Computing Chromatic Polynomials of Oriented Graphs *

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Abstract. Let $G = (V, A)$ be an antisymmetric directed graph. An oriented λ-coloring of G is defined as a mapping c from V to the set of colors $\{1, 2, \ldots, \lambda\}$ satisfying (i) $\forall$ $(x, y) \in A$, $c(x) \neq c(y)$ and (ii) $\forall$ $(x, y), (z, t) \in A$, $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$. The oriented chromatic polynomial of G is then defined as the quantity $\vec{P}(G, \lambda)$, standing for the number of oriented λ-colorings of G. We show in this paper how this polynomial can be computed and prove some properties of it.

Résumé. Soit $G = (V, A)$ un graphe orienté antisymétrique. Une λ-coloration de G est une application c de V dans l'ensemble de couleurs $\{1, 2, \ldots, \lambda\}$ satisfaisant (i) $\forall$ $(x, y) \in A$, $c(x) \neq c(y)$ et (ii) $\forall$ $(x, y), (z, t) \in A$, $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$. Le polynôme chromatique de G est alors défini comme la quantité $\vec{P}(G, \lambda)$, représentant le nombre de λ-colorations de G. Nous montrons dans cet article comment ce polynôme peut être calculé et prouvons certaines propriétés.

1 Introduction

For many years, numerous graph coloring problems have been considered in the literature [8]. However very few of them are concerned with directed graphs. Among these, is the general H-coloring problem, introduced by Maurer, Salomaa and Wood [7] in both the directed and the undirected case. This problem can be stated as follows : let $G = (V, A)$ and $H = (W, B)$ be two directed (resp. undirected) graphs ; we will say that G can be H-colored if there exists a mapping μ from V to W satisfying $(x, y) \in A \Longrightarrow (\mu x, \mu y) \in B$ (resp. $\{x, y\} \in A \Longrightarrow \{\mu x, \mu y\} \in B$). In the undirected case, this notion generalizes the usual graph coloring problem since a graph G is k-colorable if and only if it can be K_k-colored. Many authors have considered the complexity of the H-coloring problem, that is the complexity of the question "is a given graph G H-colorable ?" for some families of graphs H. This question has been recently solved in the undirected case by P. Hell [5] but is still open in the directed case.

The coloring problem we consider in this paper can be viewed as a particular case of the H-coloring problem, obtained by only considering ***antisymmetric*** directed graphs ((x, y) and (y, x) cannot both belong to the set of arcs), also called ***oriented*** graphs. We are essentially interested in answering questions of the type : "Given a family $\mathcal{F}$ of oriented graphs, find a graph H with a minimum number of vertices such that every graph in $\mathcal{F}$ is H-colorable". This question was addressed in the case of planar graphs by Courcelle [4] which studied graphs and properties of graphs definable by monadic second-order logic formulas. Some answers to that problem can also be found in [9, 12].

In this paper we begin the investigation of chromatic polynomials of oriented graphs, which generalize

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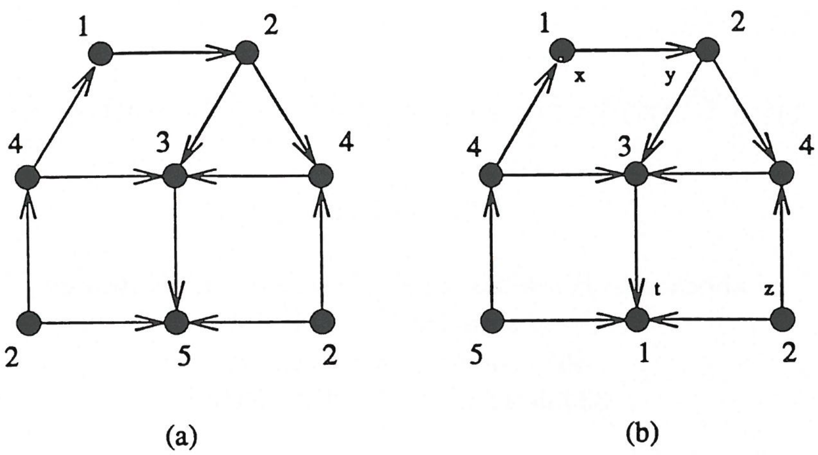


