Tableau algorithms defined naturally for pictures

Marc A. A. van Leeuwen
CWI
Kruislaan 413
1098 SJ Amsterdam
The Netherlands
email: M.van.Leeuwen@cwi.nl

Summary

Pictures are geometric realisations of the combinatorial objects counted in the Littlewood-Richardson rule; they can be seen as a generalisation of various kinds of (skew) Young tableaux, and also of permutations. It was shown by Zelevinsky that the Robinson-Schensted correspondence can be generalised to operate on (pairs of) pictures, while retaining its bijective nature; Fomin and Greene have shown that the generalised correspondence is independent of the particular choice of ordering squares of a diagram ("reading") that is used in its definition. We extend these results further in several ways. It is shown that to a large extent the choice of a "reading" can be avoided altogether in defining the correspondence, replacing a total ordering on $\mathbf{Z} \times \mathbf{Z}$ by a more natural partial ordering; thus the naturality result of Fomin and Greene is obtained in a direct way. Furthermore it is shown that the well known connection between the Robinson-Schensted algorithm and Schützenberger algorithm implies that the latter also has a natural generalisation to pictures, but in the latter case the naturality of the correspondence defined is not shared by the defining procedure. That correspondence can be defined in another way, however, namely in terms of an operation called glissement (jeu de taquin). This leads to our main result: glissement can be generalised to act on pictures, and the definition is natural in that it requires no choice of a total ordering. Moreover, due to the symmetry of the picture concept, generalised glissement can be applied in two ways (at both sides of a picture), unlike ordinary glissement. It is well known that one of the tableaux computed by the Robinson-Schensted algorithm can alternatively be found by means of glissement; in the generalised version both tableaux can. Finally it can be shown that both forms of glissement commute with each other.

Résumé

On appelle dessins les réalisations géometriques des objets combinatoires énumérés par la règle de Littlewood et Richardson; on peut les percevoir comme généralisant plusieurs types de tableaux (gauches) de Young, ainsi que les permutations. Zelevinsky a montré que la correspondence de Robinson et Schensted peut être étendue à des (couples de) dessins, tout en restant bijective; Fomin et Greene ont montré que cette généralisation de la correspondence est naturelle au sens qu'elle ne dépend pas du choix de l'ordre des carrés d'un diagramme ("lecture") qui est utilisé dans sa définition. On poursuit le dévelopement de ces résultats de plusieurs manières. On montre que le choix d'une "lecture" peut être en grande mesure évité si au lieu d'un ordre total sur $\mathbf{Z} \times \mathbf{Z}$ on considère un ordre partiel plus naturel; par conséquent on obtient une preuve directe du résultat de Fomin et Greene. De plus on montre que la relation déjà connue entre l'algorithme de Robinson et Schensted et l'algorithme de Schützenberger entraine une généralisation naturelle de l'algorithme de Schützenberger au cadre des dessins; mais la définition de la correspondence ne jouit pas de la même propriété naturelle que dans le cas de Robinson-Schensted. Toutefois, cette correspondence peut être définie aussi en termes d'une opération appellée glissement (jeu de taquin). On arrive ainsi au résultat principal de ce papier: l'opération de glissement peut être étendue aux dessins et sa définition est naturelle au sens qu'elle n'exige pas de choix d'un ordre total. De plus, grâce à la symétrie de la notion de dessin, l'opération de glissement généralisé peut s'appliquer sous deux formes (des deux côtés d'un dessin), ce qui n'est pas le cas pour le glissement habituel. On connait déjà le fait qu'un des tableaux calculés par l'algorithme de Robinson et Schensted peut être construit également en utilisant l'opération de glissement; la version généralisée permet la construction des deux tableaux. On conclut en remarquant que les deux formes d'application du glissement généralisé commutent.

§1. Introduction.

A picture between skew diagrams is a bijection of their squares satisfying certain conditions. Although the definition appears to be artificial, the concept is an important one in the theory of the symmetric group, and it provides a unifying generalisation of concepts like Young tableaux of various kinds, skew tableaux, permutations, and Littlewood-Richardson fillings. The importance of the general notion of picture was indicated in [Zel], where it was shown that the number of pictures between any pair of skew diagrams equals the intertwining number of the corresponding representations of the symmetric group. This result contains the Littlewood-Richardson rule as a special case; it is proved by reducing to that case by means of an algorithm that generalises the Robinson-Schensted algorithm from permutations to (arbitrary) pictures. The definition of pictures used in [Zel] is more symmetric than the original one in [JP], yet its use of a particular total ordering '$\leq_J$' on $\mathbf{N} \times \mathbf{N}$ (which also appears in the traditional formulation of the Littlewood-Richardson rule) seems to be somewhat artificial. This total ordering is used to view general pictures just as permutations, and it is in this way that the generalised Robinson-Schensted correspondence is obtained; this slightly obscures the significance of the pictures themselves. In [FG] it is shown that this total ordering can be replaced in the definition of pictures by a partial ordering '$\leq_{\swarrow}$', which is just the natural partial ordering rotated a quarter turn. Moreover, it is shown that in the definition of the generalised Robinson-Schensted correspondence the use of '$\leq_J$' is not essential: the same result will be obtained if instead of this ordering one uses any other total ordering extending '$\leq_{\swarrow}$', showing that the correspondence is more "natural" than it would otherwise appear to be.

In this paper we take this approach a step further. We show that the insertion procedure of generalised Robinson-Schensted algorithm (and its inverse) can be defined directly in terms of pictures (i.e., without reducing them to permutations first), using the partial ordering '$\leq_{\swarrow}$'. This leads to a simpler and more direct proof of the naturality result in [FG]. The connection with the classical Robinson-Schensted correspondence is still preserved via the choice of a total ordering. Thus we can employ known properties of the Robinson-Schensted correspondence, and its relation to the Schützenberger algorithm, an involutory operation on Young tableaux defined by "deflating" the original tableau by repeated use of a specific procedure while building up the resulting tableau (see [Schü1], [vLee]). This allows us to deduce that that operation can also be extended to pictures, and is natural in the same sense as the Robinson-Schensted correspondence is. The generalised Schützenberger correspondence, in combination with the Robinson-Schensted correspondence, enables the definition of a (non-obvious) bijection between the sets of pictures with given domain and image and those with the transposed domain and image; these sets were already known to have the same cardinality by the transposition symmetry of the character theory of the symmetric group.

