q-Bell Numbers and Polynomials

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Summary

A natural way to define q-Bell numbers is with the recurrence

$$B_{n+1}(q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} B_k(q)$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is the q-binomial coefficient. One would then want q-analogues of the generating function and the combinatorial interpretation of Bell numbers. The former follows from a notion of q-composition of functions introduced by Gessel that gives a chain rule for the q-derivative. We give the latter and some ramifications of both, which include a q-exponential formula, new q-Stirling numbers of the second kind, q-exponential polynomials, q-Bell polynomials and a q-Faà di Bruno formula.

Une manière naturelle de définir les q-nombres de Bell est par moyen de la récurrence

$$B_{n+1}(q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} B_k(q)$$

où $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ indique le q-coefficient binomial. Ceci mène à deux problèmes: trouver également des q-analogues de la fonction génératrice et de l'interprétation combinatoire des nombres de Bell. Le premier peut être résolu à partir d'une notion de q-composition de fonctions introduite par Gessel, pour laquelle il existe une règle de q-différentiation. On donne une solution au deuxième problème ainsi que d'autres résultats reliés à ces questions, parmi lesquels: une formule q-exponentielle, de nouveaux q-analogues des nombres de Stirling de la 2^{e} espèce, polynômes q-exponentiels, q-analogues des polynômes de Bell et un q-analogue de la formule de Faà di Bruno.

I q-composition of functions and the q-exponential formula

Let us first introduce some standard notation. We define $[k] = \frac{1-q^k}{1-q} = 1 + q + \ldots + q^{k-1}$ and $n!_q = [1][2]\cdots[n]$, where $0!_q = 1$. Next we define the q- (or Gaussian) binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!_q}{k!_q\,(n-k)!_q}$$

It is not difficult to see that $[n] - [k] = q^k[n-k]$, and from this it follows that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n \\ k \end{bmatrix}$$

We further define the q-exponential function

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!_q}$$

and the q-derivative of $f(x)$,

$$\mathbf{D}_q\, f(x) = \frac{f(x) - f(qx)}{x(1-q)}$$

Noting that the q-derivative of a constant is zero and that $\mathbf{D}_q \frac{x^n}{n!_q} = \frac{x^{n-1}}{(n-1)!_q}$ otherwise, we see that $e_q(x)$ is its own q-derivative. Therefore, we could prove a q-exponential formula if we had a chain rule for q-derivatives. For this, as was realized by Gessel ([Ge]), we need a different notion of functional composition. We have slightly modified Gessel's construction:

Let $f(x)$ be a function with $f(0) = 0$, and write it in the form

$$f(x) = \sum_{n=1}^{\infty} f_n \frac{x^n}{n!_q}$$

Define the 0th symbolic power of f by $f^{[0]}(x) = 1$, and for positive integer k define the kth symbolic power of f inductively by

$$\mathbf{D}_q\, f^{[k]}(x) = [k]\, f^{[k-1]}(x)\, f'(x) \tag{1.1}$$

where $f'(x)$ will mean throughout this abstract the q-derivative of f. (1.1) gives $f^{[k]}(x)$ up to an additive constant, which is determined by $f^{[k]}(0) = 0$ for $k \geq 1$. An alternative definition can be given with a q-integral ([GR])

$$\int_0^x f(t)\, d_q(t) := f(0) + x(1-q) \sum_{n=0}^{\infty} q^n\, f(xq^n) \tag{1.2}$$

(Note: for convergence of some of the quantities that we consider, it will occasionally be necessary, as here, to take $|q| < 1$. As a rule, we will proceed formally.) Since this operator inverts the q-derivative operator, we may take, for $k \geq 1$,

$$f^{[k]}(x) = x(1-q^k) \sum_{n=0}^{\infty} q^n\, f^{[k-1]}(xq^n)\, f'(xq^n) \tag{1.3}$$

