ASYMPTOTICS OF LARGE RANDOM YOUNG DIAGRAMS

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ABSTRACT. The Plancherel growth process is a Markov chain on the Young lattice, determined by transition probabilities $p(\lambda, \Lambda) = d(\Lambda) \,/\, |\Lambda|\, d(\lambda)$. Here Λ covers the Young diagram λ and $d(\lambda)$ is the number of Young tableaux of shape λ. The process originates in the character theory of the infinite symmetric group $\mathfrak{S}_\infty$. There are two basic results on the asymptotic behavior of Plancherel process: 1) the Law of Large Young Diagrams claims that the shape of a typical large random diagram is uniformly close to the graph of an explicitely known function, and 2) an analog of Central Limit Theorem describes the Gaussian limiting process for the deviations of random diagrams from the mean shape. Remarkably, the Plancherel process has deep connections far outside representation theory and combinatorics of Young tableaux. It is related to partial fractions, Markov's Moment Problem, asymptotics of interlacing roots of orthogonal polynomials, and Wigner's Semicircle Law for eigenvalues of large random matrices. Some of nontrivial combinatorial techniques, e.g. the Hook Walk algorithm, can be extended by continuity to functions which are limits of normalized Young diagrams [4].

All characters of $\mathfrak{S}_\infty$ can be interpreted as special Markov chains on the Young lattice. They are in a bijective correspondence with Polya frequency sequences. Every central spherical function on $\mathfrak{S}_\infty$ has a unique presentation as an integral of characters. Selberg integrals arise in that way in the study of harmonic analysis of pseudo-regular representations of the infinite symmetric group and infinite dimensional unitary group.

Le processus de croissance de Plancherel est une chaîne de Markov sur le treillis de Young, déterminée par les probabilités de transition $p(\lambda, \Lambda) = d(\Lambda) \,/\, |\Lambda|\, d(\lambda)$. Ici Λ recouvre le diagramme de Young λ et $d(\lambda)$ est le nombre de tableaux de Young de forme λ. L'origine du processus se trouve dans la théorie des caractères du groupe symétrique infini $\mathfrak{S}_\infty$. On a deux résultats de base concernant le comportement asymptotique du processus de Plancherel: 1) la Loi des Grands Diagrammes de Young affirme que la forme d'un grand diagramme typique aléatoire est uniformément proche au graphique d'une fonction qu'on connait de manière explicite, et 2) un analogue du Théorème de Limite Centrale décrit le processus limite de Gauss pour les déviations des diagrammes aléatoires par rapport à la forme moyenne. Il est remarquable que le processus de Plancherel ait des liens profonds bien hors de la théorie des représentations et de la combinatoire des tableaux de Young. Il est relié aux fractions partielles, au Problème du Moment de Markov, à l'asymptotique de l'entrelacement des racines de polynômes orthogonaux, et à la Loi du Demicercle de Wigner pour les valeurs propres des grandes matrices aléatoires. Certaines techniques combinatoires non triviales, par exemple l'algorithme "Hook Walk" ou Randonnée Crochet, peuvent être étendues par continuité à des fonctions qui sont des limites de diagrammes de Young normalisés [4].

Tous les caractères de $\mathfrak{S}_\infty$ peuvent être perçus comme des chaînes de Markov spéciales sur le treillis de Young. Ils se trouvent en correspondance biunivoque avec les suites de fréquences de Polya. Chaque fonction sphérique centrale sur $\mathfrak{S}_\infty$ a une présentation unique en intégrale de caractères. Les intégrales de Selberg apparaissent de cette manière dans l'étude de l'analyse harmonique des représentations pseudo-régulières du groupe symétrique infini et du groupe unitaire de dimension infinie.

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1. Introduction. The subject of the present talk is random growth of Young diagrams. We consider probability distributions on the space of infinite Young tableaux T, with the following property: the probability to obtain a Young diagram by adding boxes according to particular Young tableau depends only on the final shape of that tableau. Measures satisfying this condition are called *central.*

Central measures, ergodic with respect to the tail partition on the space T, are in one-to-one correspondence with the characters of the countable symmetric group $\mathfrak{S}_\infty$. The classification of ergodic central measures for the Young lattice is equivalent to that of Polya frequency sequences. More direct approach [11] to the classification problem is based on the study of asymptotics of shape of random Young tableaux. The main tool is the formula (6.8) for the number of standard Young tableaux of skew shape $\Lambda \setminus \lambda$, for a fixed Young diagram λ and an arbitrarily large Λ.

