A decomposition of 2–weak vertex-packing polytopes

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Abstract

Let G be a graph with d vertices. Let $\mathcal{Q}$ be the polytope which is the subset of the unit d-cube satisfying $x_i + x_j \leq 1$ whenever (i, j) is an edge of G. The dilation by 2 of $\mathcal{Q}$, denoted $\mathcal{P}$, is a polytope with integral vertices. We triangulate $\mathcal{P}$ with lattice simplices of minimal volume and label the maximal simplices with elements of the hyperoctahedral group B_d. This labeling gives rise to a shelling of the triangulation $\widehat{\mathcal{P}}$ of $\mathcal{P}$, and the h–vector of $\widehat{\mathcal{P}}$ (and the Ehrhart h^*–vector of $\mathcal{P}$) can be computed as a descent statistic on a subset of B_d determined by G. Recursive formulas are given for computing the volume of $\mathcal{P}$ and the h–vector of $\widehat{\mathcal{P}}$.

Soit G un graphe à d sommets. Soit $\mathcal{Q}$ le polytope, sous-ensemble du cube unité de l'espace à d dimensions, défini par les inégalités $x_i + x_j \leq 1$ pour tout couple (i, j) de sommets adjacents dans G. Les sommets de la dilatation de $\mathcal{Q}$ par multiplication par 2, que l'on appelle $\mathcal{P}$, ont tous des coordonnées entières. On fait une triangulation $\widehat{\mathcal{P}}$ de $\mathcal{P}$ par des simplexes dont les sommets appartiennent au treillis entier dont le volume est minimal. On attache aux simplexes maximaux des étiquettes qui sont des éléments du groupe hyperoctaèdral B_d. Cet étiquetage produit un effeuillage de la triangulation $\widehat{\mathcal{P}}$ et le vecteur h de $\widehat{\mathcal{P}}$ (ainsi que le vecteur h^* de Erhart associé à $\mathcal{P}$) peut être calculé en termes du paramètre nombre de descentes sur un sous-ensemble de B_d qui dépend de G. On donne des formules récursives pour le calcul du volume de $\mathcal{P}$ et du vecteur h de $\widehat{\mathcal{P}}$.

1 Introduction

Let G be a loopless graph, d the number of vertices in G, and label the vertices of G by the integers $1, 2, \ldots, d$. The *extended 2–weak vertex-packing polytope* $\mathcal{P}(G)$ of G is defined by

$$0 \leq x_i \leq 2, \quad 1 \leq i \leq d, \tag{1}$$

$$x_i + x_j \leq 2, \quad \text{if } (i, j) \text{ is an edge of } G. \tag{2}$$

The polytopes $\mathcal{P}(G)$ are special cases of *k–weak vertex-packing polytopes*, which in turn are approximations of vertex-packing polytopes, which have been studied from the mathematical programming point of view (see, e.g., [5] and [2]). This paper deals with the combinatorial structure of $\mathcal{P}(G)$. We triangulate $\mathcal{P}(G)$ in a certain systematic way and label the maximal simplices in the triangulation, which we denote by $\widehat{\mathcal{P}}$, with elements of the *hyperoctahedral group* B_d. This labeling allows us to *shell* $\widehat{\mathcal{P}}$ in such a way that we can compute the *h–polynomial* of $\mathcal{P}(G)$ as a *descent statistic* on a subset of B_d determined by G. Moreover, the triangulation is such that its h–polynomial equals the *Ehrhart* h^**–polynomial* of $\mathcal{P}(G)$. This gives a decomposition of $\widehat{\mathcal{P}}$ into maximal simplices, whose intersections with

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other maximal simplices we can describe explicitly. A recursive formula for the h–polynomial of $\widehat{\mathcal{P}}(G)$ can be also be given. A simplified version of this recursion yields a simple recursive formula for computing the volume of $\mathcal{P}(G)$.

2 Preliminaries

2.1 Ehrhart polynomials

Let $\mathcal{P}$ be a d–dimensional polytope (or simplicial complex (see section 2.2)) in $\mathbf{R}^n$ with integral (or lattice) vertices, i.e. $v_i \in \mathbf{Z}^n$ for all vertices v_i of $\mathcal{P}$. For $k \in \mathbf{N}$ let $k\mathcal{P} = \{kx \mid x \in \mathcal{P}\}$, i.e. $k\mathcal{P}$ is the (lattice) polytope obtained by dilating $\mathcal{P}$ by a factor of k.