Note that when $k=1$ we have $\mathbf{D}_q f^{[1]}(x) = f'(x)$, which implies that $f^{[1]}(x) = f(x)$ since they have the same q-derivative and are both zero when $x=0$. Also note that $\mathbf{D}_q x^k = [k]x^{k-1}$ and $\mathbf{D}_q x^{[k]} = [k]x^{[k-1]}$, so by induction we have $x^{[k]} = x^k$ since $x^{[1]} = x$. More generally we have

$$(x^m)^{[k]} = \frac{k!_q}{k!_{q^m}} x^{mk}$$

Using (1.1) and induction we obtain another expression which we could take as a definition of $f^{[k]}(x)$, namely

$$f^{[k]}(x) = k!_q \sum_{n=k}^{\infty} x^n \sum_{\substack{b_1+\cdots+b_k=n \\ b_i \geq 1}} \frac{f_{b_1} f_{b_2} \cdots f_{b_k}}{[b_1][b_1+b_2]\cdots[b_1+\cdots+b_k](b_1-1)!_q \cdots (b_k-1)!_q} \tag{1.4}$$

$$= k!_q \sum_{n=k}^{\infty} \frac{x^n}{n!_q} \sum_{\substack{b_1+\cdots+b_k=n \\ b_i \geq 1}} \frac{(n-1)!_q f_{b_1} f_{b_2} \cdots f_{b_k}}{[b_1][b_1+b_2]\cdots[b_1+\cdots+b_{k-1}](b_1-1)!_q \cdots (b_k-1)!_q} \tag{1.5}$$

If $g(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!_q}$, then the q-composition of g with f is defined by

$$g[f] = \sum_{n=0}^{\infty} g_n \frac{f^{[n]}}{n!_q} \tag{1.6}$$

Since $x^{[n]} = x^n$, at least we have $g[x] = g(x)$. Also note that

$$\mathbf{D}_q g[f] = \sum_{n=1}^{\infty} g_n \frac{f^{[n-1]}}{(n-1)!_q} f' = g'[f]\, f' \tag{1.7}$$

which is a q-analogue of the chain rule.

Take $g(x)$ and $f(x)$ as above and $g = e_q[f]$; then $g' = e_q'[f]\, f' = gf'$ and equating coefficients we get the q-exponential formula

$$g_{n+1} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} g_k\, f_{n-k+1}, \qquad g_0 = 1 \tag{1.8}$$

Gessel applies (1.8) to some problems in the enumeration of permutations. We give in the next section a generic combinatorial interpretation of (1.8).

II A combinatorial interpretation of the q-exponential formula

We will require a few well-known facts (see [**An**]) about partitions of numbers. A partition of a positive integer n is an unordered sum of positive integers equal to n. Each summand is called a part. For example, the partitions of 4 are $4, 3+1, 2+2, 2+1+1, 1+1+1+1$, with respectively $1, 2, 2, 3, 4$ parts. Then we have the

Lemma. *(1)* $\begin{bmatrix} n \\ k \end{bmatrix}$ *is the generating function for partitions into at most k parts not exceeding $n-k$.*
(2) $q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$ *is the generating function for partitions into exactly k parts not exceeding $n-k$.*

With this in hand, we proceed to the combinatorics of (1.8), after introducing some more notation. Let $\langle k \rangle$ denote the set $\{1, 2, \ldots, k\}$ and let $A(m, n; k)$ be the set of partitions of the number k into at most m parts each no larger than n. Let O_n denote the set of weighted objects on $\langle n\rangle$, and let f_n be the generating function for weighted objects on $\langle n\rangle$, *i.e.*

$$f_n = \sum_{\sigma\in O_n} q^{wt\,\sigma}$$

Define the weight of an assemblage of disjoint objects on $\langle n+1\rangle$ inductively as follows:

Find the object containing $n+1$. This object contains $n-k+1$ elements for some k, $0\le k\le n$, which are $a_1<a_2<\ldots<a_{n-k+1}=n+1$, say. Relabel the element a_i as i for each i, $1\le i\le n-k+1$ (we call this the **order-preserving** relabeling and abbreviate by OP), and take the weight w_1 of the relabeled object.