The most distinguished example of a central measure is the Plancherel Growth Process, corresponding to the character of the regular representation of $\mathfrak{S}_\infty$. We review in Section 7 specific versions of Law of Large Numbers and Central Limit Theorem for Plancherel measure. In particular, we describe the approximate shape of a Young diagram with n boxes which maximizes the dimension of the associated irreducible representation of the symmetric group $\mathfrak{S}_n$.

The space Δ of ergodic central measures can be considered as a boundary of the Young lattice. Every central measure has a unique presentation by an integral of ergodic central measures with respect to a mixing probability distribution on the boundary Δ (similar to Poisson Integral representation of positive harmonic functions in the unit disc, or to the integral representation of exchangeable random sequences in de Finetti theorem). In Section 8 we introduce a distinguished family of non-ergodic central Markov chains on the graph $\mathcal{Y}$, generalizing Ewens partition structures of population genetics [7]. The probability (8.5) of a Young diagram λ for these measures equals a product, over all boxes in λ, of simple expressions involving hook lengths and contents. Their integral representations coincide with the specific types of celebrated Selberg integrals.

Let us emphasize two points. First, we not only derive the Selberg integral

$$\begin{aligned}(1.1)\qquad &\int\limits_{\substack{t_1,\dots,t_k\geq 0\\ t_1+\dots+t_k=1}}\cdots\int \prod_{1\leq i<j\leq k} |t_i-t_j|^{2\theta} \prod_{j=1}^{k} t_j^{A-1}\, dt_1\dots dt_{k-1} =\\ &= \frac{1}{\Gamma(kA+(k-1)k\theta)} \prod_{j=1}^{k} \frac{\Gamma(A+(k-j)\theta)\,\Gamma(j\theta+1)}{\Gamma(\theta+1)},\end{aligned}$$

but also provide a nice discrete approximation (8.7) to the left hand side. Second, the infinite dimensional versions of Selberg integrals also arise in decompositions of some central Markov chains in the family. The study of probabilistic properties of mixing measures is a challenging open problem. Only one special case is well understood by now: that of Poisson–Dirichlet distributions representing Ewens partition structures.

2. Bratteli diagrams and branchings. Let Γ denote a graded oriented graph with the set of vertices $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$ partitioned into *levels* Γ_n, and assume that

(a) the end of every edge belongs to the level next to that of its source vertex

(b) there is a single vertex $\emptyset \in \Gamma$ with no incoming edges
(c) for every vertex there is at least one outcoming edge
(d) all the levels Γ_n are finite.

A graph satisfying (a) – (d) is called *Bratteli diagram.* We will often associate a positive number $\varkappa(\lambda, \Lambda)$, called *edge multiplicity,* to every edge (λ, Λ) of a Bratteli diagram. A pair $(\Gamma, \varkappa)$ consisting of a Bratteli diagram and a multiplicity function $\varkappa$ will be referred to as a *branching.*

The terminology and the original host of examples comes from representation theory of locally finite groups and approximately finite dimensional (AF-) algebras. Let G be a countable locally finite group, i.e. a union of an increasing sequence of finite subgroups $G_0 \subset G_1 \subset \ldots \subset G_n \subset \ldots$. We denote by $\Gamma_n = \hat{G}_n$ the finite set of equivalence classes of irreducible representations of G_n. For any two irreducibles $\lambda \in \Gamma_n$, $\Lambda \in \Gamma_{n+1}$ the coefficient $\varkappa(\lambda, \Lambda)$ of λ in the restriction $\mathrm{Res}_{G_n}^{G_{n+1}} \Lambda$ coincides with the coefficient of Λ in the induced representation $\mathrm{Ind}_{G_n}^{G_{n+1}} \lambda$. By definition, a pair (λ, Λ) for which $\varkappa(\lambda, \Lambda) \neq 0$, determines an edge of a branching, with multiplicity $\varkappa(\lambda, \Lambda)$, on the vertex set $\Gamma = \bigcup \Gamma_n$.

In this talk we will focus on three examples of branchings.

Example A (Young lattice $\mathcal{Y}$). By definition, the vertices of $\mathcal{Y}$ are Young diagrams. Two diagrams λ, Λ determine an edge, $\lambda \nearrow \Lambda$, iff Λ covers λ in the inclusion order. The edge multiplicities are trivial: $\varkappa(\lambda, \Lambda) \equiv 1$ for all $\lambda \nearrow \Lambda$. Young lattice shows up as the Bratteli diagram for the increasing sequence $\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \ldots \subset \mathfrak{S}_n \subset \ldots$ of finite symmetric groups. Another way to introduce the Young branching is Pieri formula for Schur symmetric polynomials: $s_{(1)} \cdot s_\lambda = \sum_{\Lambda:\lambda\nearrow\Lambda} s_\Lambda$. We shall also consider truncated versions $\mathcal{Y}(k)$ of Young lattice, where only diagrams with k rows or less are taken into account.

