COUNTING REDUCED WORDS

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Abstract
We study a map called *plactification* from reduced words to words. The map takes nilplactic (Coxeter-Knuth) equivalence to plactic (Knuth) equivalence, and has applications to the enumeration of reduced words, Schubert polynomials and Specht modules.

Résumé
Nous étudions une fonction appelée *plaxification*, des mots réduits aux mots. Cette fonction envoie les équivalences nilplaxiques (Coxeter-Knuth) sur les équivalences plaxiques (Knuth), et possède des applications dans l'énumération des mots réduits, les polynômes de Schubert et les modules de Specht.

Section 1. Introduction

The problem of counting the reduced words of a given permutation has received a great deal of attention since about 1980 [St]. A fundamental tool in this subject is the mysterious equivalence relation on reduced words known as *nilplactic equivalence* [LS1] or *Coxeter-Knuth equivalence* [EG], which bears a striking resemblance to the better understood equivalence relation on words known as *plactic equivalence* [LS2] or *Knuth equivalence.*

This paper describes a map (considered earlier by Lascoux and Schützenberger [La]) called *plactification.* This map takes reduced words to words and maps nilplactic equivalence to plactic equivalence, substantiating the "striking resemblance" alluded to above. The map has other pleasant properties, giving rise to applications to the enumeration of reduced words, the theory of *Schubert polynomials*, and decompositions of certain *Specht modules.*

The paper is organized as follows. Section 2 establishes terminology. Section 3 defines the plactification map and proves its main properties. Section 4 discusses applications.

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Section 2. Definitions

We will assume the reader has some familiarity with the notions of *partitions* λ and their *Ferrers diagrams*, *standard Young tableaux*, *column-strict tableaux*, and *skew column-strict tableaux*. All tableaux will be assumed to be of Ferrers shape, unless they are specifically referred to as skew. A good reference for these notions is [Sa]. We also assume the reader is somewhat familiar with the notions of *plactic (Knuth) equivalence*, the *Robinson-Schensted-Knuth correspondence*, and *jeu-de-taquin*, which we briefly review here (for more thorough discussions, see [EG, Kn, Sa]). The *plactic or Knuth equivalence* on words is the transitive closure of the relations

$$\cdots ikj \cdots \underset{K}{\sim} \cdots kij \cdots$$

if $i \leq j < k$, and

$$\cdots jik \cdots \underset{K}{\sim} \cdots jki \cdots$$

if $i < j \leq k$. We will often think of a skew column-strict tableau T as a word by reference to its *(row-) reading word*, defined to be the concatenation $\cdots \underline{b}^3 \underline{b}^2 \underline{b}^1$, where $\underline{b}^r$ is the r^{th} row of T. Thus when we speak of the Knuth equivalence class of T, we are referring to the Knuth class of this word. Reading words of column-strict tableaux of Ferrers shapes will be called *tableau words*. For T a skew tableau, it will sometimes be convenient for us to use the *column-reading word* of T, defined to be the concatenation $\underline{b}^1 \underline{b}^2 \underline{b}^3 \cdots$ where $\underline{b}^i$ is the decreasing rearrangement of the i^{th} column of T. It is well-known that the row-reading and column-reading word of T are Knuth equivalent, so we can talk about the Knuth class of T by reference to *either* reading word. Every word $\underline{b}$ is Knuth equivalent to a unique column-strict tableau $P(\underline{b})$. *Robinson-Schensted-Knuth (row-)insertion* is an algorithm for producing $P(\underline{b})$ one step at a time: let $P_r(\underline{b})$ be the unique column-strict tableau Knuth equivalent to $b_1 b_2 \cdots b_r$. $P(\underline{b})$ is often called the *insertion tableau* of $\underline{b}$. The *recording tableau* $Q(\underline{b})$ for this process is the standard Young tableaux having the same shape as $P(\underline{b})$ with entry r in the cell of $P_r(\underline{b})$ which is not in $P_{r-1}(\underline{b})$. In this case, we say

$$(\emptyset \leftarrow \underline{b}) = (P(\underline{b}), Q(\underline{b}))$$

Given a skew column-strict tableau, one can also obtain the insertion tableau for its reading word by doing *jeu-de-taquin* slides to bring T into the northwest corner (see [Sa]).