There remain k elements, say $b_1<b_2<\ldots<b_k$, in the other objects, and we relabel b_i as i for each i, $1\le i\le k$. Put $w_2=\sum_{i=1}^{k}(b_i-i)$; we call w_2 the **relabeling weight.** Observe that $\sum_{i=1}^{k}(b_i-i)$ is a partition of the number w_2 into at most k parts not exceeding $n-k$, since $b_1-1\le b_2-2\le\ldots\le b_n-k\le n-k$, and that all such partitions arise by appropriate choice of the b_i's.

After this OP relabeling, we now have an assemblage of objects on $\langle k\rangle$ with some weight w_3 (which could be found by repeating the algorithm until we come down to one object). Then we define the weight of the original assemblage to be $w=w_1+w_2+w_3$, which we call the **generic** definition of the weight, and we put

$$g_n=\sum_{\alpha\in AO_n} q^{wt\,\alpha},\qquad g_0=1.$$

where AO_n is the set of assemblages of objects on $\langle n\rangle$. We claim that $g_{n+1}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix} g_k\,f_{n-k+1}$. For, we have seen that each assemblage of objects on $\langle n+1\rangle$ comprises an object containing $n-k+1$ elements including $n+1$, an assemblage of objects on $\langle k\rangle$, and a partition into at most k parts not exceeding $n-k$ which arises from OP relabeling the assemblage of k objects, where $0\le k\le n$. Then

$$\begin{aligned}
g_{n+1} &= \sum_{\alpha\in AO_{n+1}} q^{wt\,\alpha}\\
&=\sum_{k=0}^{n}\left(\sum_{\sigma\in O_{n-k+1}}\ \sum_{\kappa\in AO_k}\ \sum_{A(k,n-k;l)} q^{wt\,\sigma+wt\,\kappa+l}\right)\\
&=\sum_{k=0}^{n}\left(\sum_{\sigma} q^{wt\,\sigma}\right)\left(\sum_{\kappa} q^{wt\,\kappa}\right)\left(\sum_{l} q^{l}\right)\\
&=\sum_{k=0}^{n} f_{n-k+1}\,g_k\begin{bmatrix}n\\k\end{bmatrix}.
\end{aligned}$$

III q-Bell numbers and q-Stirling numbers of the second kind

With the q-exponential formula in our possession, we can give a q-analogue of the theory of Bell numbers. Let $\Pi(n,k)$ denote the set of partitions of $\langle n\rangle$ into k blocks, and let $\Pi_2(n)=\sum_k\Pi(n,k)$. If we give a set weight 0 and define the weight of an assemblage of disjoint sets (*i.e.* a set partition) via the generic definition, we have $B_0(q)=1$ and

$$B_n(q)=\sum_{\pi\in\Pi_2(n)} q^{wt\,\pi} \tag{3.1}$$

$$B_{n+1}(q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} B_k(q) \tag{3.2}$$

$$e_q[e_q(x) - 1] = \sum_{n=0}^{\infty} B_n(q) \frac{x^n}{n!_q} \tag{3.3}$$

If we define q-Stirling numbers of the second kind by

$$\left\{ {n \atop k} \right\}_q = \sum_{\pi \in \Pi(n,k)} q^{wt\,\pi}, \qquad \left\{ {n \atop 0} \right\}_q = \delta_{n0} \tag{3.4}$$

then clearly we have

$$B_n(q) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q \tag{3.5}$$

and a complete q-analogue of the basic theory of Bell numbers. However, these are not the same q-Bell numbers that are given by Milne ([**M1**]), nor are these q-Stirling numbers of the second kind the same as the ones that have appeared in the literature (see, *e.g.*, [**M1**],[**M2**]). Two related sets of q-Stirling numbers of the second kind have been studied previously. One which is not too different from $\left\{ {n \atop k} \right\}_q$ may be defined by

$$S_q(n,k) = S_q(n-1,k-1) + [k]\, S_q(n-1,k), \qquad S_q(n,0) = \delta_{n0} \tag{3.6}$$