Figure 1: The oriented coloring problem.

to the oriented case the well-known notion of chromatic polynomials, introduced by Birkhoff [2] in the undirected case and studied by many authors (see e.g. [3, 10, 11] for an overview on this subject). In Section 2 we introduce and illustrate the notion of λ-colorings and of chromatic polynomials in the oriented case. In Section 3 a general method for computing the chromatic polynomials is presented. We give in Section 4 some basic properties of these chromatic polynomials leading to some computing shortcuts. In Section 5 we relate the notions of chromatic polynomials in the oriented and the undirected case and propose in Section 6 some directions for future research. Although some references to the theory of chromatic polynomials in the undirected case are made along, this paper is self-contained except for the usual basic notions of graph theory.

2 Definitions

A λ-coloring of an undirected graph $U = (V, E)$ is a mapping c from V to the finite set of colors $C_\lambda = \{1, 2, \ldots, \lambda\}$ such that any two neighbouring vertices are assigned distinct colors. Let now $G = (V, A)$ be an oriented graph. An *oriented λ-coloring* of G is a mapping c from V to the set of colors C_λ satisfying :

$$(i) \quad \forall\ (x, y) \in A,\ c(x) \neq c(y),$$

$$(ii) \quad \forall\ (x, y), (z, t) \in A,\ c(x) = c(t) \Longrightarrow c(y) \neq c(z).$$

Note that any oriented λ-coloring of G is also a λ-coloring of the underlying undirected graph of G and that the converse is not true.

Example 2.1 Figure 1.a shows an oriented graph G and an oriented 5-coloring of G. The mapping c depicted in Figure 1.b is not an oriented 5-coloring since we have $c(x) = 1$ and $c(y) = 2$ on one hand, and $c(t) = 1$ and $c(z) = 2$ on the other hand, which contradicts the condition (ii) above. Note however that the mapping c is a 5-coloring of the corresponding underlying undirected graph.

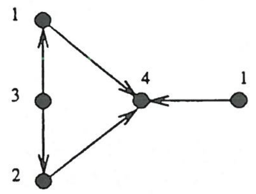
Condition (ii) of our definition essentially states that we are able to "encode" the orientation of a graph by means of some labels (the colors) associated with its vertices, provided that we keep in memory what we call the *color-graph*, which gives the relations between these labels [12]. The color-graph H corresponding to the oriented 5-coloring of the graph G in Figure 1.a is for instance given by the set of arcs $\{(1,2), (2,3), (2,4), (2,5), (3,5), (4,1), (4,3)\}$ (using the terminology of the H-coloring problem, we say that

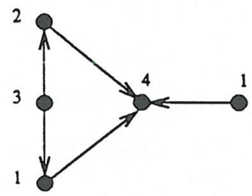
G has been H-colored). Such an encoding may be useful whenever we need to represent some "directed" notion associated with an undirected labelled graph [4, 13].

For any oriented graph G, we define the ***oriented chromatic number*** of G as the minimum value of λ such that G has an oriented λ-coloring. It has been shown in [9] that any oriented planar graph has an oriented chromatic number which does not exceed 80 and no better upper bound is known up to now. The oriented chromatic number of other families of graphs has been studied in [12].

Let us now define $\vec{P}(G,\lambda)$ as the number of oriented λ-colorings of G. This quantity is called the ***oriented chromatic polynomial*** of G since we will see in the next section that it can be expressed as a polynomial in λ. If H is an undirected graph, the chromatic polynomial of H will be denoted by $P(H,\lambda)$.

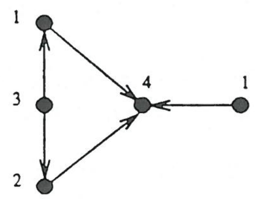
Note from our definition of oriented λ-colorings that the vertices as well as the colors are distinguishable, as is the case for undirected graphs. For further clarity we give some examples illustrating the way the oriented λ-colorings of G are counted. First the two following oriented 4-colorings

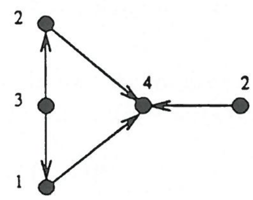




will be considered as distinct, although they are in some sense equivalent since we can obtain any one of them from the other by applying an adequate automorphism of G.

