The combinatorics of polytopes and the simplex algorithm

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Let P be a d-polyhedron with n facets, i.e. P is the set of solutions of n linear inequalities with d variables. The vertices (extreme points) and edges (1-dimensional faces) of P form a graph denoted by $G(P)$. This graph is of great interest in the combinatorial theory of polytopes and in the theory of linear programming. The simplex algorithm finds the maximum value of a linear (objective) function on P by moving from vertex to adjacent vertex in the graph of P.

An example to always have in mind is: P is the d-dimensional cube defined by the $2d$ inequalities $0 \leq x_i \leq 1$. P has $2d$ facets and 2^d vertices all 0-1 vectors. Two vertices are adjacent in the graph of P if they differs in one coordinate.

We will discuss the following open problems (all of them are wide open, but for 1-3 there are some partial results):

1. What is the maximal diameter of the graph of a d-dimensional polyhedra with n facets?
2. Is there a pivot rule for the simplex algorithm which is good in the worst case?
3. How good expanders are graphs of polytopes?
4. How hard is it to find a random vertex of a polyhedron?
5. How hard is it to approximate the number of vertices of a d-polyhedra with n facets?

Some relevant references:

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