Contrary to the insertion procedure of the Robinson-Schensted algorithm, it is easily seen that the deflation procedure of the Schützenberger algorithm cannot be defined directly in terms of the partial ordering '$\leq_{\swarrow}$' only. On the other hand, Schützenberger has shown in [Schü2] that this involution can equivalently be obtained by a different, non-deterministic construction, called *glissement* (also termed *jeu de taquin*). We show that this construction *can* be directly generalised to operate on pictures instead of skew tableaux, using only '$\leq_{\swarrow}$'. In fact it can be applied equally well on both sides of a picture (domain and image), and thus becomes a more symmetric and powerful operation than it already was. The existence of this procedure not only proves the naturality of the Schützenberger correspondence for pictures in a direct way, it can also be used as an alternative definition of the Robinson-Schensted correspond for pictures (by a direct extension of the method of [Schü2]). Here the two "tableaux" associated to a picture by the Robinson-Schensted correspond arise quite naturally from the two ways of applying glissement. Finally it is shown that these two forms of glissement for pictures commute with each other; as a consequence the bijection that transposes domain and image of a picture commutes with all the operations given here, despite its complicated definition.

§2. Pictures.

Define two partial orderings '$\leq_{\nwarrow}$' and '$\leq_{\swarrow}$' on $\mathbf{Z} \times \mathbf{Z}$ by

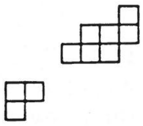
$$(i,j) \leq_{\nwarrow} (i',j') \iff i \leq i' \wedge j \leq j'$$

(the natural ordering), and

$$(i,j) \leq_{\swarrow} (i',j') \iff i \geq i' \wedge j \leq j'$$

(a transverse ordering). A skew diagram χ is a finite subset of $\mathbf{Z} \times \mathbf{Z}$ that is convex with respect to the natural ordering, i.e., if $x, z \in \chi$ and $x <_{\nwarrow} y <_{\nwarrow} z$ then $y \in \chi$; denote the set of all skew diagrams by $\mathcal{S}$. The elements of $\mathbf{Z} \times \mathbf{Z}$ are usually depicted as squares, and we shall let the first coordinate increase downwards and the second increase to the right, like matrix indices; with this convention the arrows attached to the '$\leq$'

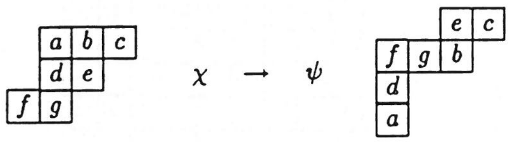
signs point towards of the smaller elements (like the '<' sign itself). Here is a typical skew diagram:



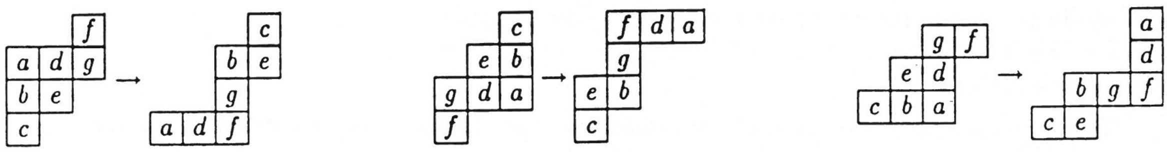
Let $\mathcal{P}$ denote its subset of Young diagrams, i.e., of finite order ideals for '$\leq_{\searrow}$' in $\mathbf{N}\times\mathbf{N}$ (these correspond bijectively to partitions); the non-empty Young diagrams are just the skew diagrams that, viewed as poset by the natural ordering, contain the origin as their minimal element (upper left corner). For each $\mu,\nu\in\mathcal{P}$ with $\mu\subseteq\nu$ the set-theoretic difference $\nu\setminus\mu$ is a skew diagram, and if a skew diagram is contained in $\mathbf{N}\times\mathbf{N}$ it can always be written in this form, not necessarily uniquely.

2.1. Definition. *Let $\chi,\psi\in\mathcal{S}$ and $f:\chi\to\psi$ a bijection; f is called a picture if it defines a morphism of partially ordered sets $(\chi,\leq_{\searrow})\to(\psi,\leq_{\swarrow})$ and if at the same time f^{-1} defines a morphism $(\psi,\leq_{\searrow})\to(\chi,\leq_{\swarrow})$.*

Note that, despite the use of the term morphism, a composition of two pictures is not a picture, because of the different orderings used; because the ordering on each of χ and ψ switches between the requirements for f and f^{-1}, we cannot speak of pictures as isomorphisms in any sense either. One way of displaying pictures is to draw its domain and image skew diagrams, and to label each point and its image with a unique letter. Here is an example of a picture displayed in this way.

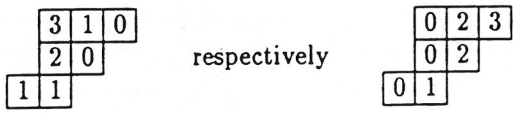


Let $\mathrm{Pic}(\chi,\psi)$ denote the set of all pictures from χ to ψ. It is immediately clear from the definition that if f is a picture, then so is f^{-1}, which provides a natural bijection between $\mathrm{Pic}(\chi,\psi)$ and $\mathrm{Pic}(\psi,\chi)$. Applying a translation to χ or ψ can obviously be matched by corresponding changes to pictures from χ to ψ, so we may consider skew diagrams up to translations when pictures are concerned (in fact the picture displayed already gives no indication of its absolute position). The set $\mathcal{S}$ is closed under the operations of transposition (given by $(i,j)\mapsto(i,j)^t=(j,i)$) and central symmetry (given by $(i,j)\mapsto-(i,j)=(-i,-j)$). By composition with these reflections one can easily show that $\mathrm{Pic}(\chi,\psi)$ is naturally in bijection with $\mathrm{Pic}(\chi^t,-\psi^t)$, $\mathrm{Pic}(-\chi^t,\psi^t)$, and $\mathrm{Pic}(-\chi,-\psi)$. Here are the results of applying these symmetries to the picture displayed above.



