \in \mathbb{Q}[u, \mathbf{x}, \mathbf{y}][[z]]$. It follows from the relation between the sets of premaps and maps, and the multiplicity associated with the edge labelling that

$$M(\mathbf{x}, \mathbf{y}, z) = 2z\frac{\partial}{\partial z} \log R(\mathbf{x}, \mathbf{y}, z)$$

where R is the generating series for rotation systems for premaps, with respect to the cycle-type of $\nu\epsilon$ marked by $\mathbf{y}$, the cycle-type of ν marked by $\mathbf{x}$, and n marked by z.

Clearly it is necessary to address the problem of determining the number of cycles in the product of ν and ϵ. Since partitions of $2n$ are a natural index for the conjugacy classes of $\mathfrak{S}_{2n}$, and since this index is precisely cycle-type, that also corresponds to vertex- and face-degree sequences, we can work in the centre of the group algebra of the symmetric group. This is spanned by K_α, the formal sum of the conjugacy class indexed by α, for all $\alpha \vdash 2n$. By using the orthogonal idempotents that span the centre, we have the following result.

Theorem 2.1 (A character representation of the genus series)

$$R(\mathbf{x}, \mathbf{y}, z) = \sum_{n \geq 0} \frac{z^n}{n!(2n)!} \sum_{\nu, \phi \vdash 2n} h^\nu h^\phi x_\nu y_\phi \sum_{\theta \vdash 2n} \frac{1}{f^\theta} \chi^\theta_\phi \chi^\theta_\nu \chi^\theta_{[2^n]}.$$

The summations are over all partitions of $2n$, h^ν is the size of the conjugacy class indexed by ν, f^θ is the degree of the irreducible representation indexed by θ, χ^θ_ν is the value of the character of this representation on the class indexed by ν, and $x_\nu = x_1^{n_1} x_2^{n_2} \cdots$, where n_j is the number of occurrences of j in the partition ν. This then gives an algebraic formulation of the genus series. The characters are not easy to deal with in this combinatorial context. For example, it is entirely unclear that Γ, defined now formally in terms of the character sums, actually belongs to $\mathbb{Q}[u, \mathbf{x}, \mathbf{y}][[z]]$. Combinatorially, it is clear that it should, but there are identities between the characters that ensure that terms with negative powers of u do not arise. Moreover, restriction to the sphere appears to be difficult, except in the case of trees, where it can be done.

Nevertheless, this form of the genus series can be used to obtain a strikingly simple relationship between the genus series for two classes of maps. Let

$$\begin{aligned} M_4(x,y,z) &= M(\mathbf{x},\mathbf{y},z) \quad \text{at } \mathbf{x}=(0,0,0,x,0,\ldots), \mathbf{y}=(y,y,\ldots), \\ M_{\mathcal{N}}(x,y,z) &= M(\mathbf{x},\mathbf{y},z) \quad \text{at } \mathbf{x}=(x,x,\ldots), \mathbf{y}=(y,y,\ldots) \end{aligned}$$

be the genus series for the sets of vertex 4-regular maps and all maps, with respect to the numbers of vertices, edges and faces (the degree distributions having been suppressed). $\Gamma_4(u,x,y,z)$ and $\Gamma_{\mathcal{N}}(u,x,y,z)$ are defined similarly. Let $H_\theta(y)$ be the polynomial in y of degree $|\theta|$ defined by $H_\theta(y) = \prod_{j=1}^{l(\theta)} (y-j+1)^{(\theta_j)}$, where $(y)^{(n)} = y(y+1)\cdots(y+n-1)$. Then

$$R_{\mathcal{N}}(x,y,z) = \sum_{n\geq 0} \frac{z^n}{n!(2n)!} \sum_{\theta \vdash 2n} f^\theta \chi^\theta_{[2^n]} H_\theta(x) H_\theta(y).$$

It can be shown by a lengthy argument that hinges on properties of particular characters, their factorisation and the factorisation of $H_\theta(y)$ that these two genus series are related as follows.

Corollary 2.2 $\Gamma_4(u^2,x,y,z) = \frac{1}{2}\{\Gamma_{\mathcal{N}}(4u^2, x+u, x, z^2y) + \Gamma_{\mathcal{N}}(4u^2, x-u, x, z^2y)\}$.

