A Non-Commutative Analogue of Weyl Denominator Formula
Extended Abstract

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Abstract

We give a non-commutative analogue of Weyl denominator formula for any Lie algebra defined by a presentation. The formula is best possible: no other (non-commutative) cancellation is possible. The proof is bijective and uses sign-reversing weight-preserving involutions on appropriate sets.

Résumé

On propose un analogue non-commutatif de la formule du dénominateur de Weyl pour toute algèbre de Lie définie par une présentation. La formule est optimale: aucune autre simplification (non-commutative) n'est possible. La démonstration est bijective et utilise de nombreuses involutions sur certains ensembles.

1 Introduction

Let A be an alphabet, A^* the corresponding monoid and $\mathbb{Q}\langle A\rangle$ the free associative algebra over A. Given an ideal I of $\mathbb{Q}\langle A\rangle$, we can construct standard bases for $\mathbb{Q}\langle A\rangle/I$, using a process similar to the construction of a Gröbner basis. It is known that if w is a word (called a standard word) in such a basis then any factor of it is also in the basis. Hence, the basis $S = S(I)$ of standard words form a lower ideal in the poset of all words ordered by the factor order (denoted by the symbol $|$).

Similarly, let $\mathcal{L}(A)$ be the free Lie algebra over A (with coefficients in $\mathbb{Q}$). Given a Lie ideal J of $\mathcal{L}(A)$, we can define a subset $SL = SL(J)$ of 'Lie-standard' Lyndon words that forms a basis of $\mathcal{L}(A)/J$ after taking standard bracketing. We know that SL is also a lower ideal in the poset L of Lyndon words ordered by the factor order.

Moreover, if I is the (algebra) ideal generated by J, we know that $SL = S \cap L$ (see [LR]). Since the Lyndon words form a complete factorization of the free monoid A^*, a word w is standard iff it can be written uniquely as a decreasing product $\ell_1 \cdots \ell_k$ of Lie-standard Lyndon words. Thus, in $\mathbb{Q}\langle\langle A\rangle\rangle$, we have the relation:

$$\sum_{w\in S} w = \prod_{\ell\in SL} (1-\ell)^{-1},$$

where the product is decreasing. If we invert this relation, we get:

$$\prod_{\ell\in SL} (1-\ell) = (\sum_{w\in S} w)^{-1}. \tag{1}$$

We can relate formula 1 to the theory of semi-simple Lie algebras (see [Bo2], [Hu]). Take $A = \{x_1 < \ldots < x_n\}$ and define J to be the Lie ideal generated by Serre's relations:

$$(ad\, x_i)^{1-c_{ji}} x_j \text{ for } 1 \le i \ne j \le n$$

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(where the c_{ji} are fixed non-positive integers defined by the so-called Cartan matrix). We can interpret Weyl denominator formula as an equality in the commutative algebra $\mathbb{Q}\,[[A]]$ which expresses the product $\prod_{\ell\in SL}(1-\ell)$ in term of a certain sum the terms of which are product of words in SL. In this interpretation, the multi-degree of each Lie-standard word correspond to a "positive root" (taking multiplicity into account).

The purpose of this note is to give an analogue of Weyl formula that will hold in $\mathbb{Q}\,\langle\langle A\rangle\rangle/I$ and that will be valid for any Lie algebra defined by a presentation. This will be done in two steps:

First, we invert the characteristic serie $\sum_{w\in S} w$ of standard words in the general setting. Of course, the inverse exists. What we give here is an explicit combinatorial inverse in a form as concise as possible: no other simplification will be possible. In fact, we show that the coefficient of each word appearing in the inverse is 1 or -1.

Next, we develop the product $\prod_{\ell\in SL}(1-\ell)$ and, using a properly devised involution, we simplify it to give the inverse formula. This will give two ways to interpret the right-hand side of the resulting analogue of Weyl denominator formula.

2 Standard and Anti-Standard Words

We will use the following notations. The empty word is denoted 1. If u and w are words: $u|_l w$ means that u is a left factor of w, $u||_l w$ means that u is a proper left factor of w (left factor but not equal). We have similar notation for right factor and proper right factor (and for proper factor). If $u|_l w$ then w can be written as $w=uv$ for a unique word v; we write $u^{-1}w=v$. We use a similar notation if $u|_r w$. Finally, if $w\neq 1$, $f(w)$ stands for the first letter of w.