There is no equally obvious bijection between $\mathrm{Pic}(\chi,\psi)$ and $\mathrm{Pic}(\chi^t,\psi^t)$, but nevertheless a bijection between these sets will be constructed below. These operations generate all symmetries of the set of pictures one could hope for, since already with $|\chi|=|\psi|=2$ one finds examples for which $|\mathrm{Pic}(\chi,\psi)|\neq|\mathrm{Pic}(\chi,\psi^t)|$.

There are other ways of representing pictures than shown above. The row encoding (respectively column encoding) of a picture $f:\chi\to\psi$ is the skew diagram χ filled with numbers, by filling each square $s\in\chi$ with the row (respectively column) coordinate of its image $f(s)$. For instance, for the picture show above these are



assuming that ψ lies in $\mathbf{N}\times\mathbf{N}$, as close to the origin as possible. The set of squares with one same entry in the row encoding of f constitute the image under f^{-1} of a row of ψ, and since that row is totally ordered by '$\leq_{\searrow}$' the morphism property of f^{-1} determines the way in which it is mapped to its image uniquely; the same holds for the column encoding. Therefore, if the image diagram ψ is known, either the row or the column encoding fixes a picture completely. The morphism property of f implies that in any row encoding the rows are weakly decreasing and the columns strictly decreasing, while in any column encoding rows are strictly increasing and columns weakly increasing; the definition of pictures poses some less obvious conditions in addition to these.

However, by selecting particular skew diagrams for χ and ψ these conditions simplify, and in this way we can get various kind of combinatorial objects as special cases of pictures. For instance, if ψ is an anti-chain for '$\leq_\searrow$' (i.e., no two distinct squares are comparable), then the morphism condition for f^{-1} is trivially satisfied, and similarly for χ and f. Hence if both χ and ψ are anti-chains for '$\leq_\searrow$', then pictures are just arbitrary bijections, or via column encoding, permutations. If only ψ is an anti-chain we similarly get the notion of a skew tableau, and if moreover χ is a Young diagram, of a (standard) Young tableau. If we interchange χ and ψ, then a Young diagram will be represented by the anti-chain χ filled with numbers such that, when read from bottom left to top right, they form a "lattice permutation" or *mot de Yamanouchi*. If we relax the condition of being an anti-chain to having at most one square in each row, then we obtain generalised Young tableaux in the sense of [Kn1], and the generalised permutations of that paper are obtained by taking such skew diagrams at both ends of the picture.

The picture condition can be made more explicit by making a table of allowed relative positions of images. To an ordered pair of distinct points in $\mathbf{Z} \times \mathbf{Z}$ we associate one of eight possible relative positions, by determining for both their coordinates whether that of the first point is less than, equal to, or greater than that of the second; these positions can be indicated by the eight compass directions. The following table expresses the allowed combinations of the relative position of a pair of points and of its image under a picture.

f	→	↘	↓	↙	←	↖	↑	↗
→	•							•
↘								•
↓							•	•
↙				•	•	•	•	•
←				•	•			
↖				•				
↑			•	•				
↗	•	•	•	•				•

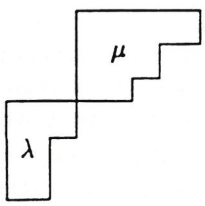
The following proposition, which is essentially equivalent to Lemma 3.4 of [FG], states that pictures can be characterised by a condition that is at first glance weaker than the definition we gave.

2.2. Proposition. *Let $f:\chi \to \psi$ be a bijection between two skew diagrams, and assume that for all pairs $x, y \in \chi$ the following two conditions hold:*
(i) we do not simultaneously have $x <_\searrow y$ and $f(y) <_\swarrow f(x)$,
(ii) we do not simultaneously have $f(x) <_\searrow f(y)$ and $y <_\swarrow x$.
Then f is a picture.

The main practical use of this proposition is that it allows us to replace the ordering '$\leq_\swarrow$' in the definition of pictures by any stronger ordering, for instance by a total one. In particular we may use an ordering '$\leq_r$' by rows, where the order among the rows is reversed:

$$(i,j) \leq_r (i',j') \iff i > i' \vee (i = i' \wedge j \leq j')$$

(this is the reverse ordering of '$\leq_J$' of [Zel]; we have chosen our transverse orderings opposite respect to those of [Zel] and [FG] since this works out better in connection with the Robinson-Schensted algorithm). When we replace '$\leq_\swarrow$' by '$\leq_r$' in the definition of pictures, and use column encoding of pictures, we arrive at a combinatorial concept that is very close to the fillings of (skew) diagrams occurring in the Littlewood-Richardson rule. In fact the coefficient $g^{\nu}_{\lambda,\mu}$ computed by the Littlewood-Richardson rule is $|\mathrm{Pic}(\nu \setminus \lambda, \mu)|$, which can be shown to be equal to $|\mathrm{Pic}(\lambda \uplus \mu, \nu)|$, where $\lambda \uplus \mu$ is the skew diagram built from λ and μ as follows:



This means that for pairs of skew diagrams of the form $\chi = \lambda \uplus \mu, \psi = \nu$, the number $|\mathrm{Pic}(\chi, \psi)|$ is equal to the inner product $\langle\{\chi\}, \{\psi\}\rangle$ of the corresponding symmetric group representations. It is shown in the next section (following [Zel]) that this remains true for arbitrary pairs of skew diagrams.

§3. The Robinson-Schensted algorithm for pictures.

The following result is due to Zelevinsky ([Zel], Theorem 2):

3.1. Theorem. *For all $\chi, \psi \in \mathcal{S}$ there is a natural bijection* $\mathrm{Pic}(\chi, \psi) \xrightarrow{\sim} \coprod_{\lambda \in \mathcal{P}} \mathrm{Pic}(\lambda, \psi) \times \mathrm{Pic}(\chi, \lambda)$.