This indicates a correspondence at the combinatorial level between vertex 4-regular maps of genus g, and all maps of genus g and lower. Some hints are offered in this result. For example the substitution $x \mapsto x+u$ suggests that a vertex can be canonically replaced by a vertex or a *handle*. For the sphere, $u=0$, and the relation reduces to one for which there is a direct combinatorial construction, namely, in dual form, the bijection between quadrangulations and maps, due to Tutte. This construction does not extend to the general relation given here.

The genus series for *monopoles* (maps with only one vertex) can be obtained from Theorem 2.1. This is equivalent to determining the number $q_k(n)$ of permutations in $\mathfrak{S}_{2n}$ with k cycles that are a product of a fixed full cycle in $\mathfrak{S}_{2n}$ and a fixed-point free involution. It can be shown that this number satisfies

$$(n+1)q_k(n) = (2n-1)(n-1)(2n-3)q_k(n-2) + 2(2n-1)q_{k-1}(n-1).$$

No direct combinatorial proof of this recurrence is known.

Vertices of degree one and two are not permitted in the physics models. It has been shown by others that vertices of degree one can be introduced into maps with none of them in all possible ways by inserting trees into each corner of the maps, a *corner* being defined by two edges with a common vertex in the boundary of the same face. Vertices of degree two can be introduced into maps with none of them by edge subdivision. A relation can therefore be obtained between the genus series M_{I_2} for maps with no vertices of degree one or two, and M_4, by a compositional argument on edges. In view of the introductory discussion, this relation implies a connexion between a pair of matrix models and, equivalently, a connexion between the ϕ^4-model (corresponding to M_4) and the Penner model (corresponding to M_{I_2}), at least at the combinatorial level of graphs embedded on surfaces, before the double scaling limit has been taken.

The same theory can be extended to ***hypermaps***, or face 2-colourable maps, to obtain similar relationships between classes of maps. No simpler argument is presently known for deriving this type of result and, again, it would be particularly interesting to have a proof based on operations on the surfaces. A further result can be obtained relating the genus series for face 2-coloured triangulations to the genus series for all face 2-coloured maps. The persistence of results of this sort in various circumstances is strong evidence that there ought to be some very nice combinatorial constructions working below the surface.

3 The analytic approach

An alternative representation for the genus series takes as its point of departure an entirely different way of determining the number of cycles in a permutation. The importance of this task has already been seen to be a consequence of the embedding theorem. One way is to track the action of a permutation $\pi \in \mathfrak{S}_n$ on an element, and thence obtain the cycle decomposition and the desired number $\kappa(\pi)$ of cycles in π. This underlies the approach that has been described above. Another approach is to note that, for a positive integer N,

$$N^{\kappa(\pi)} = \sum_{1 \le i_1, \ldots, i_n \le N} \delta_{i_1, i_{\pi(1)}} \cdots \delta_{i_n, i_{\pi(n)}}.$$

From the embedding theorem, an essential part of the genus series is the determination of

$$A_\alpha(y) = \sum_{\pi \in \nu C_{[2^n]}} y^{\kappa(\pi)},$$

for a partition α, where ν is an arbitrary but fixed element of the conjugacy class C_α of $\mathfrak{S}_{2n}$. Let the ***connector function*** ψ, for orientable surfaces, be defined by

$$\psi \colon \mathbb{N}^2 \times \mathbb{N}^2 \longrightarrow \{0,1\} \colon ((p,q),(r,s)) \longmapsto \delta_{p,s}\delta_{q,r},$$

where $\mathbb{N}$ is the set of all positive integers. Let ν be a fixed permutation in C_α, where $\alpha \vdash 2n$. Then

$$A_\alpha(N) = \sum_{1 \le i_1, \ldots, i_{2n} \le N} \; \sum_{\rho \in C_{[2^n]}} \prod_{j=1}^{2n} \psi((i_j, i_{\nu(j)}), (i_{\rho(j)}, i_{\nu\rho(j)})).$$

To sum over involutions, let r, s, be positive integers, let q be an integer such that $0 \leq q < 2$, and let the linear functional $\langle\ \rangle$ on $\mathbb{Q}[[x_1, \ldots, x_N]]$ be defined by

$$\langle x_j^r x_k^s \rangle = \langle x_j^r \rangle \cdot \langle x_k^s \rangle \quad \text{if } j \neq k, \quad \text{and} \quad \langle x_j^{2n+q} \rangle = h^{[2^n]} \delta_{q,0}.$$