Example B (Kingman's branching $\mathcal{K}$). The graph is as in Example A, and the multiplicities are defined as $\varkappa(\lambda, \Lambda) = \rho_j(\Lambda)$, where j is the length of the row of Λ containing the box $\Lambda \setminus \lambda$, and $\rho_j(\Lambda)$ is the number of rows of length j in Λ. The branching $\mathcal{K} = (\mathcal{Y}, \varkappa)$ was studied in [7]. It describes Pieri type formula for monomial symmetric functions m_λ:

$$s_{(1)} \cdot m_\lambda = \sum_{\Lambda:\lambda\nearrow\Lambda} \varkappa(\lambda, \Lambda) \cdot m_\Lambda. \tag{2.1}$$

Kingman's branching also has truncated versions $\mathcal{K}(k)$.

Example C (The family of Jack's branchings $\mathcal{J}^{(\theta)}$). Again, we take Young lattice as the underlying graph. The multiplicities are defined by the product

$$\varkappa_\theta(\lambda, \Lambda) = \prod_b \frac{(a(b) + (l(b) + 2)\theta) \cdot (a(b) + 1 + l(b)\theta)}{(a(b) + (l(b) + 1)\theta) \cdot (a(b) + 1 + (l(b) + 1)\theta)}. \tag{2.2}$$

Here $a(b)$ is the arm length and $l(b)$ is the leg length of a box $b \in \lambda$, and $\theta \geq 0$ is a parameter. The product runs over the boxes in the column of λ strictly above the new box $\Lambda \setminus \lambda$. We denote the Young lattice with edge multiplicities (2.2) by $\mathcal{J}^{(\theta)}$, and we call it *Jack branching.* Multiplicities (2.2) arise in Pieri type formula for zonal (Jack) symmetric polynomials $P_\lambda(x; \theta)$ (cf. [9]):

$$s_{(1)} \cdot P_\lambda(x; \theta) = \sum_{\Lambda:\lambda\nearrow\Lambda} \varkappa_\theta(\lambda, \Lambda) \cdot P_\Lambda(x; \theta). \tag{2.3}$$

Note that Jack polynomials $J_\lambda(x;\alpha)$ considered in [10] have a different normalization: $J_\lambda(x;\alpha) = P_\lambda(x;1/\alpha)\cdot\prod_{b\in\lambda}(\alpha a(b)+l(b)+1)$. The family of Jack branchings provides a deformation of Example B ($\theta = 0$) to the ordinary Young branching of Example A ($\theta = 1$). In fact, $P_\lambda(x;0) = m_\lambda(x)$ and $P_\lambda(x;1) = s_\lambda(x)$.

3. Combinatorial dimensions. Let Γ be a branching. For any two vertices $\lambda, \mu \in \Gamma$ we denote by $d(\lambda,\mu)$ the number of chains from λ to μ. If the edge multiplicities are nontrivial, $\varkappa(\lambda,\Lambda) \neq 1$, we assign the weight $w_u = \prod_{i=1}^n \varkappa(\lambda_{i-1},\lambda_i)$ to a chain $u = (\lambda_0 = \lambda, \lambda_1, \dots, \lambda_n = \mu)$, and define *dimension function* $d(\lambda,\mu)$ as a sum of weights $d(\lambda,\mu) = \sum_u w_u$ of all saturated chains in the interval $[\lambda,\mu]$.

Example A. For Young lattice $\mathcal{Y}$ the combinatorial dimension is equal to that of the associated irreducible of the symmetric group $\mathfrak{S}_n$. The *hook formula* holds:

$$d(\lambda) = \frac{n!}{\prod_{b\in\lambda} h(b)}; \quad \lambda \in \mathcal{Y}_n. \tag{3.1}$$

where $h(b) = a(b) + l(b) + 1$ is the hook length of the box $b \in \lambda$.