In the context of [Zel], natural means that an explicit bijection is constructed by means of an algorithm. In fact, the bijection is obtained by using the (ordinary) Robinson-Schensted algorithm, essentially as follows. The following trivial generalisation of this algorithm allows it to operate on bijections $A \to B$ of two totally ordered sets of n elements, in place of permutations of n. The elements of A are taken in increasing order and their images in B figure in place of the numbers inserted into the tableau P by the well known insertion procedure of [Sche]; the elements of A themselves are used to fill the other tableau Q. This construction can then be applied to any bijection $\chi \to \psi$, where χ and ψ are totally ordered by '$\leq_r$', and yields a pair of bijections $\lambda \to \psi$ and $\lambda \to \chi$, for some Young diagram λ; to match the statement of the theorem we take the inverse of the second bijection (since the inverse of a picture is a picture, the order of the arguments of 'Pic' is irrelevant; we have chosen the order so as to preserve χ as domain and ψ as image). The essential point of the theorem is that the bijection $\chi \to \psi$ is a picture in this way if and only if the same is true for the bijections computed from it. The theorem immediately implies $|\mathrm{Pic}(\chi, \psi)| = \langle\{\chi\}, \{\psi\}\rangle$ for arbitrary $\chi, \psi \in \mathcal{S}$, since the set of irreducible representations $\{\, \{\lambda\} \mid \lambda \in \mathcal{P} \,\}$ forms an orthonormal base with respect to the inner product of representations.

In [FG] it is shown that the construction of the theorem is natural in a stronger sense, namely that the same result is obtained if other total orderings are used on χ and ψ, as long as each of them is compatible with '$\leq_{\swarrow}$' (they call this choosing 'readings' of χ and ψ). Their proof is rather technical, showing that one can transform any reading into a standard reading (the reading by rows) by small steps that correspond to the elementary transformations of [Kn1], where at each step the correspondence between pictures is unchanged. We shall indicate here a simpler and more direct proof, which at the same time establishes the theorem and proves the naturality of the construction in the sense of [FG]. We show that one does not need the total ordering '$\leq_r$' to define the insertion and deletion procedures used in the Robinson-Schensted algorithm for pictures, but that these can be specified directly in terms of the partial ordering '$\leq_{\swarrow}$'. This is based on the simple lemma below. To facilitate its formulation define an inner (resp. outer) cocorner of $\psi \in \mathcal{S}$ to be a square $s \in \mathbf{Z} \times \mathbf{Z}$ such that $s \notin \psi$ and $\psi \cup \{s\} \in \mathcal{S}$, which diagram has s as a minimal (resp. maximal) element (the terminology is adapted from [vLee]). Also, for $\lambda \in \mathcal{P}$ and $k \in N$ let $\lambda_{[k]}$ denote the row $\{\, (i,j) \in \lambda \mid i = k \,\}$ numbered k of λ and similarly put $\lambda_{[>k]} = \lambda_{[\geq k+1]} = \bigcup_{i>k} \lambda_{[i]}$.

3.2. Lemma. *Let $\lambda \in \mathcal{P}$, $\psi \in \mathcal{S}$, $p \in \mathrm{Pic}(\lambda, \psi)$, and let s be an outer cocorner of ψ. Then '$\leq_{\swarrow}$' induces a total ordering on $p[\lambda_{[0]}] \cup \{s\}$. If moreover s is not the maximum of this totally ordered set, then it successor $\min_{\leq_{\swarrow}} \{\, y \in p[\lambda_{[0]}] \mid s <_{\swarrow} y \,\}$ is an outer cocorner of $p[\lambda_{[>0]}]$.*

Now let λ, ψ, p, s be as in the lemma. We construct a sequence $x_0, \ldots, x_r$ for some $r \in \mathrm{N}$, with $x_i \in \lambda$ for $i < r$ and $x_r \in \mathrm{N} \times \mathrm{N}$ an outer cocorner of λ, and a corresponding sequence $s_0, \ldots, s_r$ with $s_0 = s$ and $s_i = p(x_{i-1}) \in \psi$ for $i > 0$. We shall have moreover that each s_i is an outer cocorner of $p[\lambda_{[\geq i]}]$. The terms of the sequences are determined successively; assume the we have constructed all x_i for $i < k$, and consequently all s_i for $i \leq k$, and that s_k is an outer cocorner of $p[\lambda_{[\geq k]}]$. Then by restricting to $\lambda_{[\geq k]}$ and applying the lemma we find that $p[\lambda_{[k]}] \cup \{s_k\}$ is totally ordered by '$\leq_{\swarrow}$'. If s_k is the maximum of this set we complete the construction by putting $r = k$ and defining x_r to be the first square in row r that lies outside λ (which is an outer cocorner of λ) i.e., $x_r = (r, \lambda_r)$. Otherwise the set $\{\, x \in \lambda_{[k]} \mid s_k <_{\swarrow} p(x) \,\}$ is non-empty, and x_k is defined to be its leftmost element. Then $s_{k+1} = p(x_k) = \min_{\leq_{\swarrow}} \{\, y \in p[\lambda_{[k]}] \mid s_k <_{\swarrow} y \,\}$ is an outer cocorner of $p[\lambda_{[>k]}]$ by the lemma, and we may proceed to the next step of the construction.

When the construction is complete we put $\lambda' = \lambda \cup \{x_r\} \in \mathcal{P}$, and define a bijection $p' \colon \lambda' \to \{s\} \cup \psi$ by $p'(x_i) = s_i$ for $0 \leq i \leq r$ and $p'(x) = p(x)$ for $x \in \lambda \setminus \{x_0, \ldots, x_{r-1}\}$. Since the set $\{s_{k+1}\} \cup p'[\lambda'_{[k]}]$ equals $p[\lambda_{[k]}] \cup \{s_k\}$ for all $k < r$, it is totally ordered by '$\leq_{\swarrow}$', and s_k is the predecessor in this ordering of s_{k+1}; therefore if p' and x_r are given, the full sequences can be reconstructed by an inverse procedure. These constructions are identical to (the transpose of) those in [Zel], except that we use a partial ordering

'$\leq_{\swarrow}$' instead of a total ordering such as '$\leq_{\mathrm{r}}$'. However, since we use this ordering only on subsets on which it induces a total ordering, it is immediately clear that the same correspondence will be defined if we replace '$\leq_{\swarrow}$' by any ordering that extends it. In particular the proof in [Zel] that p' is again a picture, and conversely that if a picture $p': \lambda' \to \psi'$ and a corner x_r of λ' are given then the inverse construction will produce a picture $p: \lambda \to \psi$ and an outer cocorner s_0 of ψ, apply directly to our construction. In fact it would be possible to simplify that proof slightly by using proposition 2.2.