Example
Let $\underline{b} = 4252412$. Then its RSK-insertion looks like

$$P_1(\underline{b}) = 4\,, \quad P_2(\underline{b}) = \begin{array}{l} 2 \\ 4 \end{array}, \quad P_3(\underline{b}) = \begin{array}{ll} 2 & 5 \\ 4 & \end{array}, \quad P_4(\underline{b}) = \begin{array}{ll} 2 & 2 \\ 4 & 5 \end{array},$$

$$P_5(\underline{b}) = \begin{array}{lll} 2 & 2 & 4 \\ 4 & 5 & \end{array}, \quad P_6(\underline{b}) = \begin{array}{lll} 1 & 2 & 4 \\ 2 & 5 & \\ 4 & & \end{array}, \quad P_7(\underline{b}) = \begin{array}{lll} 1 & 2 & 2 \\ 2 & 4 & \\ 4 & 5 & \end{array} = P(\underline{b})$$

and hence

$$(\emptyset \leftarrow \underline{b}) = (P(\underline{b}), Q(\underline{b})) = \left(\begin{array}{lll} 1 & 2 & 2 \\ 2 & 4 & \\ 4 & 5 & \end{array} , \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & 7 & \end{array} \right).$$

Since $\underline{b}$ is the reading word of the column-strict tableau

$$T = \begin{array}{lll} & 1 & 2 \\ & 2 & 4 \\ 2 & 5 & \\ 4 & & \end{array}$$

one can also obtain $P(\underline{b})$ by jeu-de-taquin slides:

$$\begin{array}{lll} & 1 & 2 \\ & 2 & 4 \\ 2 & 5 & \\ 4 & & \end{array} \mapsto \begin{array}{lll} & 1 & 2 \\ 2 & 2 & 4 \\ 4 & 5 & \end{array} \mapsto \begin{array}{lll} 1 & 2 & 2 \\ 2 & 4 & \\ 4 & 5 & \end{array}$$

We now discuss ***reduced words***. Given a permutation w in the symmetric group S_n, a reduced word $\underline{a}$ is a sequence $a_1 a_2 \cdots a_l$ of minimal length such that $w = s_{a_1} s_{a_2} \cdots s_{a_l}$, where s_i is the adjacent transposition $(i\ i+1)$. Here $l(w)$ is called the *length* of w. Let $Red(w)$ denote the set of all reduced words for w. Since the adjacent transpositions s_i obey the *braid relations*

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

one can define the *nilplactic* or *Coxeter-Knuth* equivalence relation on $Red(w)$ to be the transitive closure of these relations:

$$\cdots ikj \cdots \underset{CK}{\sim} \cdots kij \cdots$$

$$\cdots jik \cdots \underset{CK}{\sim} \cdots jki \cdots$$

$$\cdots i\ i+1\ i \cdots \underset{CK}{\sim} \cdots i+1\ i\ i+1 \cdots$$

where $i < j < k$. [EG],[LS1] show that, miraculously, all of the constructions involving Knuth equivalence we have described carry over to Coxeter-Knuth equivalence. Thus for any reduced word $\underline{a}$, there is a unique column-strict tableau $\tilde{P}(\underline{a})$ whose reading word is Coxeter-Knuth equivalent to $\underline{a}$. One may obtain $\tilde{P}(\underline{a})$ by *Coxeter-Knuth* insertion (see [EG]), and we write

$$(\emptyset \overset{\circ}{\leftarrow} \underline{a}) = (\tilde{P}(\underline{a}), \tilde{Q}(\underline{a}))$$

Note that in this case $\tilde{P}(\underline{a})$ is not only column-strict, but also strictly increasing along *rows*, since otherwise its reading word would not be reduced. If $\underline{a}$ is a reduced word which is the reading word of a tableau T, then one can obtain $\tilde{P}(\underline{a})$ by doing *nilplactic jeu-de-taquin slides* to bring T into the northwest corner (see [LS1]).