A summation theorem for $\langle\ \rangle$ can be given as an alternative to Wick's lemma, and in a form extensible to conjugacy classes other than $C_{[2^n]}$. It is convenient to work with the product over $1, \ldots, 2n$, and the use below of the square root is simply a syntactic device to facilitate this. Let $1 \leq i_1, \ldots, i_{2n} \leq N$, let $\mathbf{x} = (x_1, \ldots, x_N)$ and let $g_k(\mathbf{x})$ be a homogeneous linear polynomial in $\mathbf{x}$ for $k = 1, \ldots, N$. Then

$$\sum_{\rho \in C_{[2^n]}} \prod_{j=1}^{2n} \langle g_{i_j}(\mathbf{x}) g_{i_{\rho(j)}}(\mathbf{x}) \rangle^{\frac{1}{2}} = \langle g_{i_1}(\mathbf{x}) \cdots g_{i_{2n}}(\mathbf{x}) \rangle.$$

The crucial task is to construct an integral representation for the ψ. Since $\langle x_j x_k \rangle = \delta_{jk}$, this can be constructed systematically by generalising the following integral representation of the linear functional from a set of singly indexed indeterminates to a set of doubly indexed indeterminates. Let $\mathbf{x} = (x_1, \ldots, x_N)$ and let $f(\mathbf{x})$ be a polynomial in $x_1, \ldots, x_N$. Then

$$\langle f(\mathbf{x}) \rangle = \frac{\int_{\mathbb{R}^N} f(\mathbf{x}) e^{-\frac{1}{2}(x_1^2 + \cdots + x_N^2)} d\mathbf{x}}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(x_1^2 + \cdots + x_N^2)} d\mathbf{x}}.$$

It is sufficient to prove the result for the monomial $x_1^{i_1} \cdots x_N^{i_N}$, where $i_1, \ldots, i_N$ are nonnegative integers. Let $I_k(\alpha) = \int_{\mathbb{R}} x^k e^{-\alpha x^2} dx$ where α is real and nonnegative. Then $I_{2k-1}(\alpha) = 0$, and $I_{2k}(\alpha) = ((2k)! \sqrt{\pi} / k! 4^k \alpha^{(2k+1)/2})$, leading to the result.

The multiplicative property of $\langle\ \rangle$ is seen to be a consequence of the multiplicativity of the kernel of the integral. To construct an integral representation of ψ, it is necessary to replace $x_1, \ldots, x_N$ by a doubly indexed set of indeterminates $a_{11}, a_{12}, \ldots, a_{NN}$. Clearly $\langle a_{pq} a_{rs} \rangle = \delta_{p,r} \delta_{q,s}$, but this is not the desired evaluation of ψ. Instead, let

$$\langle g(\mathrm{M}) \rangle = \frac{\int_{\mathcal{V}_N} g(\mathrm{M}) e^{-\frac{1}{2} \operatorname{trace} \mathrm{M}^2} d\mathrm{M}}{\int_{\mathcal{V}_N} e^{-\frac{1}{2} \operatorname{trace} \mathrm{M}^2} d\mathrm{M}},$$

where $\mathcal{V}_N$ is the set of all $N \times N$ Hermitian complex matrices. Then

$$A_\alpha(N) = \left\langle \prod_{j \geq 1} (\operatorname{trace} \mathrm{M}^j)^{a_j} \right\rangle,$$

where $\alpha = [1^{a_1}, 2^{a_2}, \ldots]$, and from this it can be shown that

$$M(\mathbf{x}, N, z) = 2z \frac{\partial}{\partial z} \log \left\langle e^{\sum_{k \geq 1} k^{-1} x_k \sqrt{z}^k \operatorname{trace} \mathrm{M}^k} \right\rangle.$$

It is well known that an integral of this form can be transformed. The integrand is invariant under the adjoint action of the unitary group so, by unitary diagonalisation, the integral is replaced by one over the spectrum, which is real, and an integration over the unitary group, which is straightforward either by constructing Haar measure or by indirect computation. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$, $p_k = \lambda_1^k + \cdots + \lambda_N^k$, let k be a positive integer, $y, z, x_1, x_2, \ldots$ be indeterminates, $\mathbf{x} = (x_1, x_2, \ldots)$, and let $V(\lambda)$ denote the Vandermonde determinant. The transformations lead to the following.

