A Symplectic Jeu de Taquin Bijection Between the Tableaux of King and of De Concini

Outline

Jeffrey T. Sheats
Mathematics, CB#3250, University of North Carolina, Chapel Hill, N.C. 27599

The main result of this paper is a bijection between two distinctly combinatorially defined sets of symplectic tableaux. The first was developed by Zhelobenko (1961) and King (1975), and the second by De Concini (1979; in explicit tableau form) and by Laksmibai and Seshadri (1979). From either of these sets one can calculate the character of finite dimensional irreducible sp(2n) modules. To provide the bijection a new symplectic jeu de taquin algorithm is developed. Schutzenberger's original jeu de taquin is a powerful tableau manipulating tool which is used to obtain combinatorial and representation theoretic results. The most famous of these is providing a bijective proof of the Littlewood-Richardson rule for decomposing tensor products of representations of gl(n,C). In the future the symplectic jeu de taqin may be used to give an analogous proof of Littelmann's (1988) results for decomposing tensor products of symplectic representations.

Le principal résultat de cet article est une bijection entre deux ensembles de tableaux symplectiques qui diffèrent au niveau combinatoire de leur definition. Le premier fut développé par Zhelobenko (1961) et King (1975), et le second par De Concini (1979; de manière explicite en termes de tableaux) et par Laksmibai et Seshadri (1979). On peut calculer les caractères des représentations irréductibles de dimension finie pour sp(2n), à partir de n'importe lequel de ces deux ensembles. Pour en arriver a cette bijection on développe un nouvel algorithme de jeu de taquin symplectique. Le jeu de taquin original, dû à Schützenberger, est un outil puissant pour obtenir des résultats aussi bien en combinatoire qu' en thèorie des représentations. Le plus connu de ces résultats est une preuve bijective de la règle de Littlewood et Richardson qui donne la décomposition des produits tensoriels de représentations de gl(n,C). On espère ainsi, qu'à l'avenir on pourra utiliser le jeu de taquin symplectique pour obtenir une preuve analogue du résultat de Littlemann (1988) sur la décomposition des produits tensoriels de représentations symplectiques.

0. Introduction

Let $\mathcal{T}(\lambda, n)$ be the set of semistandard tableaux of shape λ and with entries from $[n] := \{1,2,\cdots,n\}$. Then $\mathcal{T}(\lambda, n)$ acts as an index set for a basis of an irreducible representation of the Lie algebra gl(n). There are two distinctly combinatorially defined sets of tableaux, which we denote $\mathcal{D}(\lambda, n)$ and $\mathcal{K}(\lambda, n)$, each of which play an analogous role for sp(2n). In this paper we provide an explicit bijection between $\mathcal{D}(\lambda, n)$ and $\mathcal{K}(\lambda, n)$. In the process we introduce a new symplectic jeu de taquin (SJDT) algorithm.

The tableaux of $\mathcal{D}(\lambda, n)$ were developed by De Concini (1979; [DeC]) and by Laksmibai, Musili, and Seshadri (1979; [LSM]). The tableaux of $\mathcal{K}(\lambda, n)$ were developed by Zhelobenko (1961; [Zel]) as Gelfand patterns, and later converted by King (1975; [Kng]) to the form used below.

The construction of these sets of tableaux involve, as a first step, imposing a total order on a basis of the standard representation of the Lie algebra. Such a total order corresponds naturally to specifying the algebra's defining bilinear form. The difference in the construction of the two sets $\mathcal{D}(\lambda, n)$ and $\mathcal{K}(\lambda, n)$ stems primarily from the fact that different total orders are used. In order to make the bijection problem more manageable we define a sequence of n sets of intermediate "hybrid" tableaux $\mathcal{M}^k(\lambda, n)$, $1 \le k \le n$, where $\mathcal{D}(\lambda, n) = \mathcal{M}^n(\lambda, n)$ and $\mathcal{K}(\lambda, n) = \mathcal{M}^1(\lambda, n)$. This sequence corresponds to a gradual transformation of one total order to the other. Or, in terms of the defining skew symmetric bilinear forms for sp(2n), the sequence below.

