B_n Stanley Symmetric Functions

Extended Abstract

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Our main objective is to study the B_n Stanley symmetric functions, G_w. This has been independently investigated by Fomin and Kirillov[4] and by Stembridge[15]. It is the analogue of the Stanley symmetric functions for the symmetric group. This symmetric function also appeared in [1] in the search for an analogue of the Schubert polynomials in other classical groups. First, we will define the B_n Stanley symmetric functions using the nilCoxeter algebra. From here, it can be shown that it is symmetric and can be written as an integer combination of Schur P-functions. Using the Kraśkiewicz insertion first described in [9], we can show that they are actually nonnegative integer combinations of the Schur P-functions.(Theorem 1.5) We will described the Kraśkiewicz insertion and the B-Coxeter-Knuth relations. These are all analogues of the Edelman-Greene insertion. Next, looking into other properties of the Kraśkiewicz insertion, we are able to give some nice descriptions for G_w.

L'objet principal de cet article est d'étudier les fonctions symétriques de Stanley pour B_n, notées G_w. Celles-ci furent étudiées indépendamment par Fomin et Kirillov[4] et par Stembridge[15]. Ces fonctions symétriques ont paru également dans [1] où l'on recherche un analogue des polynômes de Schubert dans le cas d'autres groupes classiques. On commence en définissant les fonctions symétriques de Stanley pour B_n en employant l'algèbre nilCoxeter. Par la suite on peut démontrer qu'elles sont en effet symétriques et qu'elles peuvent être exprimées en combinaisons linéaires de fonctions P-Schur à coefficients entiers. En utilisant l'algorithme d'insertion de Kraśkiewicz [9], on peut montrer que les coefficients sont en fait non négatifs (Théorème 1.5). On décrit l'algorithme d'insertion de Kraśkiewicz et les relations B-Coxeter-Knuth. Ceux-ci sont tous des analogues de l'algorithme d'insertion d'Edelman et Greene. Ensuite, en considérant d'autres propriétés de l'algorithme d'insertion de Kraśkiewicz, on parvient à certaines jolies descriptions des fonctions G_w.

1 The Hyperoctahedral Group and the nilCoxeter algebra

We will spend some time on giving definitions and notations here. First, some basic facts about the hyperoctahedral group, B_n. The main reference used is [8]. B_n is the Weyl group corresponding to the root system, B_n. We represent an element of B_n as a signed permutation and write it down in 1-line notation. The simple reflections are denoted by $s_i, i = 0, 1, \cdots n$.

$$\begin{aligned} s_0 &= \bar{1}2\cdots n \\ s_i &= 12\cdots i-1\; i+1\; i\; i+2\cdots n \quad \text{for } 1 \le i < n \end{aligned}$$

Every element $w \in B_n$ can be expressed as a product of s_i's since the simple reflections generate the group. Any such expression of shortest length is called a *reduced word* and its length denoted by $l(w)$. The collection of reduced words of w is denoted by $R(w)$. When we write a reduced word, we will often only write the subscripts of the simple reflections. So, $s_3s_2s_1s_0s_3s_2s_3$ is written simply as 3210323. It is a reduced word for the signed permutation $\bar{4}321$. When we multiply s_i's, we do it from the right. Under this convention, a simple reflection s_i acts on 1-line notation by switching the numbers in positions i and $i+1$ if $i \neq 0$ and changing the sign of the first number if $i = 0$.

Following the presentation in [4], we define the nilCoxeter algebra for B_n.

Definition 1.1 *Let B_n be the non-commutative algebra generated by $u_0, u_1, \cdots, u_{n-1}$ under the relations:*

$$\begin{array}{rcll} u_i^2 &=& 0 & i \geq 0 \\ u_i u_j &=& u_j u_i & |i-j| > 1 \\ u_i u_{i+1} u_i &=& u_{i+1} u_i u_{i+1} & i > 0 \\ u_0 u_1 u_0 u_1 &=& u_1 u_0 u_1 u_0 & \end{array}$$

The nilCoxeter algebra has a vector space basis of signed permutations of B_n. This is because the last 3 relations listed above are the Coxeter relations for B_n which are satisfied by the simple reflections s_i.