The inversion of the characteristic serie of standard words depend ultimately upon the fact that the set of standard words forms a lower ideal in the poset A^* (with the factor order). Since A^* is well ordered, the set $B=\min(A^*-S)$ is well defined. This simple observation allows us to redefine the sets B,S and SL axiomatically, in a way independent but still coherent with the results of [LR].

Definition 2.1 *Let $B\subseteq A^*$ such that:*

1. $1\notin B$, $A\cap B=\emptyset$,
2. *if* $u,v\in B$ *and if* $u|v$ *then* $u=v$.

A word $w\in A^*$ is non-standard if $\exists u\in B$ such that $u|w$, otherwise, the word is standard. The set of standard words is denoted by S.

Definition 2.2 *Let $w\in A^*$. Let $w_0=1$. If $w\neq 1$, define $w_1=f(w)$. If w_{i-1} and w_i exist, define w_{i+1} (when possible) as the shortest word $w_{i+1}=au$ such that:*

1. $a\in A^*$, $u\in B$,
2. $w_{i-1}|_l a||_l w_i||_l au|_l w$.

Each of the words w_i is called anti-standard. The set of all anti-standard words will be denoted by AS. There is a maximal integer $k=\chi(w)$ (called the index of w) such that w_k exists. Define the associated sequence of anti-standard left factors (ASLF) of w to be $(w_i)_{i=0}^k$.

Observe that $k=\chi(w_k)$ and that the longest anti-standard left factor of w is w_k. Observe also that $\{1\},A,B\subseteq AS$.

Definition 2.3 *Let $w_k\in AS$ with $(w_i)_{i=0}^k$ its ASLF. Let $w\in S$. Let w' be the longest anti-standard left factor of w_kw. Define:*

$$\varphi(w_k,w)=\begin{cases}(w_{k-1},w_{k-1}^{-1}w_kw) & \text{if } w_k=w' \text{ and } k\geq 1,\\ (w',w'^{-1}w_kw) & \text{if } w_k\neq w'.\end{cases}$$

Proposition 2.1 *The function φ is an involution $(w_1,w_2)\overset{\varphi}{\longleftrightarrow}(\overline{w_1},\overline{w_2})$ over $AS\times S-\{(1,1)\}$ such that:*

1. $w_1w_2 = \overline{w_1}\,\overline{w_2}$,

2. $\chi(\overline{w_1}) = \chi(w_1) \pm 1$.

Corollary 2.1 *In* $\mathbb{Q}\langle\langle A\rangle\rangle$, *we have:*

$$\sum_{w\in AS}(-1)^{\chi(w)}w\sum_{w\in S}w = 1.$$

Similarly, we can prove that the signed characteristic serie of the anti-standard words is also a right inverse. For any word, we must define its associated sequence of anti-standard *right* factors (ASRF) working backward from the end of the word. This allows us to speak of right anti-standard words, while we had previously left anti-standard words.

Proposition 2.2 *A word is left anti-standard iff it is right anti-standard. To every ASLF* $(w_i)_{i=0}^k$ *of the anti-standard word* w *correspond a unique ASRF* $(w_i')_{i=0}^k$ *of* w, *and conversely.*

So, after all, there is no need to distinguish between a left and a right index for anti-standard words.

We can also prove this result by using a left version and a right version of an involution derived from φ on the last member of the following identity:

$$(\sum_{w\in S}w)^{-1} = (1-\sum_{w\in S-\{1\}}-w)^{-1} = \sum_{k\geq 0}(-1)^k w_1\cdots w_k,$$

where the last sum is over words $w_1,\ldots,w_k \in S-\{1\}$.

Using φ on a suitably restricted subset, we find also a way to specify the first letter in some of these series.

Proposition 2.3 *Let* $M\subseteq A$, *then in* $\mathbb{Q}\langle\langle A\rangle\rangle$, *we have:*

$$\sum_{w\in M(A-M)^*\cap S} w = \sum_{w\in M(A-M)^*\cap AS}(-1)^{\chi(w)+1}w \sum_{w\in(A-M)^*\cap S} w,$$

$$\sum_{w\in(A-M)^* M\cap S} w = \sum_{w\in(A-M)^*\cap S} w \sum_{w\in(A-M)^* M\cap AS}(-1)^{\chi(w)+1}w.$$

3 Lyndon Words

We now assume that the alphabet A is totally ordered. We will consider the lexicographical order on A^*. Assume also that $B\subseteq L$ is a set for which definition 2.1 holds. Define $SL = L\cap S$.