Example B. For the Kingman's branching $\mathcal{K}$ the dimension $d(\lambda_1, \lambda_2, \dots)$ is equal to that of the representation of $\mathfrak{S}_n$ induced by the trivial representation of the Young subgroup $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \ldots \subset \mathfrak{S}_{\lambda_n}$, namely

$$d(\lambda) = \frac{n!}{\lambda_1!\,\lambda_2!\dots}; \quad \lambda \in \mathcal{Y}_n. \tag{3.2}$$

Example C. For the Jack branching $\mathcal{J}^{(\theta)}$ there is a hook formula

$$d_\theta(\lambda) = \frac{n!}{\prod_{b\in\lambda}(a(b)+1+l(b)\theta)}. \tag{3.3}$$

4. Central Markov chains. Given a branching $(\Gamma, \varkappa)$, denote by T the compact, totally disconnected space of infinite paths $t = (\lambda_1, \dots, \lambda_n, \dots)$ in Γ, starting with the initial vertex $\emptyset$. In all examples of Section 2 the elements of T are infinite Young tableaux.

A Borel distribution M on T is determined by the measures $M(C_u)$ of cylinder sets $C_u = \{t \in T:\ \lambda_1 = \nu_1, \dots, \lambda_n = \nu_n\}$, where $u = (\nu_1, \nu_2, \dots, \nu_n = \lambda)$ is a finite path with the last vertex $\omega(u) = \lambda$.

(4.1) Definition. A probability measure M on T is *central* with respect to the branching $(\Gamma, \varkappa)$ if $M(C_u)\,/\,w_u = M(C_v)\,/\,w_v$ for any two finite paths u, v with the same end vertices $\omega(u) = \omega(v)$.

(4.2) Definition. A function $\varphi : \Gamma \to \mathbb{R}$ is called *harmonic* if it satisfies the conditions

$$\varphi(\lambda) = \sum_{\Lambda:\lambda\nearrow\Lambda} \varkappa(\lambda,\Lambda)\cdot\varphi(\Lambda); \quad \varphi(\lambda) \geq 0; \quad \lambda \in \Gamma, \qquad \varphi(\emptyset) = 1. \tag{4.3}$$

By definition (4.1), the quotient $M(C_u)\,/\,w_u = \varphi(\lambda)$, for a central measure M, depends but on the last vertex λ of a path u. Clearly, the function φ is harmonic. In the opposite direction, every harmonic function of a branching $(\Gamma, \varkappa)$ uniquely determines a central distribution with cylinder probabilities

$$M(C_u) = w_u \cdot \varphi(\omega(u)). \tag{4.4}$$

Every central measure is actually a Markov chain with transition probabilities

$$p(\lambda, \Lambda) = \frac{\varkappa(\lambda, \Lambda) \cdot \varphi(\Lambda)}{\varphi(\lambda)}. \tag{4.5}$$

All central measures M share the same conditional probabilities

$$q(\lambda, \Lambda) = \text{Prob}\ \{\lambda_n = \lambda \mid \lambda_{n+1} = \Lambda\} = \frac{d(\lambda)\ \varkappa(\lambda, \Lambda)}{d(\Lambda)}, \tag{4.6}$$

called *cotransition probabilities.*

(4.7) **Definition.** Assume there is a probability distribution M_n on the n-th level set Γ_n of a branching $(\Gamma, \varkappa)$, for all $n \geq 0$. The system $\{M_n\}_{n=0}^{\infty}$ is called *coherent*, if

$$M_n(\lambda) = \sum_{\Lambda : \lambda \nearrow \Lambda} q(\lambda, \Lambda)\ M_{n+1}(\Lambda) \tag{4.8}$$

for every $n \geq 0$ and $\lambda \in \Gamma_n$.

Level distributions $M_n(\lambda) = \text{Prob}\ \{t = (\lambda_1, \dots, \lambda_n, \dots) \in T :\ \lambda_n = \lambda\}$ of a central measure M are coherent. Vice versa, for every coherent system $\{M_n\}$ of probability distributions, the function $\varphi(\lambda) = M_n(\lambda) / d(\lambda)$ is harmonic with respect to $(\Gamma, \varkappa)$ and determines via (4.4) a central Markov chain.

5. The boundary of a branching. Given a branching $(\Gamma, \varkappa)$, assume that there is a compact topological space Δ, a map $i : \Gamma \to \Delta$, and a function $\Phi : \Gamma \times \Delta \to \mathbb{R}$ satisfying the following conditions:

(i) for every $\omega \in \Delta$ the function $\varphi_\omega(\lambda) = \Phi(\lambda, \omega)$ is harmonic

(ii) the functions $\Phi_\lambda(\omega) = \Phi(\lambda; \omega)$; $\lambda \in \Gamma$, are continuous and span a dense linear subspace in the space of continuous functions on Δ

(iii) for every $\omega \in \Delta$ consider the probability distributions $M_n^{(\omega)}(\lambda) = d(\lambda)\ \Phi(\lambda; \omega)$, $\lambda \in \Gamma_n$. Then the measures $i(M_n^{(\omega)})$ weakly converge, as $n \to \infty$, to the degenerated measure δ_ω at the point $\omega \in \Delta$.