Theorem 3.1 (Integral representation of the genus series)

$$R(\mathbf{x}, N, z) = \frac{1}{\sqrt{2\pi}^N} \prod_{j=1}^{N} \frac{1}{j!} \cdot \int_{\mathbb{R}^N} V^2(\lambda) e^{\sum_{k\geq 1} \frac{1}{k}\sqrt{z}^k x_k p_k} e^{-\frac{1}{2}p_2} d\lambda \in \mathbb{Q}[\mathbf{x}, N]\,[[z]]$$

$R(\mathbf{x}, N, z)$ is a power series in z with coefficients which are polynomial in N and in $x_1, x_2, \ldots$, so $R(\mathbf{x}, y, z)$ is defined by replacing N formally by y.

The role played by N is crucial. In using this theorem, it is necessary to carry out integrations to the point where the polynomial dependency on N is explicit. But this may obscure the parity and the degree of the polynomial. In certain cases, these can be recovered by hypergeometric analysis, but with some difficulty. It would be most useful to be able to derive the genus series in parity- and degree-respecting form, and with respect to a basis with the property that restriction to the sphere is immediate.

So far, it has not been possible to use this integral representation to provide an alternative proof of the relationship between the genus series for vertex 4-regular maps and the genus series for all maps. There are certain serious obstacles to this, of which the main one seems to be the requirement that N is to be a positive integer and not an indeterminate.

The integral representation can be used, however, to determine the genus series for vertex k-regular maps. The derivation is lengthy and is a consequence of a careful examination of the determination of the genus series for monopoles and dipoles (maps with two vertices, these being of the same degree). The integration is carried out indirectly by using the superposition of combinatorial structures associated with integrals of products of x^n and Hermite polynomials against $e^{-x^2/2}dx$, with respect to which the Hermite polynomials are orthogonal.

The set of nonnegative integers is denoted by $\mathbb{N}$, and $X = (X_1, \ldots, X_k)$ and $u = (u_1, \ldots, u_k)$ are vectors of indeterminates. For convenience, we set $X_0 = 1$. For a vector $\sigma = (\sigma_1, \ldots, \sigma_k)$, of positive integers, $m_j(\sigma)$ is the number of occurrences of j in σ. For $\alpha \in \mathbb{N}^k$, let $u^\alpha = u_1^{\alpha_1} \cdots u_k^{\alpha_k}$ and $\alpha! = \alpha_1! \cdots \alpha_k!$. Let

$$\Psi_k(X, u) = \frac{1}{1 - X_k} \exp\left\{ \frac{1}{4} \frac{1 + X_k}{1 - X_k} p_2(u) + \frac{1}{2} \frac{1}{1 - X_k} \sum_{1 \leq r < s \leq k} \left(\frac{X_s}{X_r} + X_k \frac{X_r}{X_s} \right) u_r u_s \right\}.$$

Let $[X_1^0 \cdots X_{k-1}^0] f(X, u)$ denote the constant term of f in $X_1, \ldots, X_{k-1}$, namely, the subseries of f of terms independent of $X_1, \ldots, X_{k-1}$.

Theorem 3.2 (The genus series for vertex regular maps) *Let $\alpha \in \mathbb{N}^k$ and let $P_\alpha(N)$, a polynomial in N, be such that*

$$\sum_{\alpha \in \mathbb{N}^k, N \geq 1} P_\alpha(N) \frac{u^\alpha}{\alpha!} X_k^{N-1} = [X_1^0 \cdots X_{k-1}^0] \, \Psi_k(X, u) \prod_{i=1}^{k} \left(1 - \frac{X_i}{X_{i-1}}\right)^{-1}.$$

Let n be a positive integer or a half-integer. Then the genus series for $2n$-regular maps on k vertices and nk edges, with respect to faces (marked by y), is

$$\frac{2^{nk} k}{(2n)^{k-1}} F_{n,k}(y) \quad \textit{where} \quad F_{n,k}(y) = \sum_{\alpha \vdash k} \frac{(-1)^{l(\alpha)-1}}{l(\alpha)} \frac{1}{\alpha! m(\alpha)!} \sum_{\sigma \in \mathfrak{S}_k} P_{2n\sigma(\alpha)}(y).$$

Regular maps are of interest since their genus series contains the genus series $4kF_{2,k}(y)$ for vertex 4-regular maps and these, as has been observed, are associated with the ϕ^4-model. Although it has not been possible to derive an explicit expression for the genus series itself, the form obtained in the theorem is a remarkably elementary one which fully captures information about genus.