(The first disagreement is that $S_q(4,2) = 3 + 3q + q^2$, while $\left\{ {4 \atop 2} \right\}_q = 3 + 2q + 2q^2$.) The other may be defined by $\varsigma_q(n,k) = q^{\binom{k}{2}}\, S_q(n,k)$. On the subject of notation for Stirling numbers, see [**Kn**].

$\left\{ {n \atop k} \right\}_q$ inherits several of the properties of $S(n,k)$. We have

$$\left\{ {n+1 \atop k+1} \right\}_q = \sum_{l=k}^{n} \begin{bmatrix} n \\ l \end{bmatrix} \left\{ {l \atop k} \right\}_q \tag{3.7}$$

The proof of this is much the same as before. In a partition of $\langle n+1 \rangle$ into $k+1$ blocks, the block containing $n+1$ has $n-l$ other elements in it for some l, $k \le l \le n$ ($l \ge k$ since there remain l elements in k blocks). After OP relabeling this block has weight 0. The remaining l elements are OP relabeled, with a relabeling weight that is a partition into at most l parts not exceeding $n-l$. After this relabeling, the remaining elements are a partition of $\langle l \rangle$ into k blocks. Taking weights and summing over all possibilities, (3.7) follows.

(3.7) implies the infinite generating function

$$\frac{(e_q(x)-1)^{[k]}}{k!_q} = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_q \frac{x^n}{n!_q} \tag{3.8}$$

To see this, first note that if $k=0$ both sides are 1. If $k>0$, both sides are 0 when $x=0$, and

$$\begin{aligned} \mathbf{D}_q \frac{(e_q(x)-1)^{[k]}}{k!_q} &= [k] \frac{1}{k!_q} (e_q(x)-1)^{[k-1]} (e_q(x)-1)' \\ &= \frac{(e_q(x)-1)^{[k-1]}}{(k-1)!_q} e_q(x) \end{aligned}$$

If we set

$$r_k(x) = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_q \frac{x^n}{n!_q}$$

then it suffices to show that $\mathbf{D}_q\, r_{k+1}(x) = r_k(x)\, e_q(x)$. Now

$$r_{k+1}(x) = \sum_{n=k+1}^{\infty} \left\{ {n \atop k+1} \right\}_q \frac{x^n}{n!_q} = \sum_{n=k}^{\infty} \left\{ {n+1 \atop k+1} \right\}_q \frac{x^{n+1}}{(n+1)!_q}$$

so using (3.7)

$$\begin{aligned}\mathbf{D}_q\, r_{k+1}(x) &= \sum_{n=k}^{\infty} \left\{ {n+1 \atop k+1} \right\}_q \frac{x^n}{n!_q} = \sum_{n=k}^{\infty} \sum_{l=k}^{n} \begin{bmatrix} n \\ l \end{bmatrix} \left\{ {l \atop k} \right\}_q \frac{x^n}{n!_q} \\ &= \sum_{l=k}^{\infty} \sum_{n=l}^{\infty} \left\{ {l \atop k} \right\}_q \frac{x^{n-l}}{(n-l)!_q} \frac{x^l}{l!_q} = \sum_{l=k}^{\infty} \left\{ {l \atop k} \right\}_q \frac{x^l}{l!_q} \sum_{n=l}^{\infty} \frac{x^{n-l}}{(n-l)!_q} \\ &= r_k(x)\, e_q(x)\end{aligned}$$

We may also consider the polynomial $\phi_n(x;q) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q x^k$, a q-analogue of the exponential polynomial ([RKO]). We get nice q-analogues of some of the properties of the exponential polynomials in

$$e_q\left[x\left(e_q(t) - 1\right)\right] = \sum_{n=0}^{\infty} \phi_n(x;q) \frac{t^n}{n!_q}$$

and

$$\phi_{n+1}(x;q) = x \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \phi_k(x;q)$$