$$\begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \iddots & & \\ & & 1 & & & \\ & -1 & & & & \\ & \iddots & & & & \\ -1 & & & & & \end{bmatrix}, \quad \left[\begin{array}{cccc|cc} & & & 1 & & \\ & & \iddots & & & \\ & 1 & & & & 0 \\ -1 & & & & & \\ \iddots & & & & & \\ -1 & & & & & \\ \hline & & 0 & & 0 & 1 \\ & & & & -1 & 0 \end{array}\right], \quad \dots, \quad \begin{bmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{bmatrix}$$

(These bilinear forms correspond respectively to $\mathcal{D}(\lambda, n) = \mathcal{M}^n(\lambda, n)$, $\mathcal{M}^{n-1}(\lambda, n)$, ... , $\mathcal{M}^1(\lambda, n) = \mathcal{K}(\lambda, n)$.)

Each of these intermediate sets satisfies the following basis indexing theorem:

Theorem: *Fix a positive integer* n *and a shape* λ. *Then for* $1 \le k \le n$,

$$ch_\lambda(x) = \sum_{\mathcal{T} \in \mathcal{M}^k(\lambda, n)} x^{wt(\mathcal{T})}$$

where, $ch_\lambda(x)$ *is the character of the irreducible representation of* sp(2n) *indexed by* λ. (The weight $wt(\mathcal{T})$ is defined below.)

This theorem is a consequence of our main result.

Main Result: Fix positive integers n and k, $k \le n$, and fix a shape λ. The SJDT gives an explicit weight preserving bijection from the set $\mathcal{M}^k(\lambda, n)$ to the set $\mathcal{M}^{k-1}(\lambda, n)$.

I. Definitions and Reducing the Problem

What follows are the definitions of the tableaux contained in the set $\mathcal{M}^k(\lambda,n)$ which we will call *k-admissible tableaux*. We start by defining *k-admissible columns* and then give a rules on when it is legal to glue such columns together to form larger tableaux.

Before we start we must set up some notation for finite sets. We shall denote the cardinality of any set X, by $|X|$. For two subsets X, Y, we use $X - Y$ for $X \backslash Y$ when it is known that $Y \subseteq X$. Similarly, we use $X + Y$ for $X \cup Y$ when it is known that $X \cap Y$ is empty. For subsets $X = \{ x_1 < x_2 < \cdots < x_s \}$ and $Y = \{ y_1 < y_2 < \cdots < y_t \}$ of a totally ordered set, we say that $X \leq Y$ if: (i) $s \geq t$, and (ii) $x_i \leq y_i$ for $1 \leq i \leq t$.

All columns will consist of entries from the set of symbols

$$[[n]] \;=\; \{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3, \ldots, \bar{n}, n\}.$$

A column $\mathcal{P}$ is said to be *admissible* if no entry is repeated, and for all m, $1 \leq m \leq n$,

$$|\{x \in \mathcal{P} : \|x\| \leq m\}| \leq m, \text{ where } \|\cdot\| : [[n]] \to [n], \ \|\bar{a}\| := \|a\| := a.$$

For $1 \leq k \leq n$ define the total order O^k on $[[n]]$ as

$$\bar{k} < \overline{k-1} < \ldots < \bar{1} < 1 < 2 < \ldots < k-1 < k < \overline{k+1} < k+1 < \ldots < \bar{n} < n.$$

A column is *k-admissible* if it is admissible and strictly increasing from the top w.r.t. O^k (*k-strictly increasing*).