For $k \in \mathbf{N}$ define the function $i(\mathcal{P}, k) = \#\{x \in \mathbf{R}^n \mid x \in k\mathcal{P} \cap \mathbf{Z}^n\}$. Thus, $i(\mathcal{P}, k)$ is the number of lattice points contained in $k\mathcal{P}$. By Cor. 4.6.28 in [7], $i(\mathcal{P}, k)$ is a polynomial in k of degree d, called the *Ehrhart polynomial* of $\mathcal{P}$. Now define the generating function $E(\mathcal{P}, t) = \sum_{k \geq 0} i(\mathcal{P}, k)t^k$. By Thm. 2.1 in [6], we have $E(\mathcal{P}, t) = \frac{h^*(\mathcal{P},t)}{(1-t)^{d+1}}$, where $h^*(\mathcal{P}, t)$ is a polynomial of degree at most d with non-negative integer coefficients, called the *Ehrhart h^*-polynomial of $\mathcal{P}$.*

2.2 Simplicial complexes

An *abstract simplicial complex* is a nonempty collection K of sets such that if $F \in K$ and $G \subset F$ then $G \in K$. An element of K is called a *face* of K. We will be mostly concerned with the *geometric realization* of simplicial complexes (for definitions and basic properties, see [4]) and we will, by abuse of notation, *not* distinguish between a simplicial complex and its geometric realization.

A simplicial complex K is *pure* if all its maximal faces have the same dimension $d = \dim(K)$. If K is a pure simplicial complex of dimension d, then a *facet* of K is a d–face, i.e. a d–dimensional face, of K. When a complex K triangulates a polytope $\mathcal{P}$, the facets of K are d–dimensional, but the facets of $\mathcal{P}$ have dimension $d-1$.

The *h–vector* $h(K) = (h_0, h_1, \ldots, h_d)$ of a simplicial complex K of dimension d-1 is defined as follows: Let $f_i = f_i(K)$ be the number of i–dimensional faces in K, where we set $f_{-1} = 1$, and define $h(K) = (h_0, h_1, \ldots, h_d)$ by setting

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}. \tag{3}$$

We define the *h-polynomial* $h(K, t)$ of K by $h(K, t) = h_0 + h_1 t + \cdots + h_d t^d$.

Let K be a pure simplicial lattice complex of dimension d. If all facets of K have volume $1/d!$ (see section 2.4) we say that K is *primitively triangulated.* The following theorem is essentially a consequence of Cor. 2.5 in [6], whose conclusion is expressed in greater generality in Thm. 2 in [1].

Theorem 1 *Suppose K is a primitively triangulated simplicial lattice complex. Then $h^*(K,t) = h(K,t)$, where $h^*(K,t)$ is the Ehrhart h^*-polynomial of K.* ■

For certain pure simplicial complexes K the coefficients of $h(K,t)$ can be interpreted in a way that partitions the facets of K according to how they intersect other facets.

Definition 2 *Let K be a finite pure simplicial complex of dimension d. If F is a face of K, let $\bar{F}$ be the complex consisting of F and all its faces. An ordering $F_1, F_2, \ldots, F_n$ of the facets of K is called a* shelling *if, for all k with $1 < k \leq n$, $\bar{F}_k \cap \bigcup_{i=1}^{k-1} \bar{F}_i$ is a pure complex of dimension (d-1). A complex K is said to be* shellable *if there exists a shelling of K.*

As it turns out, the h-vector of a shellable complex can be computed from the shelling. The following theorem is essentially due to McMullen [3].

Theorem 3 *Let $F_1, F_2, \ldots, F_n$ be a shelling of a d-dimensional complex K and let $c(k)$ be the number of $(d-1)$-faces of $\bar{F}_k$ contained in $\bigcup_{i<k} \bar{F}_i$. Then $h(K,t) = \sum_{i=1}^{n} t^{c(i)}$.* ■

Thus, given a shelling $F_1, F_2, \ldots, F_n$ of a simplicial complex K, we can compute the h-polynomial $h(K,t)$ of K via Theorem 3. That is, the k-th coefficient of $h(K,t)$ equals the number of F_i with $c(i) = k$.

If K is a simplicial complex and p a vertex not in K, then the *cone with apex p over K* (or *with base K*), denoted $p * K$, is the simplicial complex whose i-faces are the i-faces of K and $\{p \cup f \mid f \text{ an } (i-1)\text{-face of } K\}$. Geometrically, a cone can be defined as follows. If K is a $(d-1)$-dimensional simplicial (or polytopal) complex in $\mathbf{R}^n$ and p is a point in $\mathbf{R}^n$ such that each ray emanating from p intersects K in at most one point, then the cone $p * K$ consists of K and p and the new i-faces, for $1 \leq i \leq d$, obtained by taking, for each $(i-1)$-face f in K, the union of all line segments connecting p to points in f.