IV A q-analogue of Faà di Bruno's formula

Since Gessel's q-composition gives us a q-analogue of the chain rule, one can hope for a q-analogue of the iterated chain rule, that is, Faà di Bruno's formula ([Ch],[Ri],[Co]). For this we will require a product rule for q-derivatives. In the case of two functions we have

$$\begin{aligned}\mathbf{D}_q\, f(x)g(x) &= \frac{f(x)g(x) - f(qx)g(qx)}{x(1-q)} \\ &= \frac{f(x)g(x) - f(x)g(qx) + f(x)g(qx) - f(qx)g(qx)}{x(1-q)} \\ &= f(x)g'(x) + f'(x)g(qx) \qquad (4.1)\end{aligned}$$

The formula we require is a generalization of (4.1) to n functions:

$$\mathbf{D}_q\, f_1(x)f_2(x)\cdots f_n(x) = \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} f_j(x) \right) f_k'(x) \left(\prod_{j=k+1}^{n} f_j(qx) \right) \qquad (4.2)$$

Proof: Induction on n. (4.1) is the case $n = 2$, and we pass from n to $n+1$ by taking $f(x) = f_1(x)$ and $g(x) = f_2(x)\cdots f_{n+1}(x)$ in (4.2).

We also note that

$$\begin{aligned} \mathbf{D}_q f(q^b x) &= \frac{f(q^b x) - f(q^{b+1} x)}{x(1-q)} \\ &= q^b \frac{f(q^b x) - f(q^{b+1} x)}{q^b x(1-q)} \\ &= q^b f'(q^b x) \end{aligned} \tag{4.3}$$

For orientation, let us look at $\mathbf{D}_q^3 f[g(x)]$. There are five terms:

$$\begin{aligned} &f'[g(x)]\, g'''(x) + f''[g(x)]\, g'(x)\, g''(qx) + q\, f''[g(x)]\, g'(x)\, g''(qx) \\ &\quad + f''[g(x)]\, g''(x)\, g'(q^2 x) + f'''[g(x)]\, g'(x)\, g'(qx)\, g'(q^2 x) \end{aligned}$$

An abbreviated notation for this expression, in analogy with the notation in **[Ri]**, could be

$$f_1\, g_{30} + f_2\, g_{10} g_{21} + q\, f_2\, g_{10} g_{21} + f_2\, g_{20} g_{12} + f_3\, g_{10} g_{11} g_{12}$$

These terms correspond to the five partitions of $\langle 3 \rangle$, in the order $\{1,2,3\}$; $\{1\},\{2,3\}$; $\{2\},\{1,3\}$; $\{1,2\},\{3\}$; $\{1\},\{2\},\{3\}$. Note that the weight of $\{2\},\{1,3\}$ is 1 while the weights of the other four partitions of $\langle 3 \rangle$ are all 0.

Let π be a partition of $\langle n \rangle$ into the blocks $B_1, B_2, \ldots, B_k$. As with the partitions of $\langle 3 \rangle$ above, we put the blocks in increasing order of the maximal elements. Suppose there are b_i elements in the block B_i, for each i, and recall that we have designated the set of partitions of $\langle n \rangle$ by $\Pi_2(n)$. Then we have

Theorem (q-analogue of Faà di Bruno's formula, first form)

$$\mathbf{D}_q^n f[g(x)] = \sum_{\pi \in \Pi_2(n)} \left\{ \begin{array}{l} q^{wt\,\pi}\, f^{(k)}[g(x)]\, g^{(b_1)}(x)\, g^{(b_2)}(q^{b_1} x) \times \\ \times\, g^{(b_3)}(q^{b_1+b_2} x) \cdots g^{(b_k)}(q^{b_1+b_2+\cdots+b_{k-1}} x) \end{array} \right\} \tag{4.4}$$