Secondly, the two following oriented 4-colorings

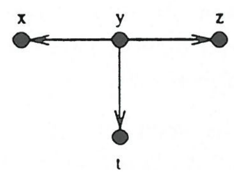




will also be considered as distinct, although we can obtain any one of them from the other by simply permuting the colors 1 and 2. In the sequel, we will speak of oriented λ-coloring ***with color indifference*** whenever we will want to consider as equivalent any two oriented colorings which only differ by a permutation of their colors. Such colorings are then defined as a partition of the set of vertices V into k subsets $V_1,\ldots,V_k$ ($k \leq \lambda$) in such a way that (i) any two vertices belonging to the same V_i are not adjacent and (ii) all the arcs linking vertices of any two subsets V_i and V_j have the same direction.

Let us now illustrate the notion of oriented chromatic polynomials.

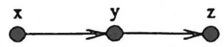
Example 2.2 Consider the following graph G_1 :



We can choose any of the λ colors for the vertex y and any of the remaining colors for x, z and t. Hence, we have :

$$\begin{aligned}\vec{P}(G_1,\lambda) &= \lambda(\lambda-1)^3 \\ &= \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda.\end{aligned}$$

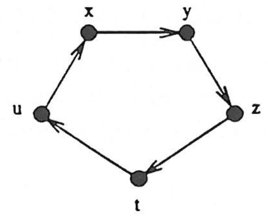
Consider now the following graph G_2 :



No two vertices in G_2 may have the same color. Hence, we have :

$$\begin{aligned}\vec{P}(G_2,\lambda) &= \lambda(\lambda-1)(\lambda-2)\\ &= \lambda^3-3\lambda^2+2\lambda.\end{aligned}$$

Similarly, it is easy to check that in the following graph G_3 :



all the vertices must have distinct colors and we have :

$$\begin{aligned}\vec{P}(G_3,\lambda) &= \lambda(\lambda-1)(\lambda-2)\ (\lambda-3)(\lambda-4)\\ &= \lambda^5-10\lambda^4+35\lambda^3-50\lambda^2+24\lambda.\end{aligned}$$

For any oriented graph $G=(V,A)$, we will denote by $\delta(G)$ the set of all unoriented pairs of vertices in G which must be assigned distinct colors in *any* oriented coloring of G. It is not difficult to see that we have

$$\begin{aligned}\delta(G) = \{\ &\{x,y\}\ /\ x,y\in V,\ (x,y)\in A \text{ or } (y,x)\in A\\ &\text{or } \exists\ z\in V,\ (x,z),(z,y)\in A \text{ or } \exists\ z\in V,\ (y,z),(z,x)\in A\ \}.\end{aligned}$$

Let $\mathcal{P}_2(X)$ denote the set of all two element subsets of X for any set X. Obviously, any oriented graph $G=(V,A)$ satisfying $\delta(G)=\mathcal{P}_2(V)$ has an oriented chromatic polynomial of the form

$$\vec{P}(G,\lambda)=\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1),$$

where n stands for the number of vertices of G. We will denote by $\lambda^{(n)}$ this factorial form. Hence, we obtain :

$$\forall\ G=(V,A) \text{ with } n=|V|,\ \delta(G)=\mathcal{P}_2(V) \implies \vec{P}(G,\lambda)=\lambda^{(n)}.$$

Note that contrary to the undirected case, the tournaments are not the only graphs having this property (see examples G_2 and G_3 above).

Finally, if G is the empty graph on n vertices, that is the graph having n vertices and no arcs, we have :

$$\vec{P}(G,\lambda)=\lambda^n.$$

In the next section, we will show that for any oriented graph G having n vertices, $\vec{P}(G,\lambda)$ can be expressed as a polynomial of degree n in λ. When $\vec{P}(G,\lambda)$ is expressed in terms of the usual monomial basis for polynomials, we will speak about the *usual form* of $\vec{P}(G,\lambda)$. We will also express it as a function of the $\lambda^{(i)}$'s, $1\le i\le n$, and then speak about its *factorial form*.