Lemma 3.1 *Let* $w\in A^*$. *Let* $w = \ell_1\cdots\ell_k$ *be its Lyndon factorization (i.e.* $\ell_i\in L$, $\ell_1\geq\cdots\geq\ell_k$*). Then:*

$$w\in S \textit{ iff } \ell_1,\ldots,\ell_k\in SL.$$

Our interpretation of Weyl denominator formula then reads:

$$\prod_{\ell\in SL}(1-\ell) = \sum_{w\in AS}(-1)^{\chi(w)}w$$

(the product is over increasing ℓ) or, after expanding the product:

$$\sum_{\ell_1<\ldots<\ell_k\in SL}(-1)^k\ell_1\cdots\ell_k = \sum_{w\in AS}(-1)^{\chi(w)}w. \tag{2}$$

Although we know this formula to hold, we will also provide the steps toward a combinatorial proof. There is a good reason for doing so: in the Weyl denominator formula, everything can be expressed in terms of roots. Accordingly, in our interpretation, we should express everything in terms of standard Lyndon words, including the right-hand side of equation 2. Hence, each anti-standard word should have a unique decomposition as a strictly *increasing* product of standard Lyndon words. This suggests to find the terms remaining in the left-hand side of equation 2 after cancellation.

We construct an involution on the set

$$Lex = \{(\ell_1, \ldots, \ell_k) | k \geq 0, \ell_1 < \ldots < \ell_k \in SL\}$$

that will kill most of the terms on the left-hand side. Of course, we could use the involution φ and the Garcia-Milne [GM] involution principle to show that:

$$\begin{aligned} \sum_{w \in AS} (-1)^{\chi(w)} w &= \sum_{w \in AS} (-1)^{\chi(w)} w \sum_{w \in S} w \sum_{(\ell_1,\ldots,\ell_k) \in Lex} (-1)^k \ell_1 \cdots \ell_k \\ &= \sum_{(\ell_1,\ldots,\ell_k) \in Lex} (-1)^k \ell_1 \cdots \ell_k. \end{aligned}$$

(The involution that shows the first equality is exactly what you think it sould be.)

However, the sought for relation between AS and Lex will be difficult to understand by this process. We need an explicit involution that shows directly the relation.

But first, we must do some work. We begin with an easily proven but not so well known statement about Lyndon words and with a generalization of the standard factorization of words.

Lemma 3.2 *Let $u, w \in A^*$ and $v \in A^+$. Suppose that $uv, vw \in L$. Then $uvw \in L$.*

Definition 3.1 *Let $w, m \in A^*$ such that $w||_l m$. Then there exists a unique pair $(n, \ell) \in A^* \times L$ such that:*

1. *$n\ell = m$,*
2. *$w|_l n$ and*
3. *ℓ is minimal (lexicographically).*

Define the standard factorization of m relatively to w as $\sigma_w(m) = (n, \ell)$. We will write also $\sigma''_w(m) = \ell$.

Remarks.

- The last condition is equivalent to "$|\ell|$ is maximal".
- The usual standard factorization of m is $\sigma_{f(m)}(m)$.
- If $m \in L$, then $\sigma_1(m) = (1, m)$.

Proposition 3.1

1. *If $w, m \in L$ and $w||_l m$, then $\sigma_w(m) \in L \times L$.*
2. *If $w, \ell, m \in L$ and if $w|_l \ell$, then $\sigma_w(\ell m) = (\ell, m)$ iff $w = \ell$ or $w||_l \ell$ and $\sigma''_w(\ell) \geq m$.*
3. *If $w \in L$; $\ell, m \in SL$; $w|_l \ell$ and $\sigma_w(\ell m) = (\ell, m)$, then $w^{-1}\ell m \in SL$.*

We recognize in the first two statements of this proposition the usual properties of the standard factorization.