Proof: Induction on n. We have verified the case $n = 3$ already, and the lower-dimensional cases are trivial. Observe that the exponent of q in the factor $g^{(b_i)}(q^{b_1+\cdots+b_{i-1}} x)$ counts the number of elements contained in $B_1 \cup \cdots \cup B_{i-1}$, *i.e.* in the blocks with a smaller maximal element than that of B_i. Assuming that (4.4) holds for n, we show it holds for $n+1$. Using (4.2) in (4.4), we have

$$\begin{aligned} &\mathbf{D}_q^{n+1} f[g(x)] \\ &= \sum_{\pi \in \Pi_2(n)} q^{wt\,\pi}\, f^{(k+1)}[g(x)] \left(\prod_{i=1}^{k} g^{(b_i)}(q^{b_1+\cdots+b_{i-1}+1} x) \right) \\ &+ \sum_{\substack{\pi \in \Pi_2(n) \\ 1 \le i \le k}} \left\{ \begin{array}{l} f^{(k)}[g(x)]\, q^{wt\,\pi + \sum_{l=1}^{i-1} b_l} \left(\prod_{j=1}^{i-1} g^{(b_j)}(q^{\sum_{l=1}^{j-1} b_l} x) \right) \times \\ \times\, g^{(b_i+1)}(q^{\sum_{l=1}^{i-1} b_l} x) \left(\prod_{j=i+1}^{k} g^{(b_j)}(q^{1+\sum_{l=1}^{j-1} b_l} x) \right) \end{array} \right\} \end{aligned} \tag{4.5}$$

where we also used (4.3).

We may obtain all the partitions of $\langle n+1\rangle$ from the partitions of $\langle n\rangle$ by relabeling the elements $1, 2, \ldots, n$ as $2, 3, \ldots, n+1$ in that order, and then either adding the element 1 as a singleton block, or adding it to one of the relabeled blocks. Doing the former increases the number of blocks by one, increases the number of elements in the blocks with a smaller maximal element than that of B_i by one for each of the original blocks, and does not change the weight. Therefore this class of partitions of $\langle n+1\rangle$ corresponds to the first summand in (4.5).

In the latter case, suppose we have adjoined 1 to the relabeled block B_i. For the B_j with $j > i$, the number of elements in the blocks of smaller index has increased by one, but this number has not changed for the B_j with $j < i$, and of course the size of B_i has increased by one. The effect on the weight is the following. The blocks B_j with $j > i$ are crossed out one by one. Each element has a label one larger than previously both before and after relabeling, so there is no net effect on the relabeling weight. When B_i is crossed out, however, the remaining elements have to be relabeled down one more than before. After this, everything has the same labels it would have had before the element 1 was adjoined, so there is no further effect on the weight. Thus the weight increases by the number of elements in the blocks with index less than i, that is, by $b_1 + \cdots + b_{i-1}$. Therefore these partitions correspond to the second summand in (4.5).

Faà di Bruno's formula is not usually stated in the form of a sum over set partitions, although it is most easily proved when so stated. One can then ask whether our q-analogue can be given without reference to set partitions. The following lemma will allow us to give such a statement.

Lemma Let π be a partition of $\langle n\rangle$ into the blocks $B_1, B_2, \ldots, B_k$, listed in increasing order of their maximal elements, with $|B_i| = b_i$, $1 \le i \le k$. If $\sum_{\pi\in(B_1,\ldots,B_k)}$ denotes the sum over all such π, then

$$\sum_{\pi\in(B_1,\ldots,B_k)} q^{wt\,\pi}$$
$$= \begin{bmatrix} b_1+b_2+\cdots+b_k-1 \\ b_k-1 \end{bmatrix} \begin{bmatrix} b_1+b_2+\cdots+b_{k-1}-1 \\ b_{k-1}-1 \end{bmatrix} \cdots \begin{bmatrix} b_1+b_2-1 \\ b_2-1 \end{bmatrix}$$

