An identity related to the enumeration of crossings of partitions

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Abstract

We consider a way of counting the number $c_o(\pi)$ of crossings of a partition π of $\{1, \ldots, n\}$, which is motivated by the study of a certain q-analogue, called *R_q-transform*, of the logarithm of the Fourier transform for probability distributions with finite moments of all orders. $c_o(\cdot)$ appears in a description of the R_q-transform via a summation formula on the set-partitions (see equation (9) below). We discuss the relation between this and two other equivalent descriptions of the R_q-transform: (a) one in terms of weighted shifts, which makes clear that the case $q = 0$ has to do with the theory of *free convolution*, as developed by D. Voiculescu; (b) another via a matrix equation related to the method of Stieltjes for expanding continued J-fractions as power series; the latter description gives an interesting connection to the q-continuous Hermite orthogonal polynomials.

Résumé

Nous considérons un mode de compter le nombre $c_o(\pi)$ de croisements d'une partition π de $\{1, \ldots, n\}$, qui est motivé par l'étude d'une q-analogue, appelée *transformation R_q*, du logarithm de la transformation de Fourier d'une distribution de probabilité avec des moments finis. $c_o(\cdot)$ apparaît dans une description de la transformation R_q faite au moyen d'une formule de sommation d'après l'ensemble de partitions de $\{1, \ldots, n\}$ (cf. l'équation (9) dans le texte dessous). Nous discutons la relation entre cette formule et deux autres descriptions équivalentes de la transformation R_q: (a) une d'elles en termes des shifts pondérés, qui montre que le cas $q = 0$ est lié à la *convolution libre* de D. Voiculescu; (b) une autre via une équation matricielle apparentée à la méthode de Stieltjes pour développer des fractions continues de Jacobi. La dernière description montre une liaison intéressante avec le q-analogue continu de polynômes d'Hermite.

Extended abstract

1. The definition of $c_o(\pi)$ For $n \geq 1$, we denote by $\mathcal{P}(\{1,\ldots,n\})$ the set of partitions of $\{1,\ldots,n\}$. For $\pi \in \mathcal{P}(\{1,\ldots,n\})$ and $1 \leq m_1, m_2 \leq n$, we will write "$m_1 \stackrel{\pi}{\sim} m_2$" for the fact that m_1 and m_2 are in the same class (block) of π.

Recall that a partition π of $\{1,\ldots,n\}$ is said to be non-crossing (notion introduced in [5]) if there is no 4-tuple (m_1, m_2, m_3, m_4) such that $1 \leq m_1 < m_2 < m_3 < m_4 \leq n$, $m_1 \stackrel{\pi}{\sim} m_3 \stackrel{\pi}{\not\sim} m_2 \stackrel{\pi}{\sim} m_4$.

We will call *left-reduced number of crossings* of $\pi \in \mathcal{P}(\{1,\ldots,n\})$ the number:

$$c_o(\pi) = \text{card} \left\{ (m_1, m_2, m_3, m_4) \;\middle|\; \begin{array}{l} 1 \leq m_1 < m_2 < m_3 < m_4 \leq n,\ m_1 \stackrel{\pi}{\sim} m_3, \\ m_2 \stackrel{\pi}{\sim} m_4, \text{ each of } m_1, m_2 \text{ is minimal} \\ \text{in the class of } \pi \text{ containing it} \end{array} \right\}. \tag{1}$$

The words "left-reduced" in the name of $c_o(\pi)$ refer to the fact that rather than counting all the 4-tuples mentioned in the preceding paragraph, we impose a more restrictive condition "on the left" (the minimality requirement on m_1 and m_2 implies of course $m_1 \stackrel{\pi}{\not\sim} m_2$). It is easy to check, however, that a partition $\pi \in \mathcal{P}(\{1,\ldots,n\})$ is non-crossing if and only if it has $c_o(\pi) = 0$.

There are, of course, many other ways of "counting the crossings" of a partition of $\{1,\ldots,n\}$, which also have the property mentioned in the previous phrase; so the choice made in formula (1) needs some justification. Our motivation for considering it came from considerations on some q-analogues of the convolution of probability distributions with finite moments of all orders. The goal of the present abstract is to present some of these considerations, and how crossings of partitions are related to them.