The weight of a tableau $\mathcal{T}$, wt($\mathcal{T}$), is the n-tuple $(v_1, \ldots, v_n)$, where v_i is the number of i's in $\mathcal{T}$ minus the number of $\bar{i}$'s in $\mathcal{T}$. For example, if $n = 5$, the weight of the tableau below is $(0,-1,3,1,0)$.

$\bar{1}$	1	$\bar{2}$	3
3	3		
$\bar{4}$	4		
4			

Remark: The set $[[n]]$ may be thought of as a basis for the standard representation for sp(2n). The order O^1 is the one imposed by King, while DeConcini uses O^n.

Now we are able to give the definition of the sets $\mathcal{K}(\lambda, n)$.

Definition: Fix positive integers n and k, $k \leq n$. Let $\mathcal{T}$ be a tableau with entries from $[[n]]$ and columns $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_c$. Then $\mathcal{T}$ is a *1-admissible tableaux* if

(i) $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_c$ are all 1-admissible columns, and

(ii) $\mathcal{T}_1 \leq \mathcal{T}_2 \leq \cdots \leq \mathcal{T}_c$ with respect to O^1.

Define $\mathcal{K}(\lambda, n)$ to be the set of all 1-admissible tableaux of shape λ.

Note that condition (i) identifies the columns to be used, and (ii) specify's how these columns may be glued together. De Concini's definition of $\mathcal{D}(\lambda, n)$ is of the same form but the condition anagous to (ii) is a lot more complicated. It involves splitting each column $\mathcal{T}_i$ into two columns, a left half $\ell\mathcal{T}_i$ and right half $r\mathcal{T}_i$. Then the column $\mathcal{T}_{i-1}$ may sit to the left of column $\mathcal{T}_i$ iff $r\mathcal{T}_{i-1} \leq \ell\mathcal{T}_i$ with respect to $\mathcal{O}^n$.

We now describe De Concini's splitting process which involves breaking an n-admissible column $\mathcal{P}$ into two parts, generating two new parts, and then assembling these four parts into the columns $r\mathcal{P}$ and $\ell\mathcal{P}$.

Definition: For each positive integer n, define the map F_n, from pairs of subsets of [n] to one column tableaux, where $F_n(X, Y)$ is the column with entries $\{\bar{x} : x \in X\} \cup \{y : y \in Y\}$ and ordered n-strictly increasing.

Let $\mathcal{P}$ be an n-strictly increasing column. Denote by $A_\mathcal{P}$ and $D_\mathcal{P}$ the subsets of [n] such that $\mathcal{P} = F_n(A_\mathcal{P}, D_\mathcal{P})$. Define $I_\mathcal{P} = A_\mathcal{P} \cap D_\mathcal{P}$ and $H_\mathcal{P} = [n] - (A_\mathcal{P} \cup D_\mathcal{P})$.

Lemma: *The column* $\mathcal{P}$ *is admissible if and only if* $H_\mathcal{P} < I_\mathcal{P}$.

Remark: De Concini uses this as his definition for an admissible column.

Let $\mathcal{P}$ be an n-admissible column and $A_\mathcal{P}, D_\mathcal{P}, H_\mathcal{P}, I_\mathcal{P}$ as above. Define the following sets:

$$J_\mathcal{P} := \max[X \subseteq H_\mathcal{P} : |X| = |I_\mathcal{P}| \text{ and } X < I_\mathcal{P}],$$

$$B_\mathcal{P} := (A_\mathcal{P} - I_\mathcal{P}) + J_\mathcal{P},$$

$$C_\mathcal{P} := (D_\mathcal{P} - I_\mathcal{P}) + J_\mathcal{P}.$$

Note that the lemma guarantees the existance of $J_\mathcal{P}$. For an n-admissible column $\mathcal{P}$ we shall refer to $A_\mathcal{P}, B_\mathcal{P}, C_\mathcal{P}, D_\mathcal{P}$, as the *associated subsets*.