Proof: To determine the weight of π, we begin by crossing out B_k and OP relabeling the elements in the other blocks with $\{1, 2, \ldots, b_1+b_2+\cdots+b_{k-1}\}$. Before relabeling, the largest element in these blocks does not exceed $b_1+b_2+\cdots+b_k-1$, so the contribution to the relabeling weight from any one element is no larger than b_k-1. For each π, then, the relabeling weight is a partition into at most $b_1+b_2+\cdots+b_{k-1}$ parts not exceeding b_k-1, and we get all such partitions from summing over all π. By the lemma of section III, we therefore have

$$\sum_{\pi\in(B_1,\ldots,B_k)} q^{wt\,\pi} = \begin{bmatrix} b_1+b_2+\cdots+b_k-1 \\ b_k-1 \end{bmatrix} \sum_{\pi\in(B_1,\ldots,B_{k-1})} q^{wt\,\pi}$$

and the lemma follows upon iteration since the weight of a single block is 0.

Observe that we may rewrite the expression in the lemma as

$$\frac{(n-1)!_q}{[b_1][b_1+b_2]\cdots[b_1+\cdots+b_{k-1}](b_1-1)!_q(b_2-1)!_q\cdots(b_k-1)!_q}$$

and that this expression has arisen already in (1.5). From this point of view, it therefore gives a natural q-analogue of the Bell polynomial ([Ri],[Co]). That is, we may put

$$\mathbf{B}_{n,k,q}(f_1, f_2, \ldots) = \sum_{\substack{b_1+\cdots+b_k=n \\ b_i\ge 1}} \frac{(n-1)!_q\, f_{b_1} f_{b_2}\cdots f_{b_k}}{[b_1][b_1+b_2]\cdots[b_1+\cdots+b_{k-1}](b_1-1)!_q\cdots(b_k-1)!_q} \tag{4.6}$$

and rewrite (1.5) as

$$f^{[k]}(x) = k!_q \sum_{n=k}^{\infty} \frac{x^n}{n!_q} \mathbf{B}_{n,k,q}(f_1, f_2, \ldots) \tag{4.7}$$

Using the shorthand notation introduced earlier, namely $g_{i,j} := g^{(i)}(q^j x)$, we may now state our q-Faà di Bruno formula in the following form:

Theorem (q-analogue of Faà di Bruno's formula, second form)

$$\mathbf{D}_q^n\, f[g(x)] = \sum_k f^{(k)}[g(x)]\, \mathbf{B}_{n,k,q}(g_{b_1,0}, g_{b_2,b_1}, g_{b_3,b_1+b_2}, \ldots)$$

V References

[An] G.E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading, MA 1976.

[Ch] William Y.C. Chen, Context-free Grammars, Differential Operators and Formal Power Series, in: M. Delest, G. Jacob, P. Leroux editors, *Series Formelles et Combinatoire Algebrique*, proceedings, Bordeaux France 1991, 145-159.

[Co] Louis Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[Ge] Ira M. Gessel, A q-analog of the exponential formula, *Discrete Math.* **40** (1982), 69-80.

[GR] George Gasper and Mizan Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, Vol. 35, Cambridge U. Press, Cambridge 1990.

[Kn] Donald E. Knuth, Two notes on notation, *Amer. Math. Monthly* **99** (1992), 403-422.

[M1] Stephen C. Milne, A q-analog of restricted growth functions, Dobinski's equality, and Charlier polynomials, *Trans. Amer. Math. Soc.* **245** (1978), 89-118.

[M2] Stephen C. Milne, Restricted growth functions, rank row matchings of partition lattices, and q-Stirling numbers, *Adv. in Math.* **43** (1982), 173-196.

[Ri] John Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.

[RKO] G.-C. Rota, D. Kahaner and A. Odlyzko, On the foundations of combinatorial theory, VIII: Finite operator calculus, *J. Math. Anal. App.* **42** (3), 1973, 684-760.