From now on, we will generally drop the word "oriented" when speaking about coloring, chromatic number or chromatic polynomial, using it only in the contexts where confusion may arise.

3 Computing the factorial form of $\vec{P}(G,\lambda)$

One of the main differences between colorings of undirected and of oriented graphs is that in the case of oriented graphs the coloring constraints are no longer local. In order to decide whether or not a color can be assigned to a vertex x, it is not sufficient to look at the immediate neighbours of x : we must consider the whole graph to check condition (ii) of the definition.

Consequently, computing the chromatic polynomial of an oriented graph will be slightly more complex than in the undirected case. In order to deal with this non locality we must use *constrained polynomials* defined as follows : let $G = (V, A)$ be an oriented graph and X be a subset of $\mathcal{P}_2(V)$; we will denote by $\vec{Q}_X(G, \lambda)$ the number of λ-colorings c of G satisfying

$$\forall \{x, y\} \in X, \ c(x) \neq c(y).$$

Intuitively speaking, the set X represents some additionnal constraints on the colorings under consideration. We will call λ-X-colorings those λ-colorings of G satisfying the set of constraints X. Note that the chromatic polynomial of G can then be expressed as

$$\vec{P}(G, \lambda) \ = \ \vec{Q}_{\emptyset}(G, \lambda).$$

In order to compute the quantity $\vec{Q}_X(G, \lambda)$, we need the following notation : let $G = (V, A)$ and a, b be any two vertices in V such that $\{a, b\} \notin \delta(G)$; we will denote by G^0_{ab} the graph obtained from G by identifying the vertices a and b. More formally, let μ be the mapping from V to $V \setminus \{b\}$ defined by (i) $\mu b = a$ and (ii) $\mu x = x$, $\forall\, x \neq b$. Then, G^0_{ab} is the oriented graph with vertex set $V^0_{ab} = V \setminus \{b\}$ and whose arcs are given by $(\mu x, \mu y) \in A^0_{ab}$ if and only if $(x, y) \in A$. It is not difficult to see that since the pair $\{a, b\}$ does not belong to $\delta(G)$, the orientation of the graph G^0_{ab} thus obtained is still antisymmetric. Similarly, for any subset X of $\mathcal{P}_2(V)$ we will denote by X^0_{ab} the subset of $\mathcal{P}_2(V \setminus \{b\})$ obtained from X by "renaming" b as a (all the constraints on b are transferred to a) and deleting the pair $\{a, b\}$ from X if it appears in X. Then we have :

Theorem 3.1 *For any oriented graph $G = (V, A)$ having n vertices and any subset X of $\mathcal{P}_2(V)$ we have*

$$\vec{Q}_X(G, \lambda) = \begin{cases} \lambda^{(n)} & \text{if } X \cup \delta(G) = \mathcal{P}_2(V), \\ \vec{Q}_{X \cup \{a,b\}}(G, \lambda) \ + \ \vec{Q}_{X^0_{ab}}(G^0_{ab}, \lambda), & \\ \qquad \forall\, \{a, b\} \notin X \cup \delta(G) & \text{otherwise.} \end{cases}$$