Example
$\underline{a} = 4253413$ is a reduced word for $w = 251643$. Then its CK-insertion looks like

$$\tilde{P}_1(\underline{a}) = 4, \quad \tilde{P}_2(\underline{a}) = \begin{array}{c} 2 \\ 4 \end{array}, \quad \tilde{P}_3(\underline{a}) = \begin{array}{cc} 2 & 5 \\ 4 & \end{array}, \quad \tilde{P}_4(\underline{a}) = \begin{array}{cc} 2 & 3 \\ 4 & 5 \end{array},$$

$$\tilde{P}_5(\underline{a}) = \begin{array}{ccc} 2 & 3 & 4 \\ 4 & 5 & \end{array}, \quad \tilde{P}_6(\underline{a}) = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \\ 4 & & \end{array}, \quad \tilde{P}_7(\underline{a}) = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & \\ 4 & 5 & \end{array} = \tilde{P}(\underline{a})$$

and hence

$$((\emptyset \overset{\circ}{\leftarrow} \underline{a}) = (\tilde{P}(\underline{a}), \tilde{Q}(\underline{a})) = \left(\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & \\ 4 & 5 & \end{array}, \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & 7 & \end{array} \right).$$

Since $\underline{a}$ is the reading word of the column-strict tableau

$$T = \begin{array}{ccc} & 1 & 3 \\ & 3 & 4 \\ 2 & 5 & \\ 4 & & \end{array}$$

one can also obtain $\tilde{P}(\underline{a})$ by nilplactic jeu-de-taquin slides:

$$\begin{array}{ccc} & 1 & 3 \\ & 3 & 4 \\ 2 & 5 & \\ 4 & & \end{array} \mapsto \begin{array}{ccc} & 1 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & \end{array} \mapsto \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & \\ 4 & 5 & \end{array}$$

We need one more tool in order to define the plactification map: the *plactic action* of the symmetric group on words [LS2], which we recall here. Given a word $\underline{b}$ and positive integer r, the r-parenthesization of $\underline{b}$ consists of replacing each occurrence of $r+1$ by a left parenthesis "(" and each occurrence of r by a right parenthesis ")". Say that an occurrence of $r+1$ and and occurrence of r are *r-paired* if they get replaced by a set of parentheses which close each other under the usual rules of parenthesization. An $r+1$ or r which is not r-paired with anyone will be called *r-unpaired*, and it follows that the subsequence of r-unpaired $r, r+1$'s in $\underline{b}$ must look like:

$$\underbrace{r\, r \cdots r}_{s}\, \underbrace{r+1\, r+1 \cdots r+1}_{t}$$

where s, t are some integers. The plactic action of s_r on $\underline{b}$ leaves all other entries fixed and replaces this subsequence by

$$\underbrace{r\, r \cdots r}_{t}\, \underbrace{r+1\, r+1 \cdots r+1}_{s}$$

to form a new word denoted $\sigma_r(\underline{b})$.

Example
If $\underline{b} = 133431312213432$ then $\sigma_2(\underline{b}) = 122431312212432$.

The following properties of r-pairing and σ_r are not hard to check [LS2]:

Proposition 1. *The number of r-paired r's and $r+1$'s is invariant under Knuth transformations, and hence constant on Knuth equivalence classes.*

Proposition 2. *σ_r commutes with with Knuth equivalence, i.e. if $\underline{b} \underset{K}{\sim} \underline{b}'$ then $\sigma_r(\underline{b}) \underset{K}{\sim} \sigma_r(\underline{b}')$.*

Proposition 3.

$$Q(\sigma_r(\underline{b})) = Q(\underline{b}).$$

Proposition 4.