Definition: Let $\mathcal{P}$ be an n-admissible column and form the associated subsets $A_\mathcal{P}, B_\mathcal{P}, C_\mathcal{P}, D_\mathcal{P}$. Define $r\mathcal{P} := F_n(A_\mathcal{P}, C_\mathcal{P})$ and $\ell\mathcal{P} := F_n(B_\mathcal{P}, D_\mathcal{P})$.

Remarks: The pairs of columns $(r\mathcal{P}, \ell\mathcal{P})$ correspond to the *admissible pairs* of extreme weights in [LSM]. It is easily seen that $r\mathcal{P} \leq \ell\mathcal{P}$ with respect to $\mathcal{O}^n$. The order $\mathcal{O}^n$ agrees with the Bruhat order on extreme weights used in [LSM].

Definition: Fix positive integers n and k, $k \leq n$. Let $\mathcal{T}$ be a tableau with entries from [[n]] and columns $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_c$. Then $\mathcal{T}$ is a *n-admissible tableaux* if

(i) $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_c$ are all n-admissible columns, and

(ii) $r\mathcal{T}_{i-1} \leq \ell\mathcal{T}_i$ with respect to $\mathcal{O}^n$, $1 < i \leq c$.

Define $\mathcal{D}(\lambda, n)$ to be the set of all n-admissible tableaux of shape λ.

The definition of the associated subsets $A_{\mathcal{P}}, B_{\mathcal{P}}, C_{\mathcal{P}}, D_{\mathcal{P}}$, has a certain symmetry. Note that $J_{\mathcal{P}} = B_{\mathcal{P}} \cap C_{\mathcal{P}}$, and

$$I_{\mathcal{P}} = \min[X \subseteq [n] - (B_{\mathcal{P}} \cup C_{\mathcal{P}}): |X| = |J_{\mathcal{P}}| \text{ and } X > J_{\mathcal{P}}],$$

$$A_{\mathcal{P}} = (B_{\mathcal{P}} - I_{\mathcal{P}}) + J_{\mathcal{P}},$$

$$C_{\mathcal{P}} = (D_{\mathcal{P}} - I_{\mathcal{P}}) + J_{\mathcal{P}}.$$

Thus, if either of the pairs of sets $(A_{\mathcal{P}}, D_{\mathcal{P}})$ or $(B_{\mathcal{P}}, C_{\mathcal{P}})$ is known, the original column $\mathcal{P}$ can be recovered. Define a column $\mathcal{P}'$ to be *n-coadmissible* if there exists an n-admissible column $\mathcal{P}$ such that $\mathcal{P}' = F_n(B_{\mathcal{P}}, C_{\mathcal{P}})$.

Lemma: *A column $\mathcal{P}'$ is n-coadmissible if and only if it is n-strictly increasing, and for all* m, $1 \le m \le n$, $|\{x \in \mathcal{P}: \|x\| \ge n - m + 1\}| \le m$.

Compare this lemma with the definition of n-admissible columns.

Definition: Define the map G_n: $\{(X,Y): F_n(X,Y) \text{ is coadmissible}\} \to \{\text{admissible columns}\}$ by $G_n(B_{\mathcal{P}}, C_{\mathcal{P}}) := \mathcal{P}$.

Heuristically, it is helpful to think of an n-admissible column $\mathcal{P}$ as the column $\begin{smallmatrix}\mathbf{A}\\ \mathbf{D}\end{smallmatrix}$, where $\mathbf{A}$ is the sub-column of barred entries of $\mathcal{P}$, and $\mathbf{D}$, the unbarred entries. Similarly, think of the split columns of $\mathcal{P}$ as the pair of columns $\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}$. For the maps F_n and G_n, we have $F_n(A,D) = \begin{smallmatrix}\mathbf{A}\\ \mathbf{D}\end{smallmatrix} = G_n(B,C)$.