(5.1) **Definition.** We say that Δ is the *boundary* of a branching $(\Gamma, \varkappa)$, and that $\Phi = \Phi(\lambda; \omega)$ is its *Poisson kernel.*

(5.2) **Theorem.** *Let $(\Gamma, \varkappa)$ be a branching with the boundary Δ and the Poisson kernel $\Phi = \Phi(\lambda; \omega)$. Then 1) every harmonic function φ can be uniquely represented by the* Poisson integral

$$\varphi(\lambda) = \int_\Delta \Phi(\lambda; \omega)\ \mu(d\omega), \tag{5.3}$$

where μ is a probability measure on Δ, and 2) the measure μ in (5.3) is the weak limit of probability distributions $i(M_n)$, where $M_n(\lambda) = d(\lambda)\,\varphi(\lambda)$ for $\lambda \in \Gamma_n$.

One can identify the boundary Δ with the set $\mathcal{E}$ of ergodic central measures. Two paths $s, t \in T$ are *tail equivalent*, if they coincide eventually. We call a central measure M on T *ergodic*, if $M(C) = 0$ or $M(C) = 1$ for every measurable set $C \subset T$ saturated with respect to the tail equivalence. The set $\mathcal{M}$ of all central measures for a given branching $(\Gamma, \varkappa)$ is compact and convex. A central measure is an extreme point of $\mathcal{M}$ iff it is ergodic. It follows from Choquet theorem that every central measure can be represented as an integral of ergodic ones. Moreover, $\mathcal{M}$ is always a simplex, i.e. integral representations above are unique. We will only consider examples where the subset $\mathcal{E}$ of ergodic central measures is closed in $\mathcal{M}$. Hence, $\mathcal{M}$ can be identified with the simplex of all probability distributions on the boundary Δ.

Problem. Find the boundary of a given branching.

We shall give the solution for Examples A, B and for truncated case of Example C. Essentially, two general methods are available.

1) Assume that $t = (\lambda_1, \dots, \lambda_n, \dots) \in T$ is such a path that the limits

$$(5.4) \qquad \lim_{n\to\infty} \frac{d(\lambda)\, d(\lambda, \lambda_n)}{d(\lambda_n)} = M_m(\lambda)$$

exist for all $\lambda \in \Gamma_m$, $m \geq 0$. It is easy to see that $\{M_m\}$ is a coherent system of probability distributions. A variant of Birkhoff Ergodic Theorem implies

(5.5) **Theorem** [11]. *Every ergodic central measure $M \in \mathcal{E}$ can be obtained, using (5.4), for appropriate path $t \in T$.*

By this result the Problem reduces to a combinatorial problem on the asymptotics of dimensions in (5.4).

2) If a branching is given by a Pieri type formula, say (2.2), we call it *multiplicative*. Every harmonic function $\varphi : \Gamma \to \mathbb{R}$ for such branching can be extended to a linear functional $\tilde{\varphi}$ on the corresponding algebra R.

(5.6) **Theorem** [6]. *Assume that a branching $(\Gamma, \varkappa)$ is multiplicative. Then*

a) the set $\mathcal{E}$ of extreme points is closed in $\mathcal{M}$

b) a harmonic function φ corresponds to an ergodic central measure iff its extension $\tilde{\varphi}$ is an algebra homomorphism.

For a multiplicative branching, determined via (2.1) by an algebra R, a linear basis $\{m_\lambda\}$ and an element $s_{(1)}$, the Problem reduces to the following one: find all algebra homomorphisms $\tilde{\varphi} : R \to \mathbb{R}$, such that $\tilde{\varphi}(m_\lambda) \geq 0$ for all λ, and $\tilde{\varphi}(s_{(1)}) = 1$. The solution for truncated versions of examples A – C is simple. In fact, consider the simplex

$$(5.7) \qquad \Delta_k = \{\alpha \in \mathbb{R}_+^k : \quad \alpha_1 \geq \dots \geq \alpha_k \geq 0, \quad \alpha_1 + \dots + \alpha_k = 1\}.$$

(5.8) **Corollary.** *For k-truncated branchings of examples A – C the boundary is $\Delta = \Delta_k$. The imbedding $i : \mathcal{Y} \to \Delta_k$ is given by the formula $i(\lambda) = (\lambda_1/n, \dots, \lambda_k/n)$. The Poisson kernel is determined by basic symmetric polynomials m_λ. For instance, $\Phi(\lambda; \alpha) = P_\lambda(\alpha; \theta)$ in Example C.*

6. The boundary of the Young lattice and related branchings. We claim that the triple (Δ, i, Φ) in the examples below satisfies all the conditions of the Definition (5.1).