(1) $\sigma_i^2 = id$.
(2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$.
(3) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Remark
We will not make use of it, but this last proposition allowed Lascoux and Schützenberger to define the ***plactic action*** of any permutation w on a word $\underline{b}$ to be

$$\sigma_{i_1} \cdots \sigma_{i_l}(\underline{b})$$

for any decomposition $w = s_{i_1} \cdots s_{i_l}$, since the symmetric group is well-known to have presentation given by the generators $\{s_i\}$ and relations (1), (2), (3).

Section 3. The plactification map

We are now in a position to define the plactification map ϕ from reduced words to words. Given a reduced word $\underline{a}$ that begins with the letter r (so $\underline{a} = r\underline{\hat{a}}$), $\phi(\underline{a})$ is defined recursively by

$$\phi(\underline{a}) = r\sigma_r\phi(\underline{\hat{a}})$$

Therefore, if $\underline{a} = a_1 \cdots a_l$ then we have

$$\phi(\underline{a}) = a_1\sigma_{a_1}(a_2\sigma_{a_2}(\cdots a_{l-1}\sigma_{a_{l-1}}(a_l)\cdots))$$

Example

$$\begin{aligned}
\phi(5345134) &= 5\sigma_5(3\sigma_3(4\sigma_4(5\sigma_5(1\sigma_1(3\sigma_3(4))))))\\
&= 5\sigma_5(3\sigma_3(4\sigma_4(5\sigma_5(1\sigma_1(33)))))\\
&= 5\sigma_5(3\sigma_3(4\sigma_4(5\sigma_5(133))))\\
&= 5\sigma_5(3\sigma_3(4\sigma_4(5133)))\\
&= 5\sigma_5(3\sigma_3(44133))\\
&= 5\sigma_5(344133)\\
&= 5344133
\end{aligned}$$

We begin with a few simple observations about the map ϕ. First, note that it is invertible: the formula

$$\phi^{-1}(r\underline{b}) = r\phi^{-1}(\sigma_r(\underline{b}))$$

recursively defines the inverse. So ϕ is an injective map from reduced words to words. The following proposition is not difficult to prove.

Proposition 5. *Let $\underline{a}$ be a reduced word which is the (row-, column-)reading word of a skew column-strict tableau T. Then $\phi(\underline{a})$ is the (row-, column-)reading word of a skew column-strict tableau of the same shape, denoted $\phi(T)$).*

For the moment, our primary goal is to characterize the image $\phi(Red(w))$. To this end, we recall from [Mac] that the *Rothe diagram* $D(w)$ is the set of cells in the plane having row and column indices

$$\{(i,j) : i < w_j^{-1} \text{ and } j < w_i\}.$$

Given any set of cells D in the plane, the *row-filled diagram* $F(D)$ is the filling of D in which every cell in row r contains the entry r. Say a word $\underline{b}$ is D-*peelable* if there is some column C of $F(D)$ which occurs flush north as a vertical segment V in the first column of $P(\underline{b})$, and $\hat{b} = P(\underline{b})/V$ is $\hat{D}$-peelable where $\hat{D} = D/C$.

Example
$w = 251643$ has

$$D(w) = \begin{array}{cccc} \times & & & \\ \times & & \times & \times \\ & & & \\ & & \times & \times \\ & & \times & \end{array}, \qquad F(D(w)) = \begin{array}{cccc} 1 & & & \\ 2 & & 2 & 2 \\ & & & \\ & & 4 & 4 \\ & & 5 & \end{array}$$