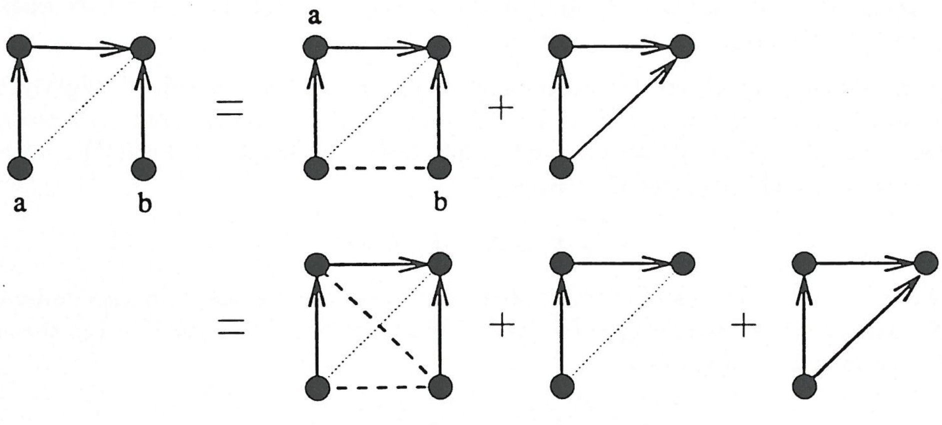
Proof. Let $G = (V, A)$ be an oriented graph and X be any set of constraints. If $X \cup \delta(G) = \mathcal{P}_2(V)$, that means that any pair of vertices in G must be assigned distinct colors and we have $\vec{Q}_X(G, \lambda) = \lambda^{(n)}$. Suppose now that $\{a, b\} \notin X \cup \delta(G)$; then the λ-X-colorings of G can be partitionned into two classes : those in which a and b are assigned distinct colors, called of type 1, and those in which a and b are assigned the same color, called of type 2. It is then not difficult to check that the colorings of type 1 and 2 are respectively counted by $\vec{Q}_{X \cup \{a,b\}}(G, \lambda)$ and $\vec{Q}_{X^0_{ab}}(G^0_{ab}, \lambda)$, and the result follows. □

Note that by applying inductively the formula of Theorem 3.1 we finally obtain an expression of $\vec{P}(G, \lambda) = \vec{Q}_{\emptyset}(G, \lambda)$ in terms of the $\lambda^{(i)}$'s, $1 \leq i \leq n$. Hence, Theorem 3.1 allows to compute the factorial form of $\vec{P}(G, \lambda)$. Such a computation will be called a *chromatic reduction*.

Example 3.2 Figure 2 shows how one can compute the chromatic polynomial of an oriented graph in its factorial form. As usually done, the chromatic polynomial of a graph is denoted by the graph itself. The pairs of the corresponding sets of constraints are joined by dashed lines (initially, the set of constraints is empty). For any graph G, the pairs of vertices belonging to $\delta(G)$ which are not induced by an arc of G are joined by dotted lines. The two vertices used in each chromatic reduction step are denoted by a and b.

By using the formula of Theorem 3.1, we can then obtain :

Proposition 3.3 *For any oriented graph $G = (V, A)$ with n vertices, we have :*
(i) $\vec{P}(G, \lambda)$ is a polynomial of order n in λ,



$$= \lambda^{(4)} + 2\lambda^{(3)} = \lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda$$

Figure 2: Computing the factorial form of the chromatic polynomial.

(ii) the coefficient of λ^n *in* $\vec{P}(G,\lambda)$ *is 1,*
(iii) $\vec{P}(G,\lambda)$ *has no constant term,*
(iv) the coefficient of λ^{n-1} *in* $\vec{P}(G,\lambda)$ *is* $-|\delta(G)|$.

As in the undirected case [10] we can interpret the coefficients of the factorial form of $\vec{P}(G,\lambda)$ as follows :

Theorem 3.4 *The coefficient of* $\lambda^{(r)}$ *in the factorial form of* $\vec{P}(G,\lambda)$ *is the number of ways of coloring* G *using exactly* r *colors with color indifference.*

Proof. Let $N(G,r)$ denote the number of ways of coloring G with exactly r colors, with color indifference. The number of ways of coloring G with exactly r colors but recognizing the different colors is then $r!N(G,r)$ since we have to assign a color to each subset. The number of λ-colorings of G using exactly r colors among the λ available ones is then $\binom{\lambda}{r}r!N(G,r)$. By summing this quantity over all possible r, we obtain :

$$\vec{P}(G,\lambda) = \sum_{r=1}^{\lambda} \binom{\lambda}{r} r!N(G,r) = \sum_{r=1}^{\lambda} \lambda^{(r)} N(G,r),$$

which concludes the proof. □

4 Some shortcuts

In this section, we will give some computationnal tricks, or shortcuts, allowing an easier computation of chromatic polynomials. These results are based on some special decompositions of the graphs under consideration.