Theorem 4 *Suppose the simplicial complex K is a cone with apex p over B, i.e. $K = p * B$. Then $h(K,t) = h(B,t)$.* ■

2.3 The hyperoctahedral group

We represent the elements of the hyperoctahedral group B_d by *signed permutation words*, i.e. ordinary permutations in which each letter has a sign. To simplify the notation, we write a_i for $+a_i$ and $\bar{a}_i$ for $-a_i$. For example, $B_2 = \{12, 21, \bar{1}2, \bar{2}1, 1\bar{2}, 2\bar{1}, \bar{1}\bar{2}, \bar{2}\bar{1}\}$.

We refer to the elements of B_d simply as permutations. We regard the letters in a permutation as integers and order them as such, i.e. $\cdots \bar{3} < \bar{2} < \bar{1} < 0 < 1 < 2 < 3 \cdots$.

Definition 5 *A* descent *in $\pi \in B_d$ is an $i \in [d]$ such that one of the following holds:*

1) $i < d$ and $a_i > a_{i+1}$,

2) $i = d$ and $a_d > 0$.

For any subset S of B_d, the descent polynomial of S *is $D(S,t) := \sum_{\pi \in S} t^{\mathrm{des}(\pi)}$, where* $\mathrm{des}(\pi)$ *is the number of descents in π.*

For example, the descents of $2\bar{3}\bar{4}1$ are 1,2 and 4, so $\mathrm{des}(2\bar{3}\bar{4}1) = 3$. If $S = \{\bar{3}\bar{2}\bar{1}, 1\bar{2}\bar{3}, 213\}$ then $D(S,t) = 1 + 2t^2$.

2.4 Volumes

When we talk about volume in $\mathbf{R}^d$ we mean the usual d–dimensional volume, which we denote $\mathrm{vol}_d(\cdot)$. If S is a subset of a d–dimensional coordinate subspace of $\mathbf{R}^n$, then by $\mathrm{vol}_d(S)$ we mean the volume of S in that subspace. If S is a union of such subsets S_i then by $\mathrm{vol}_d(S)$ we mean the sum of the volumes of the S_i. In particular, a polytope $\mathcal{P}$ of dimension less than d has $\mathrm{vol}_d(\mathcal{P}) = 0$. For convenience, we make the following definition.

Definition 6 *If $\mathcal{P}$ is a d-dimensional polytope or simplicial complex in $\mathbf{R}^n$ such that* $\mathrm{vol}_d(\mathcal{P})$ *is defined, then the* normalized volume *of $\mathcal{P}$ is* $\mathrm{Nvol}(\mathcal{P}) := d! \cdot \mathrm{vol}_d(\mathcal{P})$.

Hence, for any polytope (or simplicial complex) $\mathcal{P}$ of positive dimension, $\mathrm{Nvol}(\mathcal{P})$ is positive. The rationale behind this definition is that the least volume a lattice d-simplex can have is $1/d!$. In particular, the normalized volume of a primitively triangulated complex equals its number of maximal simplices.

3 Main Theorems

Proposition 7 *Let p be a point in the polytope $\mathcal{P}$ and let $\mathcal{P}_{\mathrm{p}}$ be the union of those facets of $\mathcal{P}$ which do not contain p. Then $\mathcal{P}$ is a cone with apex p over $\mathcal{P}_{\mathrm{p}}$.* ■

Throughout, if G is a graph, $\mathcal{P}(G)$ is the extended 2-weak vertex-packing polytope of G. By definiton, $\mathcal{P}(G)$ is a subset of $2C^d$, the dilation of the unit d–cube by 2.