Definition 1.2 *Define in the polynomial algebra $B_n[x_1, x_2, x_3, \cdots]$*

$$B(x_i) \quad = \quad (1 + x_i u_{n-1})(1 + x_i u_{n-2}) \cdots (1 + x_i u_1)(1 + x_i u_0)(1 + x_i u_1) \cdots (1 + x_i u_{n-2})(1 + x_i u_{n-1})$$

Consider the expansion of the following formal power series in terms of the basis of signed permutations.

$$B(x_1)B(x_2)\cdots = \sum_{w \in B_n} G_w(x) w$$

The $G_w(x) = G_w(x_1, x_2, \cdots)$ are called the B_n Stanley symmetric functions.

It can be shown from this definition that the G_w's are symmetric functions[4, Proposition 4.2]. Moreover, if we replace x_1 by $-x_2$, we get

$$G_w(-x_2, x_2, x_3, x_4, \cdots) = G_w(x_3, x_4, \cdots)$$

This shows that they can be expressed in terms of Schur P-functions with integer coefficients. (See [10], [12] and [16] for a definition of Schur P-functions and other properties).

To show that these coefficients are nonnegative integers, we use an alternative description of G_w involving compatible sequences.

Definition 1.3 *Let $a = a_1 a_2 \cdots a_m \in R(w)$. We call a sequence of nonnegative integers $i = (i_1, i_2, \cdots, i_m)$ an a-*compatible *sequence if*

1. $i_1 \leq i_2 \leq \cdots \leq i_m$
2. $i_j = i_{j+1} = \cdots = i_k$ *occurs only when $a_j, a_{j+1}, \cdots, a_k$ is a unimodal sequence*

Denote the set of a-compatible sequence as $K(a)$.

Let $l_0(w)$ denote the number of bars in the 1-line notation of w. Note that the number of 0's in any reduced word of w is equal to $l_0(w)$. A simple observation gives:

Theorem 1.4 ([4, Equation (6.3)])

$$G_w(x) = \sum_{a \in R(w)} \sum_{i \in K(a)} 2^{l(i) - l_0(w)} x_{i_1} x_{i_2} \cdots x_{i_m}$$

where $l(i)$ is the number of distinct integers in i.

This is an analogue of [14, Equation (1)].

Theorem 1.5 *For all $w \in B_n$,*

$$G_w(x) = \sum_{R \in \mathrm{SDT}(w)} 2^{l(R) - l_0(w)} P_{\mathrm{sh}(R)}(x)$$

↓ standard decomposition tableau

We will postpone the proof till Section 3.

1) $\pi_R = R_n R_{n-1} \cdots R_1$ is a reduced word for w.
2) R_i is unimodal
3) R_i is a unimodal subsequence of maximal length in $R_j R_{j-1} \cdots R_{i+1} R_i$

2 Kraśkiewicz Insertion

In this section, we will present Kraśkiewicz insertion. The presentation is different from that in [9]. Firstly, we have used s_0 as the special reflection instead of s_n. So, the numbers that are used in a reduced word for B_n will range from 0 to $n-1$. Secondly, our *unimodal sequence* will be a sequence of numbers that are initially strictly decreasing, then strictly increasing; that is

$$a = a_1 > a_2 > \cdots > a_k < a_{k+1} < \cdots < a_l$$

The *decreasing part* of a is defined to include the minimum, that is $a_1 > a_2 > \cdots > a_k$ and the *increasing part* is $a_{k+1} < a_{k+2} < \cdots < a_l$. We denote them by $a\downarrow$ and $a\uparrow$ respectively.

For example, 21056 is a unimodal sequence with decreasing part 210 and increasing part 56. 2489 is unimodal with decreasing part 2 and increasing part 489. Note that a unimodal sequence always has a decreasing part.

A *shifted Young diagram* of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $\lambda_i > \lambda_{i+1}$ for $1 \leq i < l$ is an arrangement of boxes such that the first row has length λ_1, the second row has length λ_2 and so on. However, each succeeding row is indented 1 box to the right. A *shifted Young tableau* P is a shifted Young diagram with the boxes filled in with numbers and we denote the ith row by P_i. Its reading word π_P is defined to be $P_l P_{l-1} \cdots P_2 P_1$ where P_i is treated as a sequence of numbers. For the rest of this text, we will sometimes treat a row of a tableau as a sequence of numbers. For example,

5	4	1	2
	4	2	3

is a shifted Young tableau of shape $(5,3)$ and reading word 4235412.