In the undirected case, the chromatic polynomial of a graph can always be expressed as the product of the chromatic polynomials of its connected components. In the oriented case, this result no longer holds in general since we have :

Theorem 4.1 *Let* G *be an oriented graph having* k *connected components* $G_1, G_2, \ldots, G_k$. *The following equality*

$$\vec{P}(G,\lambda) = \vec{P}(G_1,\lambda) \times \vec{P}(G_2,\lambda) \times \ldots \times \vec{P}(G_k,\lambda)$$

holds if and only if at most one of these components contains more than one vertex. In this case, if G_1 is the non-singleton component we have :

$$\vec{P}(G,\lambda) = \lambda^{k-1} \times \vec{P}(G_1,\lambda).$$

Proof. Suppose that any component but G_1 contains only one vertex. The $k-1$ vertices of $G_2, G_3, \ldots, G_k$ can be assigned any of the λ colors and we have $\vec{P}(G,\lambda) = \lambda^{k-1} \times \vec{P}(G_1,\lambda)$. Suppose now that $\vec{P}(G,\lambda)$ is expressed as the product of the chromatic polynomials of its components and that two of them, say G_1 and G_2, contain at least two vertices. Then we have at least one arc (x_1,y_1) in G_1 and one arc (x_2,y_2) in G_2 and the λ-colorings of G_1 and G_2 are not independent : we cannot have for instance $c(x_1) = c(y_2) = 1$ and $c(x_2) = c(y_1) = 2$. In other words the product $\vec{P}(G_1,\lambda) \times \vec{P}(G_2,\lambda)$ counts some λ-colorings which are not valid for G, which leads to a contradiction. □

The following facts allow to express the chromatic polynomial of a graph as a function of the chromatic polynomials of some of its subgraphs.

Observation 4.2 *Let $G = (V,A)$ be an oriented graph and x be a vertex of G such that either (i) $\forall\, y \in V \setminus \{x\}$, $(x,y) \in A$ or (ii) $\forall\, y \in V \setminus \{x\}$, $(y,x) \in A$. Then we have*

$$\vec{P}(G,\lambda) = \lambda \times \vec{P}(G \setminus x, \lambda - 1)$$

where $G \setminus x$ stands for the graph obtained from G by deleting the vertex x as well as all the arcs incident to x.

Proof. In any λ-coloring of G, we must choose one color for the vertex x and use the $\lambda - 1$ remaining colors to color the rest of the graph, that is $G \setminus x$. □

Let $P_1 = \sum_{i=1}^{n} a_i \lambda^{(i)}$ and $P_2 = \sum_{j=1}^{m} b_j \lambda^{(j)}$ be two chromatic polynomials expressed in their factorial form. We then define the product $P = P_1 \odot P_2$, obtained by treating the factorials as if they were powers, as :

$$P = P_1 \odot P_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \lambda^{(i+j)}.$$

Then, we have :

Theorem 4.3 *Let $G = (V,A)$ be an oriented graph and V_1, V_2 be two non-empty subsets of V such that $V_1 \cup V_2 = V$ and $\forall\, x_1 \in V_1$, $\forall\, x_2 \in V_2$, $\{x_1,x_2\} \in \delta(G)$. Let G_1 (resp. G_2) denote the subgraph of G induced by V_1 (resp. V_2). Then we have*

$$\vec{P}(G,\lambda) = \vec{P}(G_1,\lambda) \odot \vec{P}(G_2,\lambda).$$

Proof. The proof of this result is a direct translation of the proof of a similar result in the undirected case (see e.g. [10]). The main argument is that the vertices in V_1 have to be assigned colors which are distinct from those assigned to vertices in V_2 and that this property is preserved by the chromatic reduction of Theorem 3.1. □