Theorem 8 *Let G be a graph and let $\mathcal{P}'(G) = \mathcal{P}(G) \cap \partial(2C^d)$, i.e. $\mathcal{P}'(G)$ is the union of those facets of $\mathcal{P}(G)$ which lie on the boundary of $2C^d$. Let $\mathrm{p} = (1,1,\ldots,1)$. Then $\mathcal{P}(G) = \mathrm{p} * \mathcal{P}'(G)$.* ■

Theorem 9 *Let $\mathrm{v} = (v_1, v_2, \ldots, v_d)$ be a point in $\mathcal{P}$ with integral coordinates and let $S = \{i \in [d] \mid v_i = 1\}$. Let G_S be the subgraph of G induced by S. Then v is a vertex of $\mathcal{P}$* iff *each connected component of G_S contains an odd cycle (or $S = \emptyset$).* ■

To triangulate $\mathcal{P}(G)$ we first triangulate $2C^d$ in the following way. $2C^d$ is embedded in $\mathbf{R}^d$ so that its vertices are all points each of whose coordinates are either 0 or 2. In particular, its center (of symmetry) is the point $\mathbf{p} = (1, 1, ..., 1)$. We subdivide $2C^d$ into the 2^d unit cubes all of whose vertices are lattice points. Each of these small cubes contains $\mathbf{p}$ and a unique vertex which is a vertex of $2C^d$. We label each small cube by that vertex of $2C^d$ which it contains. As an example, the standard unit d-cube is labeled by $\mathbf{0} = (0, 0, \ldots, 0)$ and denoted $c_{\mathbf{0}}$.

Next, we triangulate each of these small cubes. Let $c_{\mathbf{z}}$ be the small cube labeled by $\mathbf{z}$. Then every maximal simplex in the triangulation of $c_{\mathbf{z}}$ contains $\mathbf{p}$ and $\mathbf{z}$ and is defined as the convex hull of a *path along edges of* $c_{\mathbf{z}}$ *from* $\mathbf{p}$ *to* $\mathbf{z}$, as follows.

Let $\mathbf{p}_0 = \mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_d = \mathbf{z}$ be a sequence of vertices of $c_{\mathbf{z}}$ such that $\mathbf{p}_k = \mathbf{p}_{k-1} \pm e_j$ where e_j is the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in the j–th place and 0's elsewhere. It follows that in the sequence of $\mathbf{p}_k$'s the i–th coordinate must change precisely once, from 1 to z_i, because we start out from $p = (1, 1, \ldots, 1)$ and $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ is a vertex of $2C^d$, so $z_i \in \{0, 2\}$ for each i.

The points $\mathbf{p}_i$ are geometrically independent and thus they are the vertices of a d–dimensional simplex, namely their convex hull. It is also clear that the intersection of any two maximal simplices is the convex hull of their common vertices, so this is indeed a simplicial complex.

Such a sequence of vertices defining a maximal simplex can be coded by a permutation in the hyperoctahedral group B_d. Namely, we define $\pi = a_1 a_2 \cdots a_d$ by setting $a_i = k$ if $\mathbf{p}_i - \mathbf{p}_{i-1} = e_k$ and $a_i = -k$ if $\mathbf{p}_i - \mathbf{p}_{i-1} = -e_k$. For example, the sequence $(1,1,1), (0,1,1), (0,1,2), (0,0,2)$ of points in $c_{(0,0,2)}$ corresponds to the permutation $\bar{1}3\bar{2}$. Conversely, every $\pi \in B_d$ determines a unique path from $\mathbf{p}$ to a vertex $\mathbf{z}$ of $2C^d$ and hence a unique d–simplex, which we denote by σ_π, contained in $c_{\mathbf{z}}$. The number of distinct paths from $\mathbf{p}$ to $\mathbf{z}$ is $d!$, and the following lemma is now immediate.

Lemma 10 *Let $\{\pi_i \mid 1 \le i \le d!\}$ be the permutations labeling the maximal simplices in a cube $c_{\mathbf{z}}$. Then each integer k in $[d]$ appears with the* same sign *in every π_i. More precisely, the sign of $k \in [d]$ in such a permutation is $+$ or $-$ according as the k–th coordinate of $\mathbf{z}$ is 2 or 0. Conversely, if each $k \in [d]$ appears with the same sign in two permutations π and τ, then σ_π and σ_τ belong to the same cube $c_{\mathbf{z}}$.* ■

For example, the paths in the cube $c_{(0,2)}$ are $(1,1) \to (1,2) \to (0,2)$ and $(1,1) \to (0,1) \to (0,2)$, corresponding to the permutations $2\bar{1}$ and $\bar{1}2$, respectively.