Up to the present moment we did not find references to $c_o(\cdot)$ of formula (1), in the literature. It remains possible, however, that some relation exists with other examples of "statistics on partitions" which have been studied.

2. Convolution, free convolution, and their linearizing transforms We will work with normalized linear functionals on the algebra $\mathbf{C}\langle X \rangle$ of polynomials, and we will view these functionals as a simplified algebraic way of looking to probability distributions with finite moments of all orders. Thus we will speak for example about the *convolution product* of two such functionals μ_1, μ_2, which will be denoted here by $\mu_1 \cdot \mu_2$ (because the symbol "$\star$" is used for the free product), and has the formula:

$$(\mu_1 \cdot \mu_2)(f) = (\mu_1 \otimes \mu_2)(f(X_1 + X_2)), \quad f \in \mathbf{C}\langle X \rangle. \tag{2}$$

The tensor product $\mu_1 \otimes \mu_2$ is viewed in (2) as the linear functional on the algebra of polynomials in X_1 and X_2, which has $(\mu_1 \otimes \mu_2)(X_1^m X_2^n) = \mu_1(X_1^m)\mu_2(X_2^n)$, $m, n \geq 0$.

It is well-known that the convolution product is linearized by the logarithm of the Fourier transform, i.e.

$$log\ \mathcal{F}(\mu_1 \cdot \mu_2) \ = log\ \mathcal{F}(\mu_1) + log\ \mathcal{F}(\mu_2), \tag{3}$$

for any distributions μ_1, μ_2. In the framework we have considered, the Fourier transform of a functional μ is viewed as a formal power series, $(\mathcal{F}(\mu))(z) = \sum_{n=0}^{\infty} \frac{i^n \mu(X^n)}{n!} z^n$, (obtained by formally expanding as a sum "the integral $\int e^{itz} d\mu(t)$"); moreover, $log\ \mathcal{F}$ is taken as $\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1}(\mathcal{F}(\mu)-1)^n$, a formal power series vanishing at zero.

In the work of D. Voiculescu ([11],[12]), a theory of *free convolution* of distributions was developed; the free convolution of the functionals μ_1, μ_2 is denoted by $\mu_1 \boxplus \mu_2$ and acts by a formula similar to (2), but where the tensor product of μ_1 and μ_2 is replaced by their free product $\mu_1 \star \mu_2$:

$$(\mu_1 \boxplus \mu_2)(f) \ = \ (\mu_1 \star \mu_2)(f(X_1 + X_2)), \quad f \in \mathbf{C}\langle X \rangle. \tag{4}$$

For the definition of $\mu_1 \star \mu_2$, which is a linear functional on the algebra of *non-commuting* polynomials in X_1 and X_2, see for instance Section 1.5 of [13].

The analogue of $log\ \mathcal{F}$ of (3) for the free convolution is a certain *R-transform* constructed in [11], [12]; that is, for a given distribution μ one constructs its R-transform $R(\mu)$ which is a formal power series vanishing at zero; and the formula

$$R(\mu_1 \boxplus \mu_2) \ = R(\mu_1) + R(\mu_2) \tag{5}$$

holds for every μ_1 and μ_2.

The definition of the R-transform is in terms of certain infinite Toeplitz (i.e. constant along the diagonals) matrices. More precisely, let us denote by S and S^* the matrices of the unilateral shift and its adjoint with respect to the canonical basis of the Hilbert space $l^2(\mathbf{N})$ (the (i,j)-entry of S is 1 when $i = j+1$, and 0 otherwise; S^* is the transpose of S). Given a formal power series vanishing at zero, $\theta(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, we will denote by T_θ the Toeplitz matrix

$$\begin{aligned} T_\theta \ &= \ S^* + \alpha_1 I + \alpha_2 S + \alpha_3 S^2 + \cdots + \alpha_{n+1} S^n + \cdots \\ &= \begin{pmatrix} \alpha_1 & 1 & & & \\ \alpha_2 & \alpha_1 & 1 & & \\ \alpha_3 & \alpha_2 & \alpha_1 & 1 & \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \end{aligned} \tag{6}$$

Note that it makes sense to calculate powers of T_θ (and hence polynomials of T_θ, i.e. linear combinations of its powers).