Definition 3.2 *Let $(w_i)_{i=0}^k$ be an ASLF, let $\ell \in L$ such that $w_k||_l \ell$. Define $L_{k+1} = \ell$. For $i = k, k-1, \ldots, 0$, define $(L_i, \ell_{i+1}) = \sigma_{w_i}(L_{i+1})$ (this is always possible). Define also $\sigma((w_i)_{i=0}^k, \ell) = (\ell_1, \ldots, \ell_{k+1})$ (if $k \geq 0$). (If $k < 0$, $\sigma((w_i)_{i=0}^k, \ell)$ is the empty sequence.)*

Definition 3.3 *Let $(\ell_1, \ldots, \ell_r) \in Lex$. For $i \in \{1, \ldots, r\}$, write $L_i = \ell_1 \cdots \ell_i$. Let $(w_i)_{i=0}^{k}$ be the ASLF of L_r. Let $i \in \{1, \ldots, k\}$ be maximal (if it exists) such that for all $j \le i$:*

$$w_j = L_j \text{ or } w_j ||_l L_j \text{ and } \sigma''_{w_j}(L_j) \ge \ell_{j+1}$$

(we can define $\ell_{r+1} = \infty$ if needed). Define:

$$\theta(\ell_1, \ldots, \ell_r) = \begin{cases} (\sigma((w_j)_{j=0}^{i-1}, L_{i+1}), \ell_{i+2}, \ldots, \ell_r) & \text{if } i = k < r \text{ or } i < k, r \text{ and } L_{i+1} ||_l w_{i+1}, \\ (\sigma((w_j)_{j=0}^{i+1}, L_{i+1}), \ell_{i+2}, \ldots, \ell_r) & \text{if } i < k, r;\ w_{i+1} ||_l L_{i+1} \text{ and } \sigma''_{w_{i+1}}(L_{i+1}) < \ell_{i+2}. \end{cases}$$

Proposition 3.2 *The function θ is a sign-reversing, weight-preserving involution on the set Lex. Its set of fixed points is AS.*

This involution provide a new description of AS in term of an increasing sequence of standard Lyndon words.

Proposition 3.3 *AS is the set of $(\ell_1, \ldots, \ell_k) \in Lex$ such that if $(w_i)_{i=0}^{r}$ is the ASLF of $\ell_1 \cdots \ell_k$, then: $r = k$; $w_k = \ell_1 \cdots \ell_k$ and for all $j \in \{1, 2, \ldots, k-1\}$, we have:*

$$w_j = \ell_1 \cdots \ell_j \text{ or } w_j ||_l \ell_1 \cdots \ell_j \text{ and } \sigma''_{w_j}(\ell_1 \cdots \ell_j) \ge \ell_{j+1}.$$

Clearly we get the "AS-description" from the "Lex-description" by applying the algorithm in definition 2.2. It is also possible to reverse the process:

Proposition 3.4 *Let $(w_i)_{i=0}^{k}$ be an ASLF. There is an unique element $(\ell_1, \ldots, \ell_k) \in Lex$ such that $w_k = \ell_1 \cdots \ell_k$. It is given by $\sigma((w_i)_{i=0}^{k-1}, w_k)$.*

Hence, each fixed point w has an unique "Lex-decomposition". This gives an independent proof that equality 2 is best possible.

4 Concluding Remarks

The results suggest to relate the right-hand side of equation 2 to an eventual analogue of the Weyl group. So far, even for simple Lie algebras, it is not apparent how to do so.

The set B for the simple Lie algebras are easily constructed from the list of standard Lyndon words (see [LR]). Clearly, $B \subseteq SL^2 - SL$, since any proper factor of a word in B is standard. Then, simply remove from $SL^2 - SL$ the non-Lyndon words and take the minimal elements with respect to the factor order.

Finally, it could be interesting to note that, under our interpretation, proposition 2.3 is an analogue of Weyl-Kac characters formula.

References

[Bo1] N. Bourbaki, *Groupes et algèbres de Lie*, Chap.1–3, Hermann, Paris, 1960, 1972.

[Bo2] N. Bourbaki, *Groupes et algèbres de Lie*, Chap.4–6, Masson, Paris, 1981.

[GM] A. Garcia, S. Milne, *Method for Constructing Bijections for Classical Partition Identities*, Proc. Natl. Acad. Sci. USA, **78**, 1981, 2026–2028.

[Hu] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, "Graduate Texts in Mathematics", **40**, Springer-Verlag, New York/Berlin,1972.

[LR] P. Lalonde, A. Ram, *Standard Lyndon Bases of Lie Algebras and Enveloping Algebras*, Formal Power Series and Algebraic Combinatorics, Proceedings of the 5th Conference (posters and software demonstrations), DSI, Università di Firenze, A. Bartolli, M. Delest, R. Pinzani, eds., 1993, 99–104.

[Lo] M. Lothaire, *Combinatorics on Words*, Encyc. Math, **17**, Addison Wesley, 1983.