Proposition 11 *The collection $\{\sigma_\pi \mid \pi \in B_d\}$ covers $2C^d$. Any two of these simplices are isometric, in particular each has volume $1/d!$ and hence* $\mathrm{Nvol}(\sigma_\pi) = 1$ *for each π.* ■

Corollary 12 *The triangulation $\widehat{\mathcal{P}}(G)$ is primitive. Thus,* $\mathrm{Nvol}(\mathcal{P}(G)) = \#\Pi(G)$. ■

Thus the collection $\{\sigma_\pi \mid \pi \in B_d\}$ triangulates $2C^d$. We denote this triangulation by $\widehat{2C^d}$. We can now give a succinct characterization of the permutations corresponding to the maximal simplices of $\widehat{2C^d}$ contained in $\mathcal{P}$. First a definition.

Definition 13 *Let G be a graph. The set of* permissible permutations with respect to G *is* $\Pi(G) = \{\pi \in B_d \mid \sigma_\pi \subset \mathcal{P}(G)\}$. *A permutation π is* permissible w.r.t. G *if* $\pi \in \Pi(G)$.

Theorem 14 *A permutation $\pi \in B_d$ is permissible w.r.t. G if and only if it satisfies the following condition:*

If (i, j) is an edge in G and $+i$ appears in π, then $-j$ must precede $+i$ in π. ■

Proposition 15 *Let σ_π be a maximal simplex in c_Z. If $\mathcal{P}$ intersects the interior of σ_π, then $\sigma_\pi \subset \mathcal{P}$. Hence, $\hat{\mathcal{P}} := \widehat{2C^d} \cap \mathcal{P}$ is a triangulation of $\mathcal{P}$.* ■

For the remainder of this section, fix a graph G and let $\mathcal{P}$ denote its extended 2–weak vertex-packing polytope and $\hat{\mathcal{P}}$ the triangulation of $\mathcal{P}$ described above.

Our goal is to find a shelling of $\hat{\mathcal{P}}$. To that end, we order the permutations in B_d *lexicographically*, i.e. a permutation $\pi = a_1a_2\cdots a_d$ precedes $\tau = b_1b_2\cdots b_d$ if $a_i < b_i$ for the first i at which π and τ differ. Abusing notation, we use $<$ to denote this ordering of the elements of B_d. For example, $\bar{2}3\bar{1} < 3\bar{2}\bar{1}$ and $23\bar{1} < 231$.

We will show that the ordering of maximal simplices in $\hat{\mathcal{P}}$ induced by the lexicographic ordering of their corresponding permutations is a shelling of $\hat{\mathcal{P}}$. Before proving that, we need a definition and a lemma.

Definition 16 *Let σ_π and σ_τ be two maximal simplices in $\mathcal{P}$, and $d = dim(\mathcal{P})$. We say that σ_π and σ_τ* intersect maximally *if they have a $(d-1)$–face in common.*

Lemma 17 *Suppose $\sigma_\pi \subset c_Z \cap \hat{\mathcal{P}}$, where $\pi = a_1a_2\cdots a_d$, and suppose that i is a descent in π. If i is an* internal *descent in π, i.e. $a_i > a_{i+1}$, for some $i \le d-1$, then $\sigma_{\pi'} \subset c_Z \cap \hat{\mathcal{P}}$, where $\pi' = a_1a_2\cdots a_{i+1}a_i\cdots a_d$. If $i = d$, i.e. $a_d > 0$, then $\sigma_{\pi'} \subset \hat{\mathcal{P}}$, where $\pi' = a_1a_2\cdots -a_d$. In either case, $\pi' < \pi$ and σ_π and $\sigma_{\pi'}$ intersect maximally. Moreover, if two maximal simplices σ_π and $\sigma_{\pi'}$ in $\hat{\mathcal{P}}$ intersect maximally, then π and π' either differ only by a single transposition or only by the sign of their last letter.* ■

Theorem 18 *Order the maximal simplices in $\hat{\mathcal{P}}$ so that σ_τ precedes σ_π if $\tau < \pi$. This ordering is a shelling of $\hat{\mathcal{P}}$.*

Proof: Let σ_π be a maximal simplex in $\hat{\mathcal{P}}$. If π is the (lexicographically) first permutation in $\Pi(G)$ then there is nothing to prove. Otherwise, we must show that $\sigma_\pi \cap \bigcup_{\tau<\pi} \sigma_\tau$ is a nonempty union of $(d-1)$–faces of σ_π. It suffices to show that if σ_π intersects a maximal simplex $\sigma_\tau \subset \widehat{\mathcal{P}}$ and σ_τ precedes σ_π, then $\sigma_\pi \cap \sigma_\tau$ is contained in some $(d-1)$–face f of σ_π such that $f = \sigma_\pi \cap \sigma_{\pi'}$ for some $\sigma_{\pi'} \subset \widehat{\mathcal{P}}$ with $\sigma_{\pi'}$ preceding σ_π.