With these notations, the definition of the R-transform goes as follows:

Definition 1 ([12], Section 2) Given $\mu : \mathbf{C}\langle X\rangle \to \mathbf{C}$ linear, with $\mu(1) = 1$, there exists a unique formal power series vanishing at zero, $\theta(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, such that $\mu(f) =$ the $(0,0)$-entry of $f(T_\theta)$ for every $f \in \mathbf{C}\langle X\rangle$ (T_θ defined as in (6)). This θ is denoted by $R(\mu)$, and called the R-transform of μ.

3. The R_q-transforms Let now $q \in [0,1]$ be a parameter, and denote by S_q the infinite matrix which has its (i,j)-entry equal to:

$$(S_q)_{i,j} = \begin{cases} \sqrt{[i]_q}, & \text{if } i = j+1 \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where $[i]_q$ is the q-number $1 + q + \cdots + q^{i-1}$, for $i \geq 1$. S_q is a q-deformation of the unilateral shift (corresponding to $q = 0$), which has appeared both on work on the quantum $SU(2)$ group (see for instance [15]), and in relation to the so-called q-deformation of the commutation relations (see for instance [2]). The adjoint of the matrix S_q of (7) will be denoted by S_q^*.

We define the R_q-transform by replacing S with S_q in the Definition 1 above. That is, for a given formal power series vanishing at zero, $\theta(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, we put

$$T_{\theta,q} = S_q^* + \alpha_1 I + \alpha_2 S_q + \alpha_3 S_q^2 + \cdots + \alpha_{n+1} S_q^n + \cdots \tag{8}$$

(the sum makes sense, because any two of the matrices which are summed are supported on different diagonals). Then we make the

Definition 2 Given $\mu : \mathbf{C}\langle X\rangle \to \mathbf{C}$ linear, with $\mu(1) = 1$, there exists a unique formal power series vanishing at zero, $\theta(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, such that $\mu(f) =$ the $(0,0)$-entry of $f(T_{\theta,q})$ for every $f \in \mathbf{C}\langle X\rangle$. This θ is denoted by $R_q(\mu)$ (and called the R_q-transform of μ).

The R_q-transforms so defined are in some sense interpolating (for q running in $[0,1]$) between the free and the classical situation. The point will be that the R_q-transform also has other descriptions (presented in Theorems 1,2 below) which correspond to some basic approaches used for $q = 0$ and/or $q = 1$, and which turn out to remain equivalent for arbitrary $q \in [0,1]$.

4. The relation between R_q-transforms and crossings of partitions comes from the following

Theorem 1 Let $q \in [0,1]$ be a parameter, let $\mu : \mathbf{C}\langle X\rangle \to \mathbf{C}$ be a linear functional such that $\mu(1) = 1$, and consider the formal power series $\theta = R_q(\mu)$, $\theta(z) = \sum_{n=1}^{\infty} \alpha_n z^n$. Then for every $n \geq 1$

we have

$$\mu(X^n) = \sum_{\substack{\pi\in\mathcal{P}(\{1,\ldots,n\}) \\ \pi=\{B_1,\ldots,B_k\}}} q^{c_o(\pi)} \prod_{j=1}^{k} [|B_j|-1]_q!\ \alpha_{|B_j|}. \tag{9}$$

($c_o(\pi)$ is as defined in Section 1, and the α's are the coefficients of θ. $|B_j|$ stands, as usual, for the number of elements of B_j, and we use the customary notation for q-factorials: $[0]_q! = 1$, $[l]_q! = [1]_q[2]_q\cdots[l]_q$ for $l \geq 1$.)

As it was clear from the Definition 2, The R_0-transform (obtained for $q = 0$) coincides with the R-transform of Voiculescu. We mention that the summation formula on non-crossing partitions which is obtained from (9) in this case had been observed by R. Speicher [9].