Definition 2.1 *Let P be a shifted Young tableau with l rows such that*

1. *$\pi_P = P_l P_{l-1} \cdots P_2 P_1$ is a reduced word of w*
2. *P_i is a unimodal subsequence of maximum length in $P_l P_{l-1} \cdots P_{i+1} P_i$*

Then, P is called a standard decomposition tableau *of w, and we denote the set of such tableaux by* $\mathrm{SDT}(w)$.

The previous example is a standard decomposition tableau of the permutation 351624.

Let $w \in B_n$ and $a = a_1 a_2 \cdots a_m \in R(w)$. The Kraśkiewicz insertion algorithm will give a map

$$a_1 a_2 \cdots a_m \xrightarrow{K} (P, Q)$$

where P is called the *insertion tableau* and Q is called the *recording tableau*. We will have to first construct a sequence of pairs of tableaux

$$(\emptyset, \emptyset) = (P^{(0)}, Q^{(0)}), (P^{(1)}, Q^{(1)}), \cdots, (P^{(m)}, Q^{(m)}) = (P, Q)$$

$\mathrm{sh}(P^{(i)}) = \mathrm{sh}(Q^{(i)})$ for $i = 0, 1, \cdots, m$. Each tableau $P^{(i)}$ is obtained by inserting a_i into $P^{(i-1)}$.

Insertion Algorithm:

Input: a_i and $(P^{(i-1)}, Q^{(i-1)})$. Output: $(P^{(i)}, Q^{(i)})$.
Step 1: Let $a = a_i$ and $R =$ 1st row of $P^{(i-1)}$.
Step 2: Insert a into R as follows:

- Case 0: $R = \emptyset$. If this empty row is the kth row, we write a indented $k-1$ boxes away from the left margin. This new tableau is $P^{(i)}$. To get $Q^{(i)}$, we add i to $Q^{(i-1)}$ so that $P^{(i)}$ and $Q^{(i)}$ have the same shape. Stop.

- Case 1: Ra is unimodal. Append a to R and let $P^{(i)}$ be this new tableau. To get $Q^{(i)}$, we add i to $Q^{(i-1)}$ so that $P^{(i)}$ and $Q^{(i)}$ have the same shape. Stop.
- Case 2: Ra is not unimodal. Some numbers in the increasing part of R are greater than a. Let b be the smallest number in $R\uparrow$ bigger than or equal to a.
  - Case 2.0: $a = 0$ and R contains 101 as a subsequence. We leave R unchanged and go to Step 2 with $a = 0$ and R equal to the next row.
  - Case 2.1.1: $b \neq a$. We put a in b's position and let $c = b$.
  - Case 2.1.2: $b = a$. We leave the increasing part $R\uparrow$ unchanged and let $c = a + 1$.

  We insert c into the deceasing part $R\downarrow$. Let d be the biggest number in $R\downarrow$ which is smaller than or equal to c. This number always exists because the minimum of a unimodal sequence is in its decreasing part.
  - Case 2.1.3: $d \neq c$. We put c in d's place and let $a' = d$.
  - Case 2.1.4: $d = c$. We leave $R\downarrow$ unchanged and let $a' = c - 1$.

Step 3: Repeat step 2 with $a = a'$ and R equal to the next row.

In the Kraśkiewicz insertion, there is an analogue of Knuth relations[13, Section 3.6] on the reduced words of $R(w)$ which we call the *B-Coxeter-Knuth relations.*

Definition 2.2 (B-Coxeter-Knuth relations) *Let $\mathbf{a}, \mathbf{b} \in R(w)$. We say they are B-Coxeter-Knuth related if they are in the same equivalence class generated by the following($a < b < c < d$):*

$$\begin{array}{rcll} 0101 & \sim & 1010 & \\ ab(b+1)b & \sim & a(b+1)b(b+1) & \\ ba(b+1)b & \sim & b(b+1)ab & \\ a(a+1)ba & \sim & a(a+1)ab & a+1<b \\ (a+1)ab(a+1) & \sim & (a+1)ba(a+1) & a+1<b \\ abdc & \sim & adbc & \\ acdb & \sim & acbd & \\ adcb & \sim & dacb & \\ badc & \sim & bdac & \end{array}$$

and their reverses. We denote this as $\mathbf{a} \sim \mathbf{b}$.