Suppose $\sigma_\pi, \sigma_\tau \subset \widehat{\mathcal{P}}$ and that σ_τ precedes σ_τ, so $\tau < \pi$. Let i be the first place where π and τ differ. If $i = d$ then σ_π and σ_τ intersect maximally, by Lemma 17, and we are done. Assume therefore that $i < d$. Let $\pi = a_1a_2\cdots a_d$ and $\tau = a_1a_2\cdots a_{i-1}b_i\cdots b_d$. Let k be the first descent in π after $i-1$. Such a k must exist, because otherwise we would have $a_i < a_{i+1} < \cdots < a_d < 0$ so that π was the first permutation in B_d beginning with $a_1a_2\cdots a_{i-1}$, contradicting $\tau < \pi$.

Let $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_d$ be the sequence of points defining σ_π.We claim that $\mathbf{p}_k \notin \sigma_\tau$. If σ_τ did contain $\mathbf{p}_k$ then we would have $\{a_1, a_2, \ldots, a_k\} = \{b_1, b_2, \ldots, b_k\}$, in particular $\{a_i, a_{i+1}, \ldots, a_k\} = \{b_i, b_{i+1}, \ldots, b_k\}$, so $k > i$. But then, since k was the first descent in π after $i-1$, so that $a_i < a_{i+1} < \cdots < a_k$, we must have $b_i > a_i$, contradicting the assumption $\tau < \pi$, so $\mathbf{p}_k \notin \sigma_\tau$.

If $k < d$, let $\pi' = a_1\cdots a_{k-1}a_{k+1}a_k\cdots a_d$. Then $\pi' < \pi$ and $\sigma_\pi \cap \sigma_{\pi'}$ is the convex hull of $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{k-1}, \mathbf{p}_{k+1}, \ldots, \mathbf{p}_d$. By Lemma 17, σ_π and $\sigma_{\pi'}$ intersect maximally and $\sigma_{\pi'} \subset \widehat{\mathcal{P}}$. Moreover, since $\mathbf{p}_k \notin \sigma_\tau$, $\sigma_\pi \cap \sigma_\tau \subset \sigma_\pi \cap \sigma_{\pi'}$, as desired.

If $k = d$, then $a_d > 0$. Let $\pi' = a_1a_2\cdots\bar{a}_d$. Then $\pi' < \pi$ and, by Lemma 17, σ_π and $\sigma_{\pi'}$ intersect maximally and $\sigma_{\pi'} \subset \widehat{\mathcal{P}}$. Since $\mathbf{p}_d \notin \sigma_\tau$, we have $\sigma_\pi \cap \sigma_\tau \subset \sigma_\pi \cap \sigma_{\pi'}$, as desired. ∎

Theorem 19 *The h-polynomial of $\widehat{\mathcal{P}}$ equals the descent polynomial of the set of permissible permutations with respect to G. That is, $h(\widehat{\mathcal{P}}, t) = D(\Pi(G), t)$ and hence $h^*(\mathcal{P}, t) = D(\Pi(G), t)$.*

Proof: We need to show that for each descent in $\pi \in \Pi(G)$ there is a unique maximal simplex $\sigma_\tau \in \widehat{\mathcal{P}}$ such that σ_π and σ_τ intersect in a $(d-1)$–face of each and such that $\tau < \pi$. First suppose that i is an internal descent in π, i.e. $1 \le i \le d-1$ and let $\pi = a_1a_2\cdots a_d$, so $a_i > a_{i+1}$. By Lemma 17, two maximal simplices σ_π and σ_τ in the same cube $c_{\mathbf{z}}$ intersect maximally if and only if π and τ differ by a single transposition. Let $\tau = a_1a_2\cdots a_{i+1}a_i\cdots a_d$. Then τ precedes π, $\sigma_\tau \subset \widehat{\mathcal{P}}$ and σ_π and σ_τ intersect maximally. Conversely, if σ_π and σ_τ in $c_{\mathbf{z}}$ intersect maximally then they differ by a single transposition and if $\tau < \pi$ then π has a descent at the transposition distinguishing it from τ.