In fact, the integral representation for the genus series can be derived directly from the character representation by means of the ring of symmetric functions. In this way, the construction of an integral representation of the connector function and the subsequent use of Haar measure is entirely avoided. In this sense, the derivation of the integral representation can be regarded as a formal one, although it does depend on the existence of the integral $\int_{\mathbb{R}} e^{-x^2/2} dx$.

An integral representation can also be obtained for hypermaps, and for maps on nonorientable surfaces. In fact it is necessary to deal with nonorientable and orientable surfaces together, to treat the nonorientable case. The connector function for the union of the surfaces can be shown to be

$$\psi: \mathbb{N}^2 \times \mathbb{N}^2 \longrightarrow \{0,1\} : ((p,q),(r,s)) \longmapsto \delta_{p,s}\delta_{q,r} + \delta_{p,r}\delta_{q,s},$$

and it is not difficult to follow the previous argument to see that there is a corresponding integral representation for this function. The integral in this case is over all $N \times N$ real symmetric matrices, the diagonalising group is the orthogonal group, and the Jacobian of the transformation is the absolute value of the Vandermonde determinant, rather than its square. In principle, this suggests a means of finding a generalisation of the relation for the genus series for vertex 4-regular maps in orientable surfaces to nonorientable surfaces. However, an additional symmetrisation is involved because of the presence of the absolute value of the Vandermonde, and it is not easy to see how to proceed. Moreover, there may be a direct algebraic proof of the integral representation, analogous to the one for the orientable case.

For the remaining finite dimensional real division algebra, the quaternions, much of the argument remains intact. The diagonalising group is the symplectic group, the Jacobian is the fourth power, and some of the integrations can be carried out. It is possible to work back to construct an analogue of ψ, in the absence of combinatorial information, and then to consider, in the light of the previous development, whether ψ captures anything of combinatorial significance. Of course, it would be natural, at least to speculate that the extra flip in the symplectic group looks after the "under and over" in the case of knots. A moment's reflexion will remind one that a canonical representation for a knot, which would be an essential ingredient for this to be more than speculation, is not presently known, and that invariance under the Reidermeister operations, which would be essential also, is not evident.

4 An application of maps

There is an interesting and potentially important application of the theory of maps to a general combinatorial question. In seeking to transport combinatorial questions to the ring of symmetric functions, it would be useful to have a set of symmetric functions with the same connexion coefficients under multiplication as the conjugacy classes of $\mathfrak{S}_n$ as elements K_α of the centre of the group algebra. A purely algebraic construction for certain elements in this set has been given by Macdonald in unpublished notes. For example, explicit calculation yields, for $n \geq 7$,

$$\mathsf{K}_{[1^{n-4}2^2]}\mathsf{K}_{[1^{n-3}3]} = 3(n-3)\mathsf{K}_{[1^{n-3}3]} + 4(n-4)\mathsf{K}_{[1^{n-4}2^2]} + \left\{ 4\mathsf{K}_{[1^{n-6}2\,4]} + \mathsf{K}_{[1^{n-7}2^2\,3]} + 5\mathsf{K}_{[1^{n-5}5]} \right\},$$

where the *top* terms are enclosed in large braces; the term for K_γ is *top* in $\mathsf{K}_\alpha\mathsf{K}_\beta$ if $l(\alpha) + l(\beta) = n + l(\gamma)$.

Macdonald's construction is the following. Let $H(t;x)$ be the generating series for the complete symmetric functions h_i in $x_1, x_2, \ldots$. The monomial symmetric functions are denoted by m_λ. For symmetric functions in x, the usual inner product $\langle\, ,\, \rangle_x$ on the ring of symmetric functions is defined by $\langle m_\lambda, h_\mu\rangle_x = \delta_{\lambda,\mu}$. Let u and t be related by $u = tH(t;x)$. Then t is implicitly a power series in u so we may write $t = uH^*(u;x)$ where $H^*(u;x) = \sum_{i\geq 0} h_i^*(x)u^i$ is uniquely defined. Let $h_\lambda^* = h_{\lambda_1}^* h_{\lambda_2}^* \cdots$. Then the u_λ are defined by $\langle u_\lambda, h_\mu^*\rangle = \delta_{\lambda,\mu}$.