On the other hand it is not so clear from Definition 2 why the R_1-transform (obtained for $q = 1$) should be related to the logarithm of the Fourier transform. This is, however, easy to see from (9). Indeed, by putting $q = 1$ in (9) and using the fact that for a partition $n = k_1 + 2k_2 + \cdots + nk_n$ ($k_1, \ldots, k_n \geq 0$) of the number n there are $n!/[(1!)^{k_1}\cdots(n!)^{k_n}k_1!\cdots k_n!]$ partitions of $\{1,\ldots,n\}$ which have k_1 classes of 1 element, ..., k_n classes of n elements, we get:

$$\frac{\mu(X^n)}{n!} = \sum_{\substack{k_1,\ldots,k_n\geq 0 \\ k_1+2k_2+\cdots+nk_n=n}} \frac{(\frac{\alpha_1}{1})^{k_1}\cdots(\frac{\alpha_n}{n})^{k_n}}{k_1!\cdots k_n!}, \quad n \geq 1; \tag{10}$$

in (10), $\mu : \mathbf{C}\langle X\rangle \to \mathbf{C}$ is linear with $\mu(1) = 1$, and $(\alpha_n)_{n=1}^{\infty}$ are the coefficients of $R_1(\mu)$. But, as it is easily checked, (10) means that the series $\sum_{n=0}^{\infty} \frac{\mu(X^n)}{n!} z^n$ is the exponential of $\sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^n$, and this entails the formula

$$(R_1(\mu))(z) = -iz(\log \mathcal{F}(\mu))'(-iz). \tag{11}$$

Thus, the R_1-transform differs from the logarithm of the Fourier transform only by a linear automorphism of the space of formal power series vanishing at zero (and, in particular, it shares the property of *log* $\mathcal{F}$ of linearizing usual convolution). Relation (10) is in fact equivalent to the well-known formula connecting the moments of a distribution and its so-called cumulants (see [8], Section II.12.8 for details).

We mention that one of the ingredients entering the proof of Theorem 1 is the identity stated as follows. For $n \geq 1$, we denote by C_n the set of n-tuples

$$\{\underline{\epsilon} = (\epsilon_1,\ldots,\epsilon_n) \mid \epsilon_1,\ldots,\epsilon_n \in \mathbf{N}\cup\{-1\},\ \sum_{j=1}^{m}\epsilon_j \geq 0 \text{ for every } 1\leq m\leq n, \sum_{j=1}^{n}\epsilon_j = 0\}.$$

(One thinks of the elements of C_n as of paths in the square lattice, by identifying $(\epsilon_1,\ldots,\epsilon_n)$ with the sequence of points $(0,0)$, $(1,\epsilon_1)$, $(2,\epsilon_1+\epsilon_2)$, ..., $(n,\epsilon_1+\cdots+\epsilon_n) = (n,0)$.) We denote by ρ_n :

$\mathcal{P}(\{1,\ldots,n\}) \to C_n$ the map which associates to the partition $\pi = \{B_1,\ldots,B_k\} \in \mathcal{P}(\{1,\ldots,n\})$ the n-tuple $\underline{\epsilon} = (\epsilon_1,\ldots,\epsilon_n) \in C_n$ given by:

$$\epsilon_m = \begin{cases} |B_j| - 1, & \text{if } m = \min B_j \text{ for some (uniquely determined) } 1 \leq j \leq k, \\ -1, & \text{otherwise .} \end{cases} \tag{12}$$

Then for every $\underline{\epsilon} = (\epsilon_1,\ldots,\epsilon_n) \in C_n$ we have:

$$\sum_{\pi \in \rho_n^{-1}(\underline{\epsilon})} q^{c_o(\pi)} = \Big(\prod_{\substack{1 \leq m \leq n \\ \text{such that} \\ \epsilon_m = -1}} [\epsilon_1 + \ldots + \epsilon_{m-1}]_q \Big) \cdot \Big(\prod_{\substack{1 \leq m \leq n \\ \text{such that} \\ \epsilon_m \geq 0}} [\epsilon_m]_q! \Big)^{-1}. \tag{13}$$

The case $q = 0$ of (13), which asserts that the map ρ_n defined in (12) becomes a bijection when restricted to non-crossing partitions, is a well-known remark of Poupard ([7], Section 1.3).