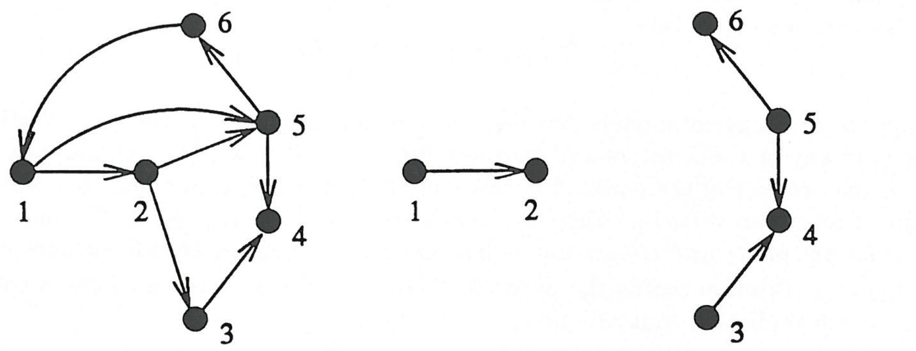
Note that the hypothesis of Theorem 3.1 amount to saying that every two vertices in G are at (directed) distance at most 2.

Example 4.4 Figure 3 shows a graph G satisfying the conditions of Theorem 4.3 and the two corresponding induced subgraphs G_1 and G_2. By using the technique introduced in the previous section, it is easy to check that we have

$$\vec{P}(G_1,\lambda) = \lambda^{(2)} \quad \text{and} \quad \vec{P}(G_2,\lambda) = \lambda^{(4)} + 3\lambda^{(3)} + \lambda^{(2)}.$$

Then, by treating these factorials as if they were powers we obtain

$$\begin{aligned} \vec{P}(G,\lambda) &= \vec{P}(G_1,\lambda) \odot \vec{P}(G_2,\lambda) &= \lambda^{(2)} \odot (\lambda^{(4)} + 3\lambda^{(3)} + \lambda^{(2)}) \\ & &= \lambda^{(6)} + 3\lambda^{(5)} + \lambda^{(4)}. \end{aligned}$$



(a) the graph G (b) the subgraph G_1 (c) the subgraph G_2

Figure 3: Illustration of Theorem 4.3.

5 Expressing $\vec{P}(G, \lambda)$ by means of chromatic polynomials of undirected subgraphs of G

In this section we give a formula relating the chromatic polynomial of an oriented graph G to the chromatic polynomials of some (underlying) undirected subgraphs of G. Let us denote by $Und(G)$ the undirected underlying graph of any oriented graph G (obtained by "forgetting" the orientation of the arcs of G). With any oriented graph $G = (V, A)$ we associate the set $\mathcal{C}(G)$ defined as :

$$\mathcal{C}(G) \;=\; \{\ \{(x,y),(z,t)\} \in A^2,\ (y,z) \notin A,\ (z,y) \notin A,\ (x,t) \notin A \text{ and } (t,x) \notin A\ \}.$$

Roughly speaking, $\mathcal{C}(G)$ represents the set of pairs of arcs which can contradict the fact that a λ-coloring of $Und(G)$ is a valid λ-coloring of G (by means of condition (ii) of the definition). Note that any pair of arcs of the form $\{(x,y),(y,z)\}$ belongs to $\mathcal{C}(G)$.

Now let $Z = \{\{(x_1,y_1),(z_1,t_1)\},\ldots,\{(x_k,y_k),(z_k,t_k)\}\}$ be any subset of $\mathcal{C}(G)$; we will denote by $Id(G,Z)$ the *undirected* graph obtained from $Und(G)$ by identifying the vertices x_i and t_i, y_i and z_i, for any $1 \le i \le k$. Note that such an operation may lead to a graph having some loops, in which case we let its chromatic polynomial to be 0. Note also that we have $Id(G,\emptyset) = Und(G)$.

Then we obtain :

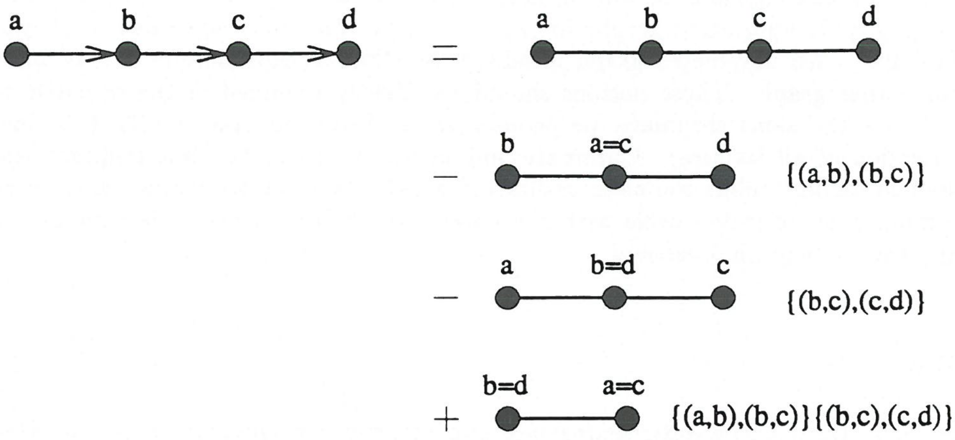
Theorem 5.1 *For any oriented graph G we have*