For a partition α, let $\alpha - 1$ be the partition obtained by subtracting one from each of the positive parts of α (and suppressing resulting zeros). Macdonald observed that the top connexion coefficients (indexed by λ) of the class algebra of $\mathfrak{S}_n$ are precisely the connexion coefficients (indexed by $\lambda - 1$) for these symmetric functions. For example, $u_{[1^2]}u_{[2]} = 4u_{[1\,3]} + u_{[1^2\,2]} + 5u_{[4]}$, where these coefficients correspond to the top terms, enclosed in braces in the above numerical example.

This algebraic construction can be motivated combinatorially by using a particularly simple class of hypermaps, namely vertex 2-coloured edge-rooted trees in the plane, and the embedding theorem for hypermaps, as follows. Let α, β be partitions of n such that $l(\alpha) + l(\beta) = n + 1$, where $l(\alpha)$ is the number of parts of α. Then there is a bijection between two-coloured plane edge-rooted trees on n edges with white vertex distribution α and black vertex distribution β, and pairs (σ, ρ), of permutations in $\mathfrak{S}_n$, with cycle distributions α, β respectively, such that $\sigma\rho = (1, 2, \ldots, n)$. Therefore, the generating series for the set of such trees is WB where W and B satisfy

$$B = \xi(w_1 + w_2W + w_3W^2 + \cdots), \quad W = \xi(b_1 + b_2B + b_3B^2 + \cdots).$$

Let $\alpha, \beta, \gamma \vdash n$ with $l(\alpha)+l(\beta) = l(\gamma)+n$. The coefficient of K_γ in $\mathsf{K}_\alpha \mathsf{K}_\beta$ is the connexion coefficient denoted by $[\mathsf{K}_\gamma]\,\mathsf{K}_\alpha \mathsf{K}_\beta$. Then

$$[\mathsf{K}_\gamma]\,\mathsf{K}_\alpha \mathsf{K}_\beta = [w_\alpha b_\beta] \prod_{i \geq 1} [\xi^{\gamma_i+1}] WB,$$

Since the $h_i^\star(z)$ are algebraically independent, let $w_i = h_{i-1}^\star(z)$ and $b_i = h_{i-1}^\star(y)$ so $B = \zeta H^\star(W; z)$ and $W = \zeta H^\star(B; y)$ It can be shown that this implies that $[\zeta^{\gamma_i+1}]WB = h_{\gamma_i-1}^\star(y, z)$. Then from the construction, $[\mathsf{K}_\gamma]\,\mathsf{K}_\alpha \mathsf{K}_\beta = [h_{\alpha-1}^\star(z) h_{\beta-1}^\star(y)] h_{\gamma-1}^\star(y, z) = [u_{\gamma-1}] u_{\alpha-1} u_{\beta-1}$, a property of dual bases, which is Macdonald's observation.

In principle, connexion coefficients other than the top ones can be determined by looking at other classes of hypermaps (or their duals) that can be counted, and other surfaces. An obvious candidate is the set of vertex 2-coloured maps, in the sphere, with exactly two faces. The corresponding graphs consist of a single cycle, incident with rooted trees, so expressions can be written down for the generating function for them. However, they appear to be unenlightening, at least at the moment. Moreover, the more remote a term is from *top*, the more complex is the corresponding map. Indeed, from this point of view, the determination of the genus series and the construction of the desired set of symmetric functions appear to be equivalent tasks.

There is a tantalising observation that can be made. The *Murphy elements* in $\mathbb{C}\mathfrak{S}_n$ are defined by $r_k = (1, k) + (2, k) + \cdots + (k-1, k)$, for $k = 2, \ldots, n$. These commute, and it is known that symmetric functions of them generate the centre of $\mathbb{C}\mathfrak{S}_n$. By using power sum symmetric functions of them it is possible to perform computations that generate the desired symmetric functions. However, nothing appears to be known about these symmetric functions beyond the numerical. Still, it is not unreasonable to anticipate that properties of the Murphy elements may hold the key to the new symmetric functions, and that once found it ought to be feasible to develop some of their properties. Perhaps a trenchant flash of insight into the combinatorics of transpositions would help.