As an example let n = 6, and $\mathcal{P} = \begin{array}{|c|}\hline \bar{6}\\ \hline \bar{5}\\ \hline 2\\ \hline 5\\ \hline\end{array}$, then $\ell\mathcal{P} = \begin{array}{|c|}\hline \bar{6}\\ \hline \bar{5}\\ \hline 2\\ \hline 4\\ \hline\end{array}$ and $r\mathcal{P} = \begin{array}{|c|}\hline \bar{6}\\ \hline \bar{4}\\ \hline 2\\ \hline 5\\ \hline\end{array}$.

Now we define the sets of hybrid tableaux $\mathcal{M}^k(\lambda, n)$, $1 \le k \le n$.

Definition: Fix positive integers n and k, $k \le n$. Let $\mathcal{T}$ be a tableau with entries from $[[n]]$ and columns $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_c$. Then $\mathcal{T}$ is a *k-admissible tableaux* if

(i) $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_c$ are all k-admissible columns,

(ii) $\mathcal{T}_1 \le \mathcal{T}_2 \le \cdots \le \mathcal{T}_c$ with respect to O^k, and

(iii) form the subtableau $\mathcal{T}_\mu$, consisting of those entries less than or equal to $\mathbf{k}$; require that $\mathcal{T}_\mu \in \mathcal{D}(\mu, k)$, where μ is the shape of $\mathcal{T}_\mu$.

Define $\mathcal{M}^k(\lambda, n)$ to be the set of all k-admissible tableaux of shape λ.

In essence, what we are doing here, is using De Concini's definition for the subtableau with entries $\leq \mathbf{k}$, and King's definition for the remainder.

It is clear that $\mathcal{D}(\lambda, n) = \mathcal{M}^n(\lambda, n)$ and $\mathcal{K}(\lambda, n) = \mathcal{M}^1(\lambda, n)$. So, with this construction, we have reduced the problem of finding a bijection from $\mathcal{D}(\lambda, n)$ to $\mathcal{K}(\lambda, n)$, to finding a bijection from $\mathcal{M}^k(\lambda, n)$ to $\mathcal{M}^{k-1}(\lambda, n)$ for $k = n, n-1, \cdots, 2$. We can reduce the problem further by using the following easy lemma.

Lemma: (1) *Suppose* $\mathcal{T} \in \mathcal{M}^k(\lambda, n)$ *and let* $\mathcal{T}_\mu \in \mathcal{M}^k(\mu, k)$ *be as in the definition. Let* $\tilde{\mathcal{T}}_\mu$ *be any element of* $\mathcal{M}^{k-1}(\mu, k)$ *and form the tableau* $\tilde{\mathcal{T}}$ *by replacing* $\mathcal{T}_\mu$ *with* $\tilde{\mathcal{T}}_\mu$ *within the tableau* $\mathcal{T}$. *Then* $\tilde{\mathcal{T}} \in \mathcal{M}^{k-1}(\lambda, n)$.

(2) *Conversely, suppose* $\tilde{\mathcal{T}} \in \mathcal{M}^{k-1}(\lambda, n)$ *and* $\tilde{\mathcal{T}}_\mu$ *is the subtabeau of consisting of those entries less than or equal to* $\mathbf{k}$. *Then* $\tilde{\mathcal{T}}_\mu \in \mathcal{M}^{k-1}(\mu, k)$ *where* μ *is the shape of* $\tilde{\mathcal{T}}_\mu$. *Let* $\mathcal{T}_\mu$ *be any element of* $\mathcal{M}^k(\mu, k)$ *and form the tableau* $\mathcal{T}$ *by replacing* $\tilde{\mathcal{T}}_\mu$ *with* $\mathcal{T}_\mu$ *in the tableau* $\tilde{\mathcal{T}}$. *Then* $\mathcal{T} \in \mathcal{M}^k(\lambda, n)$.

This reduces the problem to finding a bijection from $\mathcal{M}^n(\lambda, n)$ to $\mathcal{M}^{n-1}(\lambda, n)$.