Example A ([11]). The boundary Δ is the space of pairs $(\alpha;\beta)$ of sequences of non-negative real numbers $\alpha = (\alpha_1, ..., \alpha_n, ...)$, $\beta = (\beta_1, ..., \beta_n, ...)$ such that

$$\alpha_1 \geq \ldots \geq \alpha_n \geq \ldots \geq 0; \quad \beta_1 \geq \ldots \geq \beta_n \geq \ldots \geq 0; \quad \sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leq 1. \tag{6.1}$$

The topology is that of coordinatewise convergence.

In order to describe the imbedding $i : \mathcal{Y} \to \Delta$, denote by $(f_1, ..., f_d; g_1, ..., g_d)$ the Frobenius parameters of a Young diagram $\lambda = (\lambda_1, \lambda_2, ...) \in \Gamma_n$ (i.e. d is the diagonal length of λ, $f_k = \lambda_k - k + 1/2$, and $g_k = \lambda'_k - k + 1/2$, where λ'_k is the length of the kth column of λ). By definition,

$$i(\lambda) = \left(\frac{f_1}{n}, \frac{f_2}{n}, \ldots; \frac{g_1}{n}, \frac{g_2}{n}, \ldots\right) \in \Delta,$$

where both sequences are tailed by zeros. The Poisson kernel $\Phi(\lambda; \alpha, \beta) = s_\lambda(\alpha; \beta; \gamma)$, where $\gamma = 1 - \sum \alpha_i - \sum \beta_i$, is determined by *extended Schur functions* s_λ.

The extended power sum symmetric functions p_n are defined as follows:

$$\begin{aligned} p_1(\alpha;\beta;\gamma) &= \sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i + \gamma, \\ p_n(\alpha;\beta;\gamma) &= \sum_{i=1}^{\infty} \alpha_i^n + (-1)^{n+1} \sum_{i=1}^{\infty} \beta_i^n \quad \text{if} \quad n \geq 2. \end{aligned} \tag{6.2}$$

Other symmetric functions are polynomials in power sum functions $p_1, p_2, ...$. For instance, $s_\lambda(\alpha;\beta;\gamma) = \sum_\rho \chi^\lambda_\rho p_\rho(\alpha;\beta;\gamma) \,/\, z_\rho$. Here χ^λ_ρ is the value of a character of the symmetric group $\mathfrak{S}_n$, indexed by λ, on a conjugacy class indexed by $\rho = (1^{\rho_1}, 2^{\rho_2}, ...)$.

(6.3) **Theorem** [11]. *Let $t = (\lambda_1, \ldots, \lambda_n, \ldots)$ be an infinite Young tableau. The following conditions are equivalent:*

$$\lim_{n\to\infty} i(\lambda_n) = (\alpha;\beta) \in \Delta \tag{6.4}$$

$$\lim_{n\to\infty} \frac{d(\lambda, \lambda_n)}{d(\lambda_n)} = s_\lambda(\alpha;\beta; 1 - \Sigma\alpha_i - \Sigma\beta_i); \quad \lambda \in \mathcal{Y}. \tag{6.5}$$

(6.6) **Corollary.** *Let M_ω denote the ergodic central measure associated to a point $\omega \in \Delta(\mathcal{Y})$. Then $\lim i(\lambda_n) = \omega$ for a.a. Young tableaux $t = (\lambda_1, \ldots, \lambda_n, \ldots) \in T$.*

We shall write simply $r(\lambda)$ for the value of an extended symmetric function $r \in R$ at the point $(f_1, f_2, \ldots; g_1, g_2, \ldots) = n \cdot i(\lambda)$, $\lambda \in \mathcal{Y}_n$. The main tool in the proof of Theorem (6.3) is

(6.7) **Lemma.** *For every Young diagram $\lambda \in \mathcal{Y}_m$ there is a symmetric polynomial Q_λ of degree* $\deg Q_\lambda < m$, *such that*

$$\frac{d(\lambda,\Lambda)}{d(\Lambda)} = \frac{s_\lambda(\Lambda) + Q_\lambda(\Lambda)}{|\Lambda|(|\Lambda|-1)\dots(|\Lambda|-m+1)}. \tag{6.8}$$

Example B ([2], [7]). The boundary Δ is the simplex of nonincreasing sequences $\alpha = (\alpha_1, ..., \alpha_n, ...)$ of nonnegative numbers, with a restricted sum: $\gamma = 1 - \Sigma\alpha_i \geq 0$. The imbedding $i : \mathcal{Y} \to \Delta$ is given by the formula $i(\lambda) = (\lambda_1/n, \lambda_2/n, \dots)$. The Poisson kernel $\Phi(\lambda;\alpha) = m_\lambda(\alpha;0;\gamma)$ is determined by extended monomial symmetric functions.