If we number the elements of ψ by $1, \ldots, n$ according to some total ordering extending '$\leq_{\swarrow}$', then the procedure becomes just the insertion procedure of the Robinson-Schensted algorithm in the formulation of [Sche]. This suggests that we define the full analogue of the Robinson-Schensted algorithm (which constructs the bijection of theorem 3.1) as follows. Let a picture $f: \chi \to \psi$ be given, and put $n = |\chi| = |\psi|$. Order the elements of χ into a list $c_1, \ldots, c_n$ in such a way that $c_i <_{\swarrow} c_j$ only occurs if $i < j$. Then for $i = 1, 2, \ldots, n$, succesively compute pictures $p_i: \lambda^{(i)} \to f[\{c_1, \ldots, c_i\}]$ by starting with $\lambda^{(0)} = \emptyset$, and computing p_i from p_{i-1} for $i > 0$ by performing the construction above with $p = p_{i-1}$ (and therefore $\lambda = \lambda^{(i-1)}$, $\psi = f[\{c_1, \ldots, c_{i-1}\}]$) and $s = f(c_i)$; we define p_i to be the resulting picture p', and we also define d_i to be the last square x_r determined in the construction (so that $\lambda^{(i)} = \lambda^{(i-1)} \cup \{d_i\}$). Since $\{c_1, \ldots, c_i\}$ is an order ideal of χ with respect to '$\leq_{\swarrow}$' and f is a picture, $f(c_i)$ is indeed an outer cocorner of $f[\{c_1, \ldots, c_{i-1}\}]$, so that the construction is legitimate. When p_n is finally determined put $\lambda = \lambda^{(n)}$ and $p = p_n$ (which is clearly a picture $\lambda \to \psi$), and also define a bijection $q: \chi \to \lambda$ by $q(c_i) = d_i$. Since the construction mimics the ordinary Robinson-Schensted algorithm, it follows from a well known symmetry property of that construction (see for instance [vLee], Theorem 3.1) that if it is applied to $f^{-1}: \psi \to \chi$, it will compute the pair (q^{-1}, p^{-1}) instead of (p, q). From this it follows directly that q is a picture, and that the choice involved in ordering the list $c_1, \ldots, c_n$ does not affect the pair (p, q) computed. The invertibility of each step guarantees that the picture f can be reconstructed from (p, q), and hence that we have established a bijection as stated in the theorem above.

As an illustration we show here the last few steps of the algorithm applied to the picture that has been our working example (the first steps are hardly illustrative). Recall that we had labeled its domain as follows

$$\begin{array}{cccc} & a & b & c \\ & d & e & \\ f & g & & \end{array}$$

For the list $c_1, \ldots, c_7$ we choose the sequence labeled f, g, d, e, a, b, c (the other legitimate choice would be to interchange e and a). Then the pictures $p_4, \ldots, p_7$ are succesively

$$p_4 = \begin{array}{ccc} d & g & e \\ f & & \end{array} \to \begin{array}{ccc} & & e \\ f & g & \\ d & & \end{array}, \quad p_5 = \begin{array}{ccc} a & g & e \\ d & & \\ f & & \end{array} \to \begin{array}{ccc} & & e \\ f & g & \\ d & & \\ a & & \end{array}, \quad p_6 = \begin{array}{ccc} a & g & b \\ d & e & \\ f & & \end{array} \to \begin{array}{ccc} & & e \\ f & g & b \\ d & & \\ a & & \end{array}, \quad p_7 = p = \begin{array}{cccc} a & g & b & c \\ d & e & & \\ f & & & \end{array} \to \begin{array}{cccc} & & e & c \\ f & g & b & \\ d & & & \\ a & & & \end{array}.$$

For a proper understanding it should be pointed out that the symbols $a, \ldots, g$ do not denote mathematical values, but are just used to indicate matching squares within a single picture; however, for subdiagrams of χ and ψ (like the images of the p_i, which are subdiagrams of ψ) the same labels have been used as for displaying f. To illustrate a single insertion, consider the transition from p_5 to p_6. First the point labeled b is added to the image, and is compared with the images of the first row, together forming the chain $a <_{\swarrow} g <_{\swarrow} b <_{\swarrow} e$. It has a successor, e, which therefore moves on to the next step; in the chain of images $d <_{\swarrow} e$ it is maximal so that the insertion stops at this step, and the image labeled e becomes the image of a new square at the end of the second row that is added to the domain. The other picture computed is

$$q = \begin{array}{cccc} & a & b & c \\ & d & e & \\ f & g & & \end{array} \to \begin{array}{cccc} f & g & e & c \\ d & b & & \\ a & & & \end{array} \qquad \text{where the image diagram corresponds to} \qquad \begin{array}{cccc} 1 & 2 & 4 & 7 \\ 3 & 6 & & \\ 5 & & & \end{array}$$

displaying the order in which the images were determined. Note that the point-image pairs are determined one by one, but the intermediate stages do not always correspond to pictures: after p_5 is computed the pairs of q labeled a, d, e, f, g are determined, but the corresponding subset of the domain χ is not a skew diagram.

§4. The Schützenberger algorithm for pictures.

As was mentioned above, the set $\mathrm{Pic}(\chi, \psi)$ is in bijection with $\mathrm{Pic}(-\chi, -\psi)$, $\mathrm{Pic}(\chi^t, -\psi^t)$, and $\mathrm{Pic}(-\chi^t, \psi^t)$, by composing a picture with the indicated reflections in domain and image; we shall denote the counterparts of a picture f so obtained by $-f$, f^t, and $-f^t$. An obvious question is what happens to the pair of pictures computed by the Robinson-Schensted algorithm when we apply these symmetries to f; the answer cannot simply be that the same symmetry is applied to the pair, since shapes that should be Young diagrams are turned upside down. An answer can be given using an algorithmic operation related to the Robinson-Schensted algorithm: the Schützenberger algorithm, which defines a shape preserving transformation of Young tableaux; it was first defined in [Schü1], see also [Kn2] and [vLee]. It has similar properties to the Robinson-Schensted algorithm, and there is a strong connection between the two, due to Schützenberger, as expressed by [vLee], theorem 5.1; this theorem precisely describes, for the case of permutations (which we view as pictures whose domain and image are anti-chains for '$\leq_\swarrow$'), the effects of our symmetries on the tableaux computed by the Robinson-Schensted algorithm.