The reverse of a word, $\mathbf{a} = a_1a_2\cdots a_m$ is defined to be $\mathbf{a}^r = a_ma_{m-1}\cdots a_2a_1$. These relations are a refinement of the Coxeter relations for B_n. This set of relations also appeared in [6]. There, they were obtained by considering promotion sequences.

The properties of the Kraśkiewicz insertion parallel that of the Edelman-Greene insertion. We will state them below.

Theorem 2.3 ([9, Theorem 5.2]) *The Kraśkiewicz insertion is a bijection between $R(w)$ and pairs of tableaux (P, Q) where $P \in \mathrm{SDT}(w)$ and Q is a standard shifted Young tableau.*

Theorem 2.4 *Let $\mathbf{a}, \mathbf{b} \in R(w)$. They have the same insertion tableaux iff $\mathbf{a} \sim \mathbf{b}$.*

Theorem 2.5 ([9, Lemma 4.8]) *Let $\mathbf{a} \xrightarrow{K} (P, Q)$ and λ_1 be the length of the first row of P. Then the length of the longest unimodal subsequence in $\mathbf{a}$ is λ_1.*

3 Proof of Theorem 1.5

In order to proof Theorem 1.5, we need to know how a sequence of consecutive terms in $\boldsymbol{a}$ behaves under the Kraśkiewicz insertion.

Theorem 3.1 *Let $\boldsymbol{a} = a_1 a_2 \cdots a_m \in R(w)$ and $\boldsymbol{a} \xrightarrow{K} (P, Q)$ under Kraśkiewicz insertion. If the subsequence $a_j a_{j+1} \cdots a_l$ of $\boldsymbol{a}$ is unimodal, then in Q the boxes with the entries $j, j+1, \cdots, l$ form a rim hook. Moreover, the way the entries appear in the rim hooks is as follows:*

1. *the entries $j, j+1, \cdots, k$ form a vertical strip where k is the entry in the leftmost and lowest box of the rim hook*
2. *these entries are increasing down the vertical strip*
3. *the entries $k+1, \cdots, l-1, l$ form a horizontal strip*
4. *these entries are increasing from left to right*

Theorem 3.2 *Let $\boldsymbol{a} \xrightarrow{K} (P, Q)$. Let the boxes with entries $j, j+1, \cdots, l$ form a rim hook in Q satisfying the conditions listed in the previous theorem. Then the corresponding subsequence $a_j a_{j+1} \cdots a_l$ is a unimodal sequence.*

These two theorems can be proved by a tedious process of checking all the possible cases of the insertion.

Proof of Theorem 1.5: We will show that

$$2^{l_o(w)} G_w(\boldsymbol{x}) = \sum_{R \in \mathrm{SDT}(w)} Q_{\mathrm{sh}(R)}(\boldsymbol{x})$$

since $Q_\lambda = 2^{l(\lambda)} P_\lambda$. Fix $w \in B_n$ and let $m = l(w)$. To achieve this, we generalize the idea of $\boldsymbol{a}$-compatible sequence to include sequences with barred and unbarred numbers. Let $\boldsymbol{f} = (f_1, f_2, \cdots, f_m)$ be a sequence where $f_j \in \{\bar{1}, 1, \bar{2}, 2, \cdots\}$. This is called a *generalized* sequence. We give the barred and unbarred numbers the linear order

$$\bar{1} < 1 < \bar{2} < 2 < \cdots.$$

We say that $\boldsymbol{f}$ is *$\boldsymbol{a}$-compatible* if

1. $f_1 \leq f_2 \leq \cdots \leq f_m$
2. $f_j = f_{j+1} = \cdots = f_k = \bar{l}$ occurs only when $a_j > a_{j+1} > \cdots > a_k$
3. $f_j = f_{j+1} = \cdots = f_k = l$ occurs only when $a_j < a_{j+1} < \cdots < a_k$

Let $K'(\boldsymbol{a})$ be the set of all $\boldsymbol{a}$-compatible generalized sequences. In what follows, $\boldsymbol{i}$ will always be a sequence of unbarred numbers and $\boldsymbol{f}$ will denote a generalized sequence. Also, we will use $|f_j|$ to mean

$$|f_j| = l \quad \text{if } f_j = \bar{l} \text{ or } f_j = l.$$

and

$$|\boldsymbol{f}| = (|f_1|, |f_2|, \cdots, |f_m|)$$

It is not difficult to show that

$$2^{l_o(w)} G_w(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{f} \in K'(\boldsymbol{a})} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|}$$

Now, we will exhibit a bijection Φ from $\{(\boldsymbol{a}, \boldsymbol{f}) : \boldsymbol{a} \in R(w), \boldsymbol{f} \in K'(\boldsymbol{a})\}$ to $\{(P, T) : P \in \mathrm{SDT}(w), T$ a Q-semistandard Young tableau such that $\mathrm{sh}(T) = \mathrm{sh}(P)\}$.