There are three $D(w)$-peelable column-strict tableau words, with "peelings" exhibited below:

$$\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & \\ 4 & & \\ 5 & & \end{array} \mapsto \begin{array}{ccc} . & 2 & 2 \\ . & 4 & \\ 4 & & \\ 5 & & \end{array} \mapsto \begin{array}{cc} 2 & 2 \\ 4 & 4 \\ 5 & \end{array} \mapsto \begin{array}{cc} . & 2 \\ . & 4 \\ . & \end{array} \mapsto \begin{array}{c} 2 \\ 4 \end{array} \mapsto \emptyset$$

$$\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & \\ 4 & 5 & \end{array} \mapsto \begin{array}{ccc} . & 2 & 2 \\ . & 4 & \\ 4 & 5 & \end{array} \mapsto \begin{array}{cc} 2 & 2 \\ 4 & 4 \\ 5 & \end{array} \mapsto \begin{array}{cc} . & 2 \\ . & 4 \\ . & \end{array} \mapsto \begin{array}{c} 2 \\ 4 \end{array} \mapsto \emptyset$$

$$\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 5 & & \end{array} \mapsto \begin{array}{ccc} . & 2 & 2 \\ . & 4 & 4 \\ 5 & & \end{array} \mapsto \begin{array}{cc} 2 & 2 \\ 4 & 4 \\ 5 & \end{array} \mapsto \begin{array}{cc} . & 2 \\ . & 4 \\ . & \end{array} \mapsto \begin{array}{c} 2 \\ 4 \end{array} \mapsto \emptyset$$

It will eventually be shown below that $\phi(Red(w^{-1}))$ is exactly the set of $D(w)$-peelable words $\underline{b}$. First we note a few consequences of $D(w)$-peelability. For any word $\underline{b}$, the content of $\underline{b}$ is the sequence $(c_1, c_2, \ldots)$, where c_i is the number of occurrences of i in $\underline{b}$. For a permutation w, the *code* of w is the sequence $(c_1, c_2, \ldots)$ where

$$c_i = \#\{(i,j) : i < j, w_i > w_j\}.$$

The following proposition is evident:

Proposition 6. *If $\underline{b}$ is $D(w)$-peelable, then the content of $\underline{b}$ is the same as the code of w.*

Proposition 7. *If $\underline{b}$ is $D(w)$-peelable, then*

(1) *all r's in $\underline{b}$ are r-paired if $w_r < w_{r+1}$,*
(2) *all $r+1$'s in $\underline{b}$ are r-paired if $w_r > w_{r+1}$.*

The proof proceeds by induction on the number of columns of $D = D(w)$.

Here are two important properties of plactification:

Theorem 8.

(A) *If $\underline{a}, \underline{a}'$ in Red(w) satisfy $\underline{a} \underset{CK}{\sim} \underline{a}'$ then $\phi(\underline{a}) \underset{K}{\sim} \phi(\underline{a}')$.*
(B) *For $\underline{a}$ in Red(w^{-1}), $\phi(\underline{a})$ is $D(w)$-peelable.*

Proof. Both assertions are proven simultaneously by induction on $l(w)$. so let A_l, B_l be assertions A, B for all permutations w with $l(w) = l$. The theorem immediately follows from the next two lemmas, which require some work to prove. □

Lemma 9. *B_k for $k < l$ implies A_l.*

Lemma 10. *A_k for $k \leq l$ and B_k for $k < l$ imply B_l.*

We can now characterize the image $\phi(Red(w))$:

Theorem 11. *$\underline{b} = \phi(\underline{a})$ for some $\underline{a}$ in Red(w^{-1}) if and only if $\underline{b}$ is $D(w)$-peelable.*

Assertion B of the previous theorem gives one implication. The other direction follows by induction on the number of columns of $D(w)$.

Interestingly, the plactification map not only takes Coxeter-Knuth to Knuth equivalence, but also preserves recording tableaux:

Proposition 12.

$$Q(\phi(\underline{a})) = \tilde{Q}(\underline{a})$$

This result can be proven by induction, using the column insertion versions of the CK and RSK correspondences.