$$\vec{P}(G,\lambda) \;=\; \sum_{Z \subseteq \mathcal{C}(G)} (-1)^{\#Z} \times P(Id(G,Z),\lambda).$$

Proof. This result is obtained by using a standard inclusion/exclusion argument. □

Example 5.2 Figure 4 illustrates the above Theorem on an oriented graph G with $\mathcal{C}(G) = \{\{(a,b),(b,c)\},\{(b,c),(c,d)\}\}$. The set of considered pairs of arcs is precised besides each corresponding undirected graph.

Note that Theorem 5.1 does not give an efficient way of computing $\vec{P}(G,\lambda)$ but only relates chromatic polynomials of oriented and undirected graphs.



$$= (\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda) - 2(\lambda^3 - 2\lambda^2 + \lambda) + (\lambda^2 - \lambda)$$

$$= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda$$

Figure 4: Illustration of Theorem 5.1.

6 Discussion

In this paper, we have introduced the notion of chromatic polynomials of oriented graphs and shown how these polynomials can be computed. Many questions which are solved in the case of undirected graphs remain open in the oriented case and give natural directions for future work. For instance, we do not have up to now any simple mechanism allowing to derive the usual form of the chromatic polynomial, that is allowing to express the chromatic polynomial of a graph G as a function of the chromatic polynomials of some empty graphs. In the same vein, we do not have any general interpretation of the coefficients in the usual form of $\vec{P}(G, \lambda)$ (only the first two coefficients are interpreted by Proposition 3.3). Unfortunately, it seems that such questions are quite difficult in the oriented case, due to the non locality of the coloring requirements. The property that the coefficients of the chromatic polynomial alternate in sign is for instance no longer satisfied in the oriented case as shown by the following graph :

$$= \lambda^5 - 4\lambda^4 + 5\lambda^3 - \lambda^2 - \lambda$$

Similarly, the absolute values of the coefficients no longer have the unimodality property, as shown by the following graph :

$$= \lambda^8 - 13\lambda^7 + 66\lambda^6 - 155\lambda^5 + 127\lambda^4 + 121\lambda^3 - 281\lambda^2 + 134\lambda$$

It would also be interesting to study the relation between the chromatic polynomial of an undirected graph H and the chromatic polynomials of its orientations (i.e. the graphs obtained from H by giving any orientation to its edges). Note that since any λ-coloring of any orientation $\vec{H}$ of an undirected graph H is a λ-coloring of H itself, we always have $\vec{P}(\vec{H}, \lambda) \leq P(H, \lambda)$. A characterization of the undirected graphs for which one can find one orientation having the same chromatic polynomial remains to be done (it is not difficult to check that complete multipartite graphs belong to that class).

The notions of chromatic equivalence and chromatic uniqueness [1, 6] could also be asked in the oriented case. Two non-isomorphic undirected graphs are said to be chromatically equivalent if they have identical chromatic polynomials. An undirected graph is said to be chromatically unique if it is not chromatically equivalent to any other graph. These notions should be slightly modified in the oriented case since any oriented graph G has the same chromatic polynomial as its "reversed graph" G^{-1} (obtained from G by reversing the direction of all its arcs). Chromatic uniqueness seems to be "less frequent" in the oriented case. As mentionned earlier, unlike complete undirected graphs, oriented tournaments, for instance, are not chromatically unique : the directed cycle with 5 vertices (see the graph G_3 in Section 2) is chromatically equivalent to any tournament on 5 vertices.

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