The only other maximal simplices σ_π can intersect maximally are those belonging to other cubes than $c_{\mathbf{z}}$. By Lemma 17, if σ_τ is such a simplex and $\pi = a_1a_2\cdots a_d$, then $\tau = a_1a_2\cdots\bar{a}_d$, so, for τ to precede π, we must have $a_d > 0$, i.e. d is a descent in π. Conversely, if d is a descent in π then $a_d > 0$, so if $\tau = a_1a_2\cdots\bar{a}_d$ then $\tau < \pi$, $\sigma_\tau \subset \widehat{\mathcal{P}}$ and σ_π and σ_τ intersect maximally. ∎

4 Applications

Definition 20 $\mathcal{B} := \mathcal{P} \cap \partial(2C^d)$ *and* $\widehat{\mathcal{B}} := \widehat{\mathcal{P}} \cap \partial(2C^d)$.

Theorem 21 $h(\widehat{\mathcal{P}}, t) = h(\widehat{\mathcal{B}}, t)$. *Hence,* $h^*(\mathcal{P}, t) = h(\widehat{\mathcal{B}}, t)$.

Proof: $\widehat{\mathcal{P}}$ is a cone over $\widehat{\mathcal{B}}$, which yields the equality of h–polynomials, by Theorem 4. The equality $h^*(\mathcal{P}, t) = h(\widehat{\mathcal{B}}, t)$ is then implied by Theorem 1 and the fact that $\mathcal{P}$ (and hence $\mathcal{B}$) is primitively triangulated. ∎

Corollary 22 $\mathrm{vol}_d(\mathcal{P}) = \mathrm{vol}_{d-1}(\mathcal{B})$. *Equivalently,* $\mathrm{Nvol}(\mathcal{P}) = \mathrm{Nvol}(\mathcal{B})$.

Proof: $\mathrm{Nvol}(\mathcal{P})$ equals the number of maximal simplices in $\widehat{\mathcal{P}}$, which in turn equals the number of maximal simplices in $\widehat{\mathcal{B}}$, since $\widehat{\mathcal{P}}$ is a cone over $\widehat{\mathcal{B}}$. ∎

4.1 Volumes

Corollary 22 yields a recursive formula for the volume of $\mathcal{P}$, because each facet of $\mathcal{B}$ (i.e. a facet of $\mathcal{P}$ contained in $\mathcal{P}$) is an extended 2–weak vertex-packing polytope. More precisely, the facet of $\mathcal{B}$ obtained by setting $x_i = 0$ (which we denote $\mathcal{B}_{x_i=0}$) is the extended 2–weak vertex-packing polytope of the graph obtained by removing x_i from G. If x_i is an isolated vertex of G then $\mathcal{B}_{x_i=2}$ is isometric to $\mathcal{B}_{x_i=0}$ (since then $\mathcal{P} = \mathcal{P}_{x_i=0} \times [0,2]$), but otherwise $\mathcal{B}_{x_i=2}$ has dimension less than $d-1$ and thus $\mathrm{vol}_{d-1}(\mathcal{B}_{x_i=2}) = 0$.

If $d = a_1 + \cdots + a_k$, let $\binom{d}{a_1,\ldots,a_k} = \frac{d!}{a_1!\cdots a_k!}$. Abusing notation, we will write $\mathrm{Nvol}(G)$ instead of $\mathrm{Nvol}(\mathcal{P}(G))$, where G is a graph and $\mathcal{P}(G)$ its extended 2–weak vertex-packing polytope.

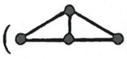
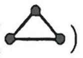
Theorem 23 *Let* $C_1, C_2, \ldots, C_k$ *be the connected components of* G, *with* $a_i = \#C_i$ *for each* i, *and* $d = \#G$. *Then* $\mathrm{Nvol}(G) = \binom{d}{a_1,a_2,\ldots,a_k} \prod_{i=1}^k \mathrm{Nvol}(C_i)$. *In particular, if* G *has an isolated vertex* x *and* G_x *is the graph obtained by removing* x *from* G, *then* $\mathrm{Nvol}(G) = 2 \cdot d \cdot \mathrm{Nvol}(G_x)$. ∎

Theorem 24 *Let* G *be a graph without isolated vertex,* $\#G = d$, *and let* G_x *denote the graph obtained by removing* x *from* G. *Then*

$$\mathrm{Nvol}(G) = \sum_{x \in G} \mathrm{Nvol}(G_x).$$ ∎

We now give a few examples of how to use the recurrence of Theorems 23 and 24 to compute the volume of extended 2–weak vertex-packing polytopes. To get the recursion off the ground, observe that if G consists of a single vertex, then $\mathcal{P}(G) = [0,2] \subset \mathbf{R}$, so $\mathrm{Nvol}(G) = 2$.