Example C. The description of the boundary for the Jack branching reduces to the following problem: find all homomorphisms of the symmetric function algebra to $\mathbb{R}$, non-negative at all Jack polynomials $P_\lambda(x;\theta)$ for a fixed θ. According to the Conjecture of Section 7.3 in [3], for all $\theta > 0$ the boundary is the same as for the Young lattice. See (5.8) for the solution in truncated case.

7. Plancherel Growth Process. The most central of all central measures on the Young lattice is the Plancherel measure M, corresponding to the point $\alpha = \beta = 0$ of the boundary $\Delta(\mathcal{Y})$. The transition probabilities for this Markov chain are $p(\lambda,\Lambda) = d(\Lambda)\,/\,|\Lambda|\,d(\lambda)$, and its level distribution $M_n(\lambda) = d^2(\lambda)\,/\,n!$ is the ordinary Plancherel measure of the symmetric group $\mathfrak{S}_n$. In what follows, it will be convenient to consider a Young diagram λ as a continuous piecewise linear function $v = \lambda(u)$ with the derivative ± 1. In combinatorial terms, $\lambda(u)$ is the length of the diagonal of λ with the fixed content u.

(7.1) **Theorem** [13]. *Let $\lambda_n \in \mathcal{Y}_n$ denote any Young diagram of the maximal dimension $d(\lambda)$. Then the limit*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \lambda_n(u\sqrt{n}) = \Omega(u) \tag{7.2}$$

exists uniformly in u, where

$$\Omega(u) = \begin{cases} \frac{2}{\pi}(u \arcsin\frac{u}{2} + \sqrt{4-u^2}), & \text{if } |u| \leq 2 \\ |u|, & \text{if } |u| \geq 2. \end{cases} \tag{7.3}$$

(7.4) **Theorem** [13]. *The uniform limit (7.2) exists for a.a., with respect to Plancherel measure, Young tableaux $t = (\lambda_1, \dots, \lambda_n, \dots) \in T$.*

A weaker version of this Law of Large Young Diagrams appeared in [8], [12]. The following result can be compared to the Central Limit Theorem.

(7.5) **Theorem** [1]. *Consider the random function $G_n(x) = \lambda(x\sqrt{n}) - \sqrt{n}\,\Omega(x)$, where a diagram $\lambda \in \mathcal{Y}_n$ has Plancherel probability $d^2(\lambda)\,/\,n!$. Then $G_n(x)$ weakly converges to a Gaussian random process $G(x) = \sum_{n\geq 1} \xi_n\, u_n(x)/\sqrt{n+1}$, where ξ_n are independent standard normal random variables, and $u_n(2\cos\theta) = \sin(n+1)\theta\,/\,\sin\theta$ is the Tchebychef polynomial of the second kind.*

An equivalent version is this. Define random variables $\varphi_k(\lambda) = \chi^\lambda_{(k,1^{n-k})} n^{k/2}\,/\,d(\lambda)$ on the set $\mathcal{Y}_n$ endowed with the Plancherel measure. Then the functionals $\varphi_k(\lambda)$ are asymptotically independent and have Gaussian limiting distributions. More precisely,

$$\lim_{n\to\infty} M_n\{\lambda \in \mathcal{Y}_n : \varphi_k(\lambda) < x_k,\ \ 2 \le k \le m\} = \prod_{k=2}^{m} \frac{1}{\sqrt{2\pi k}} \int_{-\infty}^{x_k} \exp(-\frac{y_k^2}{2k})\, dy_k \tag{7.6}$$

for every sequence $x_2, \dots, x_m \in \mathbb{R}$.