Step 1: Apply Kraśkiewicz insertion to $\boldsymbol{a}$. Let $\boldsymbol{a} \xrightarrow{K} (P, Q)$. This P is the standard decomposition tableau we want.

Step 2: Take a Young diagram of the same shape as Q. We fill each box with $|f_j|$ when the corresponding box in Q has the entry j. Then the new tableau is weakly increasing along the columns and rows because

$$j < k \Rightarrow |f_j| \leq |f_k|$$

Step 3: For each constant subsequence, $f_j f_{j+1} \cdots f_k$ such that $|f_j| = |f_{j+1}| = \cdots = |f_k| = l$ with $|f_{j-1}| < |f_j|$ and $|f_k| < |f_{k+1}|$, we know that $a_j a_{j+1} \cdots a_k$ is unimodal. Let a_h be the smallest number in it. Furthermore, from Theorem 3.1 the entries $j, j+1, \cdots k$ form a rim hook in Q. Let g be the entry of the box in the lowest row and leftmost column among all the boxes in these rim hooks.

1. If f_h is unbarred, we add a bar to all the new entries from f_j to f_{g-1}.

2. If f_h is barred, we add a bar to all the new entries from f_j to f_g.

This will give us a Young tableau with barred and unbarred numbers. This will be our T. Clearly, $\mathrm{sh}(T) = \mathrm{sh}(Q) = \mathrm{sh}(P)$. It can be verified that T is a Q-semistandard Young tableau.

Now for the inverse map. Given (P, T), we first construct Q.

Step 1: Take a Young diagram of the same shape as T. We fill all the boxes with distinct numbers $1, 2, \cdots m$ as follows:

1. The entries in the Young diagram preserve the order of the entries in T.

2. For all the boxes in T with the same barred number, these form a vertical strip and we fill the corresponding boxes in an increasing order from top to bottom.

3. For all the boxes in T with the same unbarred number, these form a horizontal strip and we fill the corresponding boxes in an increasing order from left to right.

This will be our Q. It is clearly a standard shifted Young tableau with the same shape as P.

Step 2: $\boldsymbol{a}$ is obtained from (P, Q) by the inverse Kraśkiewicz insertion. This is the reduced word that we want.

Step 3: To get $\boldsymbol{f}$, remove all the bars in T and let $\boldsymbol{i} = i_1 i_2 \cdots i_m$ be the content of this new tableau laid out in weakly increasing order. Fix a number l. We know that in T all the boxes with the entry $\bar{l}$ or l form a rim hook in T. The corresponding boxes in Q are filled with consecutive numbers $j, j+1, \cdots, k$. Also, they satisfy the hypothesis in Theorem 3.2. Hence, $a_j a_{j+1} \cdots a_k$ is a unimodal sequence with smallest number a_h. Now, let (I, J) be the coordinates of the lowest and leftmost boxes in the rim hook which has entry $\bar{l}$ or l.

1. If (I, J) has the entry $\bar{l}$, then we add bars to $i_j, i_{j+1}, \cdots, i_h$.

2. If (I, J) has the entry l, we add bars to $i_j, i_{j+1}, \cdots, i_{h-1}$.

This generalized sequence will be our $\boldsymbol{f}$. By construction, it is a-compatible and $(\boldsymbol{a}, \boldsymbol{f}) = \Phi^{-1}(P, T)$. Hence, if we look at the associated monomials for generalized sequence and the Q-semistandard Young tableau, we see that they have to be equal. This gives

$$\begin{aligned} 2^{l_0(w)} G_w(\boldsymbol{x}) &= \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{f} \in K'(\boldsymbol{a})} x_{|f_1|} x_{|f_2|} \cdots x_{|f_m|} \\ &= \sum_{P \in \mathrm{SDT}(w)} \sum_{\mathrm{sh}(T) = \mathrm{sh}(P)} \boldsymbol{x}^T \\ &= \sum_{P \in \mathrm{SDT}(w)} Q_{\mathrm{sh}(P)}(\boldsymbol{x}) \end{aligned}$$