To handle the general case, we need a generalisation of the Schützenberger algorithm in the form of an operation S that bijectively maps $\mathrm{Pic}(\lambda,\psi)$ to $\mathrm{Pic}(\lambda,-\psi)$ whenever the domain λ is a Young diagram. It is derived from the ordinary Schützenberger algorithm by choosing a total ordering on ψ extending '$\leq_\nearrow$', and transferring it to $-\psi$ by symmetry; the former is used to view a picture $\lambda \to \psi$ as a Young tableau, and the latter to interpret the Young tableau computed from it as bijection $\lambda \to -\psi$. The reversal here is in fact quite natural, since the Schützenberger algorithm extracts entries from the original tableau in increasing order while computing the entries of the new tableau in decreasing order; therefore its picture counterpart removes a point with image $x \in \psi$ in the same step as it defines a point of the computed picture to have image $-x \in -\psi$.

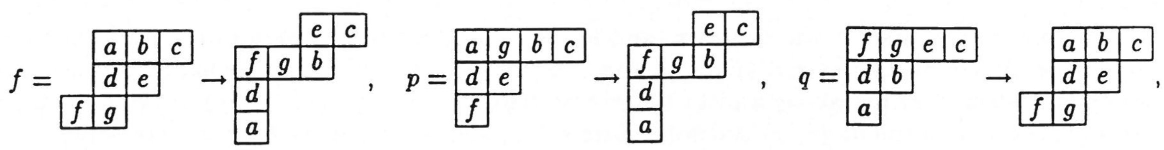
The following theorem gives the fundamental properties of the operation S. To facilitate its formulation, we use the proposition $RS(f,p,q)$ to express the fact that the Robinson-Schensted algorithm applied to the picture f yields the pictures p and q^{-1} (the inverse is taken to make q of the proper form for application of S).

4.1. Theorem. *For every picture $p\colon \lambda \to \psi$ with $\lambda \in \mathcal{P}$, the bijection $S(p)\colon \lambda \to -\psi$ is a picture, it value is independent of the choice of total ordering in the definition of S, and $S(S(p)) = p$. Furthermore, if f is any picture and p and q are such that $RS(f,p,q)$, then*

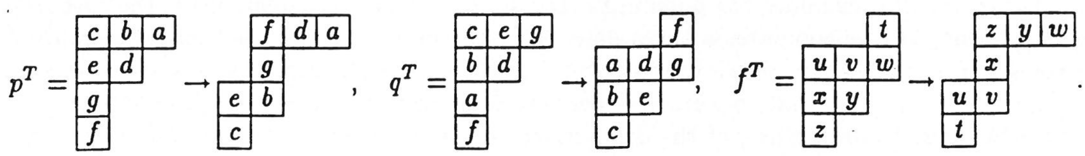
$$RS(-f, S(p), S(q)),$$
$$RS(f^t, p^t, S(q^t)), \text{ and}$$
$$RS(-f^t, S(p^t), q^t)$$

The proof is quite simple, given the mentioned theorem for the ordinary Schützenberger algorithm. By choosing fixed total orderings on χ and ψ extending '$\leq_\nearrow$', we view f as a permutation, to which we can apply the theorem. We know that the outcomes of the ordinary and generalised Robinson-Schensted algorithms are consistent, and if we choose the proper ordering on the image diagram for the computation of the various applications of S, then the theorem gives us the three relations stated in the theorem. Since the bijections computed by the generalised Robinson-Schensted algorithm are pictures, and are independent of the chosen total orderings, the same must be true for the generalised Schützenberger algorithm (this already follows from any one of the three relations). The fact that $S(S(p)) = p$ follows directly from the corresponding property of the ordinary Schützenberger algorithm.

This correspondence allows us to construct a bijection $\mathrm{Pic}(\chi,\psi) \to \mathrm{Pic}(\chi^t,\psi^t)$, which we shall write $f \mapsto f^T$. In the special case that χ is a Young diagram we put $f^T = S(f^t)$, and in the general case we define it by $RS(f,p,q) \Rightarrow RS(f^T, p^T, q^T)$ (it is fairly easy to see that these definitions do not conflict). As an illustration we take up our running example for which we had



and doing the required computations we get



Note that for f^T we switched to a different set of letters, as there is no obvious correspondence between its individual point-image pairs and those of f. In fact, there is no easily understood connection between f and f^T that does not involve the Robinson-Schensted algorithm. Nevertheless the operation seems to have some significance, in view of the fact that it can be easily shown to satisfy the identities $f^{TT} = f$, $(f^{-1})^T = (f^T)^{-1}$, $(-f)^T = -(f^T)$, $(f^t)^T = (f^T)^t$, and when applicable $S(f^T) = S(f)^T$.

Since we get independence of the choice of the total ordering almost for free, it is natural to ask whether the operation S could be defined without choosing a total ordering at all, like we showed for the Robinson-Schensted insertion procedure. The reason that that procedure could be adapted without problems is that, if we regard a picture $\lambda \to \psi$ as a Young tableau of shape λ with entries in a set that is only partially ordered, then no comparison is ever needed of entries that are incomparable by '$\leq_{\swarrow}$'. The hope that the same might be true for the basic deflation procedure of the Schützenberger algorithm is vain, however, for two reasons. This procedure starts with creating an empty square by removing the entry at the origin, and then repeatedly slides its neighbour to the right or below into the empty square, whichever is smaller; when it is finished the procedure is repeated for the entry that is now at the origin, etc. The first problem in doing this with pictures is that that the image of a point that is a minimum for '$\leq_{\nwarrow}$' is minimal for '$\leq_{\swarrow}$', not for '$\leq_{\nwarrow}$', so after removing it the image may no longer be a skew diagram. The other problem is that the entries are removed from the tableau strictly in increasing order, so apparently every pair of distinct entries is compared either explicitly or implicitly (by transitivity), in the process. We must therefore conclude that, although the choice of total ordering does not affect the outcome of S, it does greatly affect the sequence of steps by which this result is obtained. Without the connection with the Robinson-Schensted algorithm, it would also be hard to understand why S preserves the picture property. However, we shall show in the next section that a direct definition without choosing total orderings is still possible, provided we use an alternative construction by so-called *glissements*, which is given in [Schü2], section 2.