The primary difference between the n-admissible tableaux and the (n–1)-admissible tableaux is the location of the $\bar{\mathbf{n}}$'s. If $\bar{\mathbf{n}}$'s are present in an n-admissible tableau they must occur at the extreme left of the top row, while the $\bar{\mathbf{n}}$'s in an (n–1)-admissible tableau will occur near the southeast boundary. In the SJDT algorithm the $\bar{\mathbf{n}}$'s of an n-admissible tableau are replaced by empty boxes. Just as the jeu de taquin, the SJDT has sliding operations which specify how the empty boxes are moved to the southeast perimeter of the tableau. In the SJDT, however, when an entry is slid from one column to another, the two columns involved must be "recalculated" using the functions F_n and G_n defined above.

We say a column $\mathcal{P}$ is *punctured* if it contains a single empty box. The definitions for n-admissibility and associated subsets apply to punctured columns by ignoring the empty box. We use the notation $\mathcal{P} = \mathcal{Q} + \square_p$ to mean that the column $\mathcal{P}$ is obtained by inserting empty box into $\mathcal{Q}$ at the p^{th} position. If $\mathcal{P} = \mathcal{Q} + \square_p$ is a punctured n-admissible column then we define $\ell\mathcal{P} := \ell\mathcal{Q} + \square_p$ and $r\mathcal{P} := r\mathcal{Q} + \square_p$.

Let $\mathcal{T}$ be an n-admissble column with columns $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_c$. The entries of $\mathcal{T}$ are indexed matrix style. That is, we write $\mathcal{T} = \{t_{ij}\}$ where $t_{1,1}$ is the top entry of the first column. For the p^{th} entries from the top of $r\mathcal{T}_q$ and $\ell\mathcal{T}_q$ we use the notation $r_{p,q}$ and $\ell_{p,q}$, respectively.

For example if n = 6, let

$$\mathcal{P} = \begin{array}{|c|}\hline \bar{6} \\ \hline \bar{5} \\ \hline 2 \\ \hline 5 \\ \hline \end{array} \quad \text{and} \quad \mathcal{T}_q = \mathcal{P} + \square_3, \quad \text{then} \quad \ell\mathcal{T}_q = \begin{array}{|c|}\hline \bar{6} \\ \hline \bar{5} \\ \hline \\ \hline 2 \\ \hline 4 \\ \hline \end{array}, \quad r\mathcal{T}_q = \begin{array}{|c|}\hline \bar{6} \\ \hline \bar{4} \\ \hline \\ \hline 2 \\ \hline 5 \\ \hline \end{array},$$

and the subets associated to $\mathcal{T}_q$ are

$$A_q = \{6,5\}, \quad B_q = \{6,4\}, \quad C_q = \{2,4\}, \quad D_q = \{2,5\}.$$

Suppose $\ell\mathcal{T}_{q+1}$ has a $\bar{2}$ as its third element: $\ell_{3,q+1} = \bar{2}$. The SJDT demands that this element be slid into the empty space. The column $\mathcal{T}_q$ would then be recalculated:

$$\mathcal{T}_q := G_n(B_q + \{2\}, C_q) = \begin{array}{|c|}\hline \bar{6} \\ \hline \bar{5} \\ \hline \bar{3} \\ \hline 3 \\ \hline 5 \\ \hline \end{array}.$$

The column $\mathcal{T}_{q+1}$ would similarly be recalculated: $\mathcal{T}_{q+1} := F_n(A_{q+1} - \{2\}, D_{q+1}) + \square_3$.

II. The SJDT-algorithm

Input: $\mathcal{T} = \{t_{ij}\}$, an n-admissable tableau of shape λ and columns $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_c$.
Output: an (n–1)-admissable tableau of shape λ.

All inequalities below are with respect to $\mathcal{O}^n$.