Section 4. Applications

The first application gives a new, efficient way of counting the number of reduced words of a permutation.

Theorem. *For any permutation w, the cardinality of $Red(w)$ is*

$$\sum_{P} f_{shape(P)}$$

where P runs over all $D(w)$-peelable column-strict tableaux and f_λ is the number of standard Young tableaux of shape λ.

Proof. The fact that the cardinality of $Red(w)$ is

$$\sum_{\tilde{P}} f_{shape\tilde{P}}$$

where $\tilde{P}$ runs over all tableau which are reduced words for w^{-1} was proven by [EG],[LS1]. We have seen (Theorem 11, Proposition 5) that ϕ is a shape-preserving bijection between this set of $\tilde{P}$ and the set of $D(w)$-peelable tableaux. □

Remark
This theorem is quite practical, since the definition of peelability admits an efficient algorithm for producing all $D(w)$-peelable column-strict tableaux.

The second application is the expansion of a Schubert polynomial as a sum of key polynomials, indexed by the $D(w)$-peelable tableaux.

Theorem 13. *For any permutation w, we have*

$$\mathfrak{S}_w = \sum_{P} \kappa_{content(K_-(P))}$$

where $\mathfrak{S}_w$ and κ_α are the ***Schubert polynomials*** *and* ***key polynomials*** *respectively of Lascoux and Schützenberger [LS3],[LS4], and P runs over all $D(w)$-peelable column-strict tableaux.*

The last application is again closely related to the first two, and deals with decomposing *Specht modules* of the symmetric group over $\mathbb{C}$ into irreducible representations. Given a diagram D, i.e. an n-element subset of the plane $\mathbb{Z} \times \mathbb{Z}$, and a field $\mathbb{F}$, the *Specht module* S^D is a representation of the symmetric group S_n over the field $\mathbb{F}$ defined using the *Young-symmetrizer construction* (see [JP] for a definition). When D is a Ferrers diagram λ, S^λ is irreducible for $\mathbb{F} = \mathbb{C}$, and leads to a construction of all irreducibles over arbitrary fields $\mathbb{F}$. When D is a skew diagram, the *Littlewood-Richardson rule* decomposes S^D into Specht modules S^λ [JP]. It is a natural question to ask about such decompositions for more general diagrams D.

It was known (although never written down) that using results of Kraskiewicz and Pragacz [KP] and [LS1],[EG] along with *Schur-Weyl duality*, one can prove the following:

Theorem 14. *Let $\mathbb{F}$ be a field of characteristic zero. Then for any permutation w*

$$S^{D(w)} = \bigoplus_{\tilde{P}} S^{shape(\tilde{P})}$$

where $\tilde{P}$ runs over all tableau reduced words for w^{-1}.

Therefore, we immediately deduce

Corollary 15. *If $\mathbb{F}$ is a field of characteristic zero and w any permutation, then*

$$S^{D(w)} = \bigoplus_P S^{shape(P)}$$

where P runs over all $D(w)$-peelable column-strict tableaux.

This corollary is interesting for the following reason. In [RS2], it is shown that there is *another* class of diagrams D for which the decomposition of S^D into irreducibles over $\mathbb{C}$ is given by the shapes of the D-peelable tableaux. This other class is the class of *column-convex* diagrams D, i.e. those D for which each column has no gaps between its cells. Both Rothe diagrams $D(w)$ and column-convex diagrams D have the property that one can rearrange the columns of D (which does not affect S^D as an S_n-representation) to make D have the *northwest property*: if $i_1 < i_2$ and $j_1 < j_2$ and $(i_2, j_1), (i_1, j_2)$ are in D then (i_1, j_1) is in D. This suggests the following conjecture:

Conjecture 16. *If $\mathbb{F}$ is a field of characteristic zero, and D has the northwest property, then*

$$S^D = \bigoplus_P S^{shape(P)}$$

as P runs over all D-peelable column-strict tableaux.

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