Therefore

$$G_w(\boldsymbol{x}) = \sum_{R \in \mathrm{SDT}(w)} 2^{l(R) - l_0(w)} P_{\mathrm{sh}(R)}(\boldsymbol{x})$$

Note that $l(R) > l_0(w)$ since every row of R can have at most one 0. This shows that G_w is a nonnegative integer linear combination of Schur P-functions. □

Example: SDT($\bar{3}21$) contains two standard decomposition tableaux,

2	1	0	2

2	1	0
	1	

Therefore,

$$G_{\bar{3}21}(x) = P_{(4)}(x) + 2P_{(3,1)}(x)$$

4 Properties

In this section, we will describe some properties of Kraśkiewicz insertion and how they relate to G_w.

1. Using Theorem 2.5, we are able to prove a result conjectured by Stembridge([15]).

   Theorem 4.1 *Let $w = w_1w_2\cdots w_{n-1}\bar{n}$ and $v = w_1w_2\cdots w_{n-1}n$. Then there is a bijection between* SDT(w) *and* SDT(v) *given by removing the top row of any $P \in$ SDT(w). Therefore,*

   $$G_w(x) = \sum_{R\in \mathrm{SDT}(v)} 2^{l(R)-l_0(v)} P_{(2n-1,\mathrm{sh}(R))}(x)$$

   where $(2n-1, \lambda) = (2n-1, \lambda_1, \lambda_2, \cdots)$.

   In particular, $G_{w_B}(x) = P_{(2n-1,2n-3,\cdots,3,1)}(x)$ where $w_B = \bar{1}\bar{2}\cdots\bar{n}$ is the signed permutation of longest length in B_n.

2. Focusing on the insertion tableau of the Kraśkiewicz insertion, we managed to show:

   Theorem 4.2 *Let P be a standard decomposition tableau and let $P\Downarrow$ be the tableau that is obtained when we delete the increasing parts of each row of P. Then, $P\Downarrow$ is a shifted tableau which is strictly decreasing in each row and in each top-left to bottom-right diagonal.*

3. Using the above, we are able to get an analogous result on the length of the longest decreasing subsequence in $\boldsymbol{a}$.

   Theorem 4.3 *Let $\boldsymbol{a} \xrightarrow{K} (P, Q)$. Then the longest strictly decreasing subsequence in $\boldsymbol{a}$ has length equal to the length of the decreasing part of the first row.*

   Theorem 4.4 *Let $w = \bar{n}w_2w_3\cdots w_n$ and $v = w_2w_3\cdots w_nn$ be 2 elements of B_n. Also, let* $\mathrm{SDT}_n(w) = \{P \in \mathrm{SDT}(w) : |P_1| = n\}$. *Then, there exists an injection Φ from* $\mathrm{SDT}_n(w)$ *into* SDT(v). *Furthermore, if $\forall \boldsymbol{a} \in R(v)$, the length of the longest unimodal subsequence in $\boldsymbol{a}$ is strictly less than n, then Φ is a bijection.*

   Using this, we are able to prove a result of Billey and Haiman that claims that every Schur P-function can be expressed as a B_n Stanley symmetric function. Here, $\hat{k}$ means omitting k.

   Corollary 4.5 ([1, Proposition 3.14])
   Let $w = \bar{\lambda}_1\bar{\lambda}_2\cdots\bar{\lambda}_l 123\cdots\widehat{\lambda_l}\cdots\widehat{\lambda_{l-1}}\cdots\cdots\widehat{\lambda_1}\cdots$ where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is a shifted shape. Then,

   $$G_w(x) = P_\lambda(x)$$

4. We are also to give a combinatorial proof of the following:

Theorem 4.6 **([4, Corollary 8.1],[15])** *Let $w \in S_n$, then*

$$G_w(x) = F_w(x/x)$$

where $F_w(x)$ is the S_n Stanley symmetric function of w described in [5] and the right-handside is the superfication of that function.

This shows a connection between the Edelman-Greene insertion and the Kraśkiewicz insertion. Another connection is given by:

Theorem 4.7 *Let $a \in S_n$. Suppose*

$$a \xrightarrow{K} (P, Q)$$

and let

$$a \xrightarrow{EG} (R, S)$$

denote the Edelman-Greene insertion of a. Then S is shifted jeu de taquin equivalent to Q.