Set $\lambda' := \{(i,j): t_{ij} < n\}$.
Denote by $\{A_j, B_j, C_j, D_j\}$ the associated subsets of $\mathcal{T}_j$.

Begin
While $\mathcal{T}$ has an $\bar{n}$ in its first row **do**$_1$,

Remove the right most $\bar{n}$ from its box and let (p,q) be the position of the resulting empty box.

While (p,q) is not an outer corner of λ', **do**$_2$

If$_1$ $r_{p+1,q} \le \ell_{p,q+1}$ or the length of $\mathcal{T}_{q+1}$ is less that p **then**

slide $t_{p+1,q}$ up into the empty box. Set (p,q) := (p+1,q).

Else

If$_2$ $\ell_{p,q+1} = \bar{a}$ (a barred element) **then**

set $B_q^+ := B_q + \{a\}$ and replace $\mathcal{T}_q$ with $G_n(B_q^+, C_q)$.
Set $A_{q+1}^- := A_{q+1} - \{a\}$ and replace $\mathcal{T}_{q+1}$ with $F_n(A_{q+1}^-, D_{q+1}) + \Box_p$.

Else ($\zeta_{p,q+1}$ = **a** an unbarred element),
set $D_q^+ := D_q + \{a\}$ and replace $\mathcal{T}_q$ with $F_n(A_q, D_q^+)$.
Set $C_{q+1}^- := C_{q+1} - \{a\}$ and replace $\mathcal{T}_{q+1}$ with $G_n(B_{q+1}, C_{q+1}^-) + \Box_p$.

end If$_2$.
Set $(p,q) := (p,q+1)$.
end If$_1$.
end do$_2$.
end do$_1$.
Fill any empty box in $\mathcal{T}$ with an $\bar{\mathbf{n}}$.
Output $\mathcal{T}$.
End.

Theorem: *The SJDT algorithm is a well defined, weight preserving, bijection from* $\mathcal{M}^n(\lambda, n)$ *to* $\mathcal{M}^{n-1}(\lambda, n)$.

Most of the proof is checking that the algorithm (and its inverse $(SJDT)^{-1}$) is well defined. This is a long and tedious process, but not difficult. The weight preserving property follows easily after noting that $wt(F_n(A_{\mathcal{T}}, D_{\mathcal{T}})) = wt(F_n(B_{\mathcal{T}}, C_{\mathcal{T}}))$. Checking that the algorithm is bijective is an obvious procedure of verifying that $(SJDT)^{-1}$ reverses the steps of SJDT.

III. Conclusion

Schutzenberger's original jeu de taquin operates on skew-shaped semistandard tableaux. That is, tableaux which have a subshape of empty boxes in their upper left hand corner. A powerful aspect of the jeu de taquin is that it produces the same normal shaped tableau independent of the order in which the empty boxes are removed. In order to apply the SJDT as defined above to a skew-shaped (n-admissible) tableaux one must first fill in the empty boxes with large barred entries, say $\{\overline{n+1}, \overline{n+2}, \ldots\}$, thus prescribing an order of removal. However, it is easy to modify the SJDT to operate on skew shaped tableaux with the boxes left empty. Empirical evidence exists supporting the conjecture that this modified SJDT also operates independent of box removal order.

One of the main applications of the jeu de taquin is in giving a bijective proof of the Littlewood-Richardson rule which solves the tensor product problem for (f.d, polynomial) irreducible gl(n) modules. In 1988, using the algebraic geometric work of Laksmibai and Seshadri, Littelmann [Lit1] gave analogues to the Littlewood-Richardson rule for all the classical Lie algebras. In 1992, he expanded his results to all symmetrizable Kac-Moody algebras [Lit2]. Again, empirical evidence suggests that the SJDT might be used to provide a bijective combinatorial proof of Littelmann result for sp(2n). If the order independence of box removal for SJDT was known this would be easier, but one could probably make do without it.

IV. References

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