§5. Glissement.

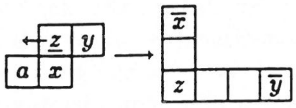
The notion of glissement is based on an operation very similar to the deflation procedure of the Schützenberger algorithm. Instead of operating on Young tableaux it acts on skew tableaux, and instead of removing the entry at the origin to create an empty square it starts with designating an inner cocorner of the skew tableau as initial position of the "empty square"; from then on however it is guided by exactly the same rule for moving entries into the empty square. In view of the negative observations of the previous section, the following therefore comes as a bit of a surprise.

5.1. Theorem. *The notion of glissement can be generalised in a natural way to pictures, in the following sense. With a picture $f:\chi \to \psi$ and a inner cocorner s of χ are associated a picture $f':\chi' \to \psi$, called the domain-glissement of f into the square s, with $\{s\} \cup \chi = \chi' \cup \{s'\}$ for some outer cocorner s' of χ'. Choosing any total ordering compatible with '$\leq_{\swarrow}$' on ψ we may consider f and f' as skew tableaux, and as such f' is the glissement of f into s, as defined in [Schü2].*

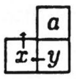
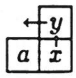
Clearly for any chosen total ordering the glissement f' of f is uniquely defined; we first show that f' does not depend on this choice, nor indeed does any step of its computation.

5.2. Lemma. *With f and f' as in the theorem, any pair of images $x, y \in \psi$ that are compared with each other during the computation of the glissement are already comparable by '$\leq_{\swarrow}$'.*

To prove this, assume the contrary, and let (i, j) be the first coordinates (i.e., minimal for '$\leq_{\nwarrow}$') for which the images of squares $x = (i+1, j)$ and $y = (i, j+1)$ of χ are being compared but are incomparable for '$\leq_{\swarrow}$'. Then $f(x)$ lies above and to the left of $f(y)$, i.e., $f(x) = (k, l)$, $f(y) = (k', l')$ with $k < k'$ and $l < l'$. Let z be the square (k', l) which lies below $f(x)$ and to the left of $f(y)$; we have $f(x) <_{\nwarrow} z <_{\nwarrow} f(y)$ and hence $z \in \psi$. Since f is a picture we have $x <_{\swarrow} f^{-1}(z) <_{\swarrow} y$ and $f^{-1}(z) \neq (i+1, j+1)$, so necessarily $f^{-1}(z) = (i, j)$. This excludes the possibility that $f(x)$ and $f(y)$ are compared at the first step of computing the glissement, so this comparison takes place after the entry z was moved into the empty square, leaving the square (i, j) empty. By possibly replacing f by f^t we may assume that the move was a horizontal one into $(i, j-1)$. Then at that move z was compared against the image $f(a)$ where $a = (i-1, j+1)$, and apparently found to be smaller; as the comparison was made before $f(x)$ and $f(y)$ were compared, we must in fact have $z <_{\swarrow} f(a)$. But this contradicts the fact that, because a lies directly to the left of x, $f(a)$ lies to the left of the column of $f(x)$, thus proving the lemma. The reasoning can be illustrated as follows, where for the sake of compactness images are indicated by overlining and inverse images by underlining; the empty squares indicate that there might, but need not, be intermediate squares.



To prove the theorem it suffices to verify that f' will always be a picture, which is done by establishing the conditions of proposition 2.2. The fact that for some (indeed, any) chosen total ordering on ψ f' is a skew tableau immediately gives us condition (i), so we only need to consider the possibility of a pair $x', y' \in \chi'$ that violates condition 2.2(ii) for f', i.e., for which $f'(x') <_\nwarrow f'(y')$ while $y' <_\nearrow x'$. Now consider the squares $x, y \in \chi$ for which $f(x) = f'(x')$ and $f(y) = f'(y')$; since f is a picture and $f(x) <_\nwarrow f(y)$ we have $x <_\nearrow y$. But from the definition of glissement x and x' can be at most one place apart, and similarly for y and y'; the only way that we can have $x <_\nearrow y$ and $y' <_\nearrow x'$ is when x and y are horizontally adjacent while x' and y' are vertically adjacent, or vice versa. In the former case the image of x must have been compared against that of the square a directly above y, and by the lemma we must have $f(x) <_\nearrow f(a)$, but this is in contradiction with the fact that $f(x) <_\nwarrow f(y)$ and $f(a)$ lies below the row of $f(y)$. The other case is handled similarly, completing the proof; the following illustrates the two cases (the arrows point from x to x' and from y to y').


respectively


The generalised version of glissement was named domain-glissement because the invertibility of pictures allows another operation to be derived from it: a picture f' is called the image-glissement of f into the square s if f'^{-1} is the domain-glissement of f^{-1} into the square s. As most of the theory of glissement in [Schü2] generalises directly to the picture case, this theory becomes more symmetric and acquires added significance by the possibility to apply glissement at both sides of a picture. For instance, it was shown in [Schü2] that for any permutation the first Young tableau associated to it by the Robinson-Schensted algorithm can be alternatively computed by viewing the permutation as a skew tableau whose shape is an anti-chain for $\leq_\nwarrow$' and repeatedly forming glissements (in any order) until the shape is a Young diagram; it now follows that for any picture $f: \chi \to \psi$ *both* pictures obtained from it by the generalised Robinson-Schensted algorithm can be similarly obtained from f by using forming domain-glissements respectively by forming image-glissements. One can easily check this for the picture for which demonstrated the Robinson-Schensted algorithm.