8. Multiplicative central measures. Let $c(b) = j - i$ denote the content of a box.

(8.1) **Lemma** [5]. *For every complex number z the formula*

$$p^{(z)}(\lambda, \Lambda) = \frac{|c(\Lambda \setminus \lambda) + z|^2}{|\lambda| + |z|^2} \cdot \frac{d(\Lambda)}{|\Lambda|\, d(\lambda)}; \quad \lambda \nearrow \Lambda \tag{8.2}$$

determines transition probabilities of a central Markov chain. Moreover, for natural k and real $A > 0$ transition probabilities

$$p_k^{(A)}(\lambda, \Lambda) = \frac{(k + c(\Lambda \setminus \lambda)) \cdot (A + k - 1 + c(\Lambda \setminus \lambda))}{|\lambda| + kA + k(k-1)} \cdot \frac{d(\Lambda)}{|\Lambda|\, d(\lambda)} \tag{8.3}$$

determine a central Markov chain for the truncated branching $\mathcal{Y}(k)$.

More generally, transition probabilities

$$p(\lambda, \Lambda) = \frac{(k\theta + c_\theta(\Lambda \setminus \lambda)) \cdot (A + (k-1)\theta + c_\theta(\Lambda \setminus \lambda))}{|\lambda| + kA + k(k-1)\theta} \cdot \frac{d_{1/\theta}(\Lambda')}{|\Lambda|\, d_{1/\theta}(\lambda')}, \tag{8.4}$$

where $d_\theta(\lambda)$ is given by (3.3), determine a central Markov chain for the truncated Jack branching $J^{(\theta)}(k)$. Its level distributions are multiplicative:

$$M_n(\lambda) = \frac{n!}{(k(k-1)\theta + kA)_n} \prod_{b \in \lambda} \frac{(k\theta + c_\theta(b)) \cdot (A + (k-1)\theta + c_\theta(b))}{(a(b) + (l(b) + 1)\theta) \cdot (a(b) + 1 + l(b)\theta)}. \tag{8.5}$$

Here $c_\theta(b) = (j-1) - (i-1)\theta$ is an analog of the content of a box $b = (i,j) \in \mathbb{R}^2$, and $(x)_n = x(x+1)\dots(x+n-1)$ is the Pochhammer symbol.

(8.6) **Theorem.** *Let $\{\lambda^{(n)}\}_{n=1}^{\infty}$ be such a sequence of Young diagrams that the limits $\lim_n \lambda_i^{(n)} / n = \alpha_i$ exist for all $i = 1, 2, \dots, k$. Then*

$$\lim_{n\to\infty} n^{k-1} M_n(\lambda^{(n)}) = \frac{k!\, \Gamma(kA + k(k-1)\theta)\, \Gamma^k(\theta+1)}{\prod_{j=1}^{k} \Gamma(A + (j-1)\theta)\, \Gamma(j\theta + 1)} \prod_{1 \le i < j \le k} |\alpha_i - \alpha_j|^{2\theta} \prod_{i=1}^{k} \alpha_i^{A-1}. \tag{8.7}$$

Selberg's integral (1.1) follows immediately from the fact that M_n is a probability distribution. Theorem (5.2) and Corollary (5.8) imply then that

$$\begin{aligned} &\int_{\Delta_k} P_\lambda(t;\theta) \prod_{1 \le i < j \le k} |t_i - t_j|^{2\theta} \prod_{j=1}^{k} t_j^{A-1}\, dt_1 \dots dt_{k-1} = \\ &= k! \frac{P_\lambda(1, \dots, 1; \theta)}{\Gamma(n + kA + (k-1)k\theta)} \prod_{j=1}^{k} \frac{\Gamma(\lambda_j + A + (j-1)\theta)\, \Gamma(j\theta + 1)}{\Gamma(\theta + 1)}. \end{aligned} \tag{8.8}$$

For $\theta = 0$ the distribution (8.5) reduces to a truncated version of Ewens sampling formula

(8.9) $$M_n(\lambda) = \frac{n!}{\lambda_1!\,\lambda_2!\dots} \cdot \frac{k!}{r_0!\,r_1!\,r_2!\dots} \cdot \frac{(A)_{\lambda_1}\dots(A)_{\lambda_k}}{(kA)_n},$$

where $r_j = \rho_j(\lambda)$ is the number of rows of length j in λ. Ewens formula

(8.10) $$M_n(\lambda) = \frac{n!}{z_\lambda} \cdot \frac{t^{l(\lambda)}}{(t)_n}$$

is its limit as $k \to \infty$, $A = t/k \to 0$. In this case the mixing measure is the Poisson - Dirichlet distribution on an infinite dimensional simplex. There are deep results on the properties of this distribution (cf. [14]). Much less is known on the mixing distributions of central Markov chains determined by (8.2) or its θ-analog (see [5]).

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