5. The relation with the q-continuous Hermite polynomials Another characterization of the R_q-transform is provided by the following

Theorem 2 Let $q \in [0,1]$ be a parameter, let $\mu : \mathbf{C}\langle X\rangle \to \mathbf{C}$ be a linear functional such that $\mu(1) = 1$, and consider the formal power series $\theta = R_q(\mu)$, $\theta(z) = \sum_{n=1}^{\infty} \alpha_n z^n$. Then the matrix equation

$$\begin{pmatrix} \gamma_{1,0} & 1 & & & \\ \gamma_{2,0} & \gamma_{2,1} & 1 & & \\ \gamma_{3,0} & \gamma_{3,1} & \gamma_{3,2} & 1 & \\ \gamma_{4,0} & \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,3} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \gamma_{1,0} & 1 & & \\ \gamma_{2,0} & \gamma_{2,1} & 1 & \\ \gamma_{3,0} & \gamma_{3,1} & \gamma_{3,2} & 1 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \cdot \tag{14}$$

$$\cdot \begin{pmatrix} \alpha_1[0]_q!/[0]_q! & 1 & & & \\ \alpha_2[1]_q!/[0]_q! & \alpha_1[1]_q!/[1]_q! & 1 & & \\ \alpha_3[2]_q!/[0]_q! & \alpha_2[2]_q!/[1]_q! & \alpha_1[2]_q!/[2]_q! & 1 & \\ \alpha_4[3]_q!/[0]_q! & \alpha_3[3]_q!/[1]_q! & \alpha_2[3]_q!/[2]_q! & \alpha_1[3]_q!/[3]_q! & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(where $(\alpha_n)_{n=1}^{\infty}$ are the coefficients of θ, and $(\gamma_{i,j})_{i>j\geq 0}$ are unknowns) has a unique solution, and we have $\mu(X^n) = \gamma_{n,0}$ for every $n \geq 1$.

The equation (14) is very similar (especially if we also take into account the significance of the first column of Γ) to what one has when converting a continued J-fraction into a power series, after the method of Stieltjes (see [14], Section 53). The difference between (14) and the matrix equation of Stieltjes is that the third matrix in (14) may have non-zero (i,j)-entries for all pairs (i,j) such that $j \leq i+1$ (and not only for those (i,j) with $|i-j| \leq 1$).

Note that if the considered power series θ happens to be a quadratic polynomial vanishing at zero, $\theta(z) = \alpha_1 z + \alpha_2 z^2$ (i.e. $\alpha_3 = \alpha_4 = \ldots = 0$), then the theorem of Stieltjes can indeed be applied, and gives

$$\sum_{n=0}^{\infty} \mu(X^n) z^n = \cfrac{1}{1-\alpha_1 z - \cfrac{[1]_q \alpha_2 z^2}{1-\alpha_1 z - \cfrac{[2]_q \alpha_2 z^2}{1-\alpha_1 z - \cfrac{[3]_q \alpha_2 z^2}{1-\alpha_1 z - \cdots}}}}, \tag{15}$$

with $\mu = R_q^{-1}(\theta)$. In particular, by putting $\alpha_1 = 0$ and $\alpha_2 = 1$ in (15), we have that the generating function for the moments of $R_q^{-1}(z^2)$ is the expansion of

$$\cfrac{1}{1-\cfrac{[1]_q z^2}{1-\cfrac{[2]_q z^2}{1-\cfrac{[3]_q z^2}{1-\ldots}}}} \tag{16}$$

This continued fraction is known to be associated to the q-continuous Hermite polynomials (see for instance [4], equation (4.4), or Sections 2 and 3.5 of [1]); therefore, $R_q^{-1}(z^2)$ is the measure associated with this set of orthogonal polynomials.

We mention that various facts concerning the particular case discussed in the preceding paragraph were known. On one hand, the formula for the moment of order $2n$ of $R_q^{-1}(z^2)$ provided by Theorem 1 above is $\sum_\pi q^{co(\pi)}$, with summation after all the matchings (i.e. partitions into classes with exactly two elements) of $\{1, \ldots, 2n\}$. The relation between this sum and the continued fraction (16) was pointed out by Touchard [10] (see also Flajolet [3]). On the other hand, the operator $S_q + S_q^*$ (with S_q as in equation (7) above) was studied by Bożejko and Speicher [2] Part II, and its connection with the q-continuous Hermite polynomials was found. (By the Definition 2 above, the linear functional $R_q^{-1}(z^2)$ sends a polynomial f into the $(0,0)$-entry of $f(S_q + S_q^*)$.)

We also mention, without entering into details, that the importance of the particular example discussed above comes from the fact that it plays the role of "central limit" in a certain analogue of the central limit theorem for the "q-convolution" operation which is linearized by the R_q-transform (see Section 4 of [6] for an exact statement of this).

The proofs of the above Theorems 1 and 2 are presented in the paper [6].

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