Using glissements of pictures, and results of [Schü2] on ordinary glissements, we can also give the following definition of S that does not require the choice of total orderings. Let $p: \lambda \to \psi$ be a picture with $\lambda \in \mathcal{P}$; take $-p: -\lambda \to -\psi$ (whose domain is a anti-Young diagram), and form domain-glissements to transform $-\lambda$ into (a translation of) a Young diagram; this Young diagram will always be λ and the picture $\lambda \to -\psi$ so obtained is $S(p)$. With these descriptions of the Robinson-Schensted and Schützenberger correspondences, theorem 4.1 becomes quite obvious.

The finite partial order called C_ϕ in [Schü2], which plays a crucial rôle there, also has a counterpart for pictures: its underlying set is the set of pairs $\{\,(x, f(x)) \mid x \in \chi\,\}$, with a partial ordering defined by $(x, f(x) \leq (y, f(y)) \iff x \leq_\nearrow y \wedge f(x) \leq_\nearrow f(y)$; the significance of this poset associated to f has been shown in [FG], sections 6 and 7. It appears that nearly all of Schützenberger's long paper can be generalised easily to the case of pictures, and some results (e.g., 4.7) even become simpler, since they seem to deal with pictures *avant la lettre*.

The two-sided nature of the theory of glissement raises a question about the interaction between domain-glissement and image-glissement, which could of course not be considered for ordinary glissement.

5.3. Theorem. *The operations of domain-glissement and image-glissement commute. i.e., for any picture $f: \chi \to \psi$ and inner cocorners s and t of χ and ψ respectively, the image-glissement into t of the domain-glissement into s of f equals the domain-glissement into s of the image-glissement into t of f.*

We omit here the slightly technical proof, but do mention that it is not always true that the individual steps in the computation of one glissement are unaffected by performing glissement at the opposite side; the equality holds only for the final pictures. In other words, the sequence of moves may follow another path, although the theorem does imply that the paths must necessarily end in the same square. Taking multiple glissements at the opposite side, we deduce that for any pair of pictures for which the second of the pictures (q) computed by the Robinson-Schensted algorithm are the same, and which therefore have the same domain, the domains stay equal after we form the glissement into the same square of both pictures; this is remarkable since the two pictures can differ considerably in other respects. Another consequence of the theorem is that the operation $f \mapsto f^T$ of the previous section commutes with both domain-glissement and image-glissement; again this is surprising in view of the complicated change produced by the operation.

§6. Concluding remarks.

The fact that the Robinson-Schensted algorithm can be generalised in a natural way to pictures, while retaining its remarkable properties, provides a great amount of new insight; e.g., versions of the algorithm with repeated entries should really be understood as special instances of the general picture case. In this light it is important that the Schützenberger algorithm, which is so intimately related to it, can also be generalised to pictures. The most significant fact seems to be that glissement can be generalised to pictures, in a two-sided way, especially since both other correspondences can be defined in terms of glissement.

A question remains what would be the best way to set up the theory if one does not assume the facts known for the non-picture versions of the algorithms. In the approach taken here, we used results proved for the ordinary Robinson-Schensted algorithm (especially its symmetry with respect to inverses) in order to establish that the generalised version is well defined and natural. This makes it difficult to do everything directly for pictures, and also the proofs known for the symmetry property (see for instance [Fom] or [vLee]) all seem to depend essentially on a total ordering of the steps and entries. The approach of [Schü2] that starts with glissement does not suffer from such difficulties. A drawback of this setup is however, that it requires a lot of work before one obtains any well defined correspondences to begin with, which includes reasoning about finite posets and adapting several results from the classical formulation of the theory in a more complicated form; in particular, the invertibility of the Robinson-Schenstedcorrespondence is much less obvious when formulated in terms of glissements. For the moment it seems best to mix the various approaches as convenient for each result.

References.

[Fom] S. V. Fomin, "Generalised Robinson-Schensted-Knuth correspondence", *J. of Soviet Mathematics* **41**, (1988), 979–991. (Translation from Записки научных семинаров ЛОМИ **155** (1986) 156–175; authorised translation available from the author).

[FG] S. Fomin and C. Greene, "A Littlewood-Richardson Miscellany", *Europ. J. Combin.* **14**, (1993), 191–212.

[Gre] C. Greene, "An extension of Schensted's Theorem", *Advances in Mathematics* **14**, (1974), 254–265.

[JP] G. D. James and M. H. Peel, "Specht series for skew representations of symmetric groups", *J. of Algebra* **56**, (1979), 343–364.

[Kn1] D. E. Knuth, "Permutations, matrices and generalized Young tableaux", *Pacific J. of Math.* **34**, (1970), 709–727.

[Kn2] D. E. Knuth, *The art of Computer programming, Vol. III Sorting and Searching*, pp. 48–72, Addison-Wesley, 1975.

[vLee] M. A. A. van Leeuwen, *The Robinson-Schensted and Schützenberger algorithms, Part I: New combinatorial proofs*, CWI report AM-R9208, (1992). (Electronically available in `ftp.cwi.nl:/pub/CWIreports/AM`).

[Roby] T. W. Roby, *Applications and extensions of Fomin's generalization of the Robinson-Schensted correspondence to differential posets*, Thesis (M. I. T.), (1991).

[Sche] C. Schensted, "Longest increasing and decreasing subsequences", *Canadian J. of Math.* **13**, (1961), 179–191.

[Schü1] M. P. Schützenberger, "Quelques remarques sur une construction de Schensted", *Math. Scandinavica* **12**, (1963), 117–128.

[Schü2] M. P. Schützenberger, "La correspondence de Robinson", pp. 59–113 in *Combinatoire et Représentation du Groupe Symétrique* (D. Foata, ed.), Lecture Notes in Mathematics 579, 1976.

[Whi] D. White, "Some connections between the Littlewood-Richardson rule and the construction of Schensted", *J. Combin. Theory, Ser. A* **30**, (1981), 237–247.

[Zel] A. V. Zelevinsky, "A Generalisation of the Littlewood-Richardson Rule and the Robinson-Schensted-Knuth Correspondence", *J. of Algebra* **69**, (1981), 82–94.