For more information on the Edelman-Greene insertion, please refer to [3]. The shifted jeu de taquin operation is a shifted analogue of jeu de taquin. A description can be found in [16], [12] and [6].

5. Another interesting result is that the Edelman-Greene insertion manifests itself as a special case of the Kraśkiewicz insertion. Let U be the unique standard decomposition tableau of $\bar{n} \cdots \bar{2}\bar{1}$. It has shape $(n, n-1, \cdots, 2, 1)$ and the ith row is $U_i = n-i \; n-i-1 \cdots 1 \; 0$.

Theorem 4.8 *Let $a \in S_n$. Then the Kraśkiewicz insertion of a into U is the same as the Edelman-Greene insertion of a in the following sense:*
If we ignore U, then

(a) the recording tableau of the Kraśkiewicz insertion is the same as the recording tableau of the Edelman-Greene insertion, and

(b) the insertion tableau of the Kraśkiewicz insertion is just the insertion tableau of the Edelman-Greene insertion with $i-1$ subtracted from each box in the ith row.

Using this observation, we manage to prove a conjecture of Stembridge[15]. Let $\mathrm{SDT}_S(w)$ denote the set of Edelman-Greene insertion tableaux for the reduced words of w(see [3]) and let $w_S = n \cdots 21$ be the permutation of longest length in S_n.

Theorem 4.9 *Let $w \in S_n$. Denote $\bar{w}$ as the element of B_n obtained from w by putting a bar over all w_i. There exists a bijection Φ from $\mathrm{SDT}_S(w_S w)$ to $\mathrm{SDT}(\bar{w})$. Furthermore,*

$$G_{\bar{w}}(x) = \sum_{\tilde{P} \in \mathrm{SDT}_S(w_S w)} P_{\delta_n + \mathrm{sh}(\tilde{P})}(x)$$

where $\delta_n = (n, n-1, \cdots, 2, 1)$.

6. Most of the previous results come from properties of the insertion tableau of the Kraśkiewicz insertion. The next few come from properties of the recording tableau.

Theorem 4.10 *Let $a_1 a_2 \cdots a_m \xrightarrow{K} (P, Q)$ then*

$$a_2 \cdots a_m \xrightarrow{K} (R, \Delta(Q)) \qquad (1)$$

$$a_m \cdots a_2 a_1 \xrightarrow{K} (S, \mathrm{ev}(Q)) \qquad (2)$$

Using the language in [7], $\Delta(Q)$ is the standard shifted Young tableau that is obtained by subtracting 1 from every box in $Q(1 \to \infty)$. A description of evacuation operation $\text{ev}(Q)$ can be found in [7, Definition 8.1].

7. In [6], Haiman described the *short promotion sequence* which gives a bijection from the set of standard shifted Young tableau of shape $(2n-1, 2n-3, \cdots, 3, 1)$ to $R(w_B)$, the set of reduced words of the longest element in B_n. Using (1), we are able to show:

Theorem 4.11 *Let $a \in R(w_B)$ and $a \xrightarrow{K} (P, Q)$. Define $\Psi(a) = Q$. Then Ψ is the inverse of $\dot{p}$.*

8. We also attempted to answer the question as to when $\text{SDT}(w)$ contains only one element. This corresponds to finding conditions as to when $G_w(x) = 2^{l(\lambda)-l_0(w)} P_\lambda(x)$ where λ is the shape of the single standard decomposition tableau of w. We were able to get the following:

Theorem 4.12

If $w \in S_n$,

(a) $\text{SDT}(w)$ contains only 1 element iff w is obtained by taking 2 consecutive segments in $12 \cdots n$ and switching them.

(b) $\text{SDT}(\bar{w})$ contains only 1 element iff w is 3412-avoiding

If $w \in B_n$ is such that $w_i = i$ or $-i$, then $\text{SDT}(w)$ contains 1 element.

There are a number of open problems in this area that we would like to explore.

1. Is there a theory of shifted balanced tableaux?
2. Is there an analogue of the jeu de taquin here?
3. What are the conditions for $2^{l_0(w)} G_w = Q_{\lambda/\mu}$?
4. Extend this theory to D_n.

We will end this abstract by remarking that the last problem is being worked on and there are some preliminary results.

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