Example 25 $\operatorname{Nvol}(\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet) = 2\cdot\operatorname{Nvol}(\bullet\!\!-\!\!\bullet) + \operatorname{Nvol}(\bullet \quad \bullet) =$
$2\cdot 2\cdot\operatorname{Nvol}(\,\bullet\,) + \binom{2}{1}\cdot(\operatorname{Nvol}(\,\bullet\,))^2 = 8 + 2\cdot 2^2 = 16.$

Example 26 $\operatorname{Nvol}($  $) = 2\cdot\operatorname{Nvol}(\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet) + 2\cdot\operatorname{Nvol}($  $) =$
$2\cdot 16 + 6\cdot\operatorname{Nvol}(\,\bullet\!\!-\!\!\bullet) = 32 + 6\cdot 4 = 56.$

Example 27 Let K_d be the complete graph on d vertices. Then Theorem 24 gives $\operatorname{Nvol}(K_d) = d\cdot\operatorname{Nvol}(K_{d-1}) = \cdots = d!\cdot\operatorname{Nvol}(\,\bullet\,) = 2d!$.

Example 28 If G_d is the graph $\bullet\!\!-\!\!\bullet\!\!- \;\cdots\; -\!\!\bullet\!\!-\!\!\bullet$ with d vertices, so G_d is the comparability graph of the *fence poset* on elements $x_1,\ldots,x_d$ (with relations $x_1 < x_2 > x_3 < x_4 > \cdots$), then it is well known (see [8]) that the volume of the 2-weak vertex-packing polytope of G_d (of which $\mathcal{P}(G)$ is the dilation by 2) is given by the d-th coefficient of the Taylor series of $\tan x + \sec x$. We can compute the corresponding result for $\mathcal{P}(G_d)$ in the following way. Using Theorems 23 and 24, we get this recurrence for $A_d = \operatorname{Nvol}(G_d)$: For $d \geq 1, A_{d+1} = \sum_{i=0}^{d}\binom{d}{i}A_i\cdot A_{d-i}$ and hence $\sum_{d\geq 1}A_{d+1}\frac{x^d}{d!} = \left(\sum_{d\geq 0}A_d\frac{x^d}{d!}\right)^2$. Setting $F(x) = \sum_{d\geq 0}A_d\frac{x^d}{d!}$ yields $F'(x) - 1 = (F(x))^2$, which, together with $F(0) = 1$, has the unique solution $F(x) = \tan(2x) + \sec(2x)$.

This means, by Theorem 3.37 in [9], that $\operatorname{Nvol}(G_d)$ equals the number of *weakly alternating* permutations in B_d, i.e. those permutations for which $i \in \{1,2,\ldots,d-1\}$ is a descent if and only if i is odd. However, this set of permutations does not coincide with $\Pi(G_d)$. It might be interesting to find a bijection between these two sets of permutations in B_d.

4.2 The h–polynomial of $\widehat{\mathcal{P}}$

Example 29 Let K_d be the complete graph on d vertices. Then $\Pi(K_d)$ consists of all permutations $\pi = a_1a_2\cdots a_d \in B_d$ such that $a_i < 0$ for all i except, perhaps, for $i = d$. Let Π_- be the set of permutations in $\Pi(G)$ all of whose letters are negative, and let Π_+ be the set of those permutations in $\Pi(G)$ whose last letter is positive but all others negative. It is easy to see that $D(\Pi_-, t)$ equals the *usual* descent polynomial of the symmetric group S_d, consisting of all permutations of the letters $\{1,2,\ldots,d\}$ (no signs involved and never a descent at d). This polynomial is well known and called the d-th *Eulerian polynomial* (see, e.g. [7]) and often denoted by $A_d(t)$. In fact, $A_d(t)$ equals the h-polynomial of our triangulation of c_0. As for Π_+, we see that there is always a descent at d, never a descent at $d-1$, and the first $d-1$ letters in a $\pi \in \Pi_+$ behave just like a permutation in S_{d-1}. Hence each permutation in Π_+ corresponds to a permutation in S_{d-1}, but has an extra descent, namely the one at d. There are d possible choices for the last letter of $\pi \in \Pi_+$ and the descent polynomial of Π_+ is thus equal to $d\cdot tA_{d-1}(t)$. Hence, $h(\widehat{\mathcal{P}}(K_d), t) = A_d(t) + d\cdot tA_{d-1}(t)$. Moreover, the exponential generating function of $A_d(t)$ is $\sum_{d\geq 0}A_d(t)\frac{x^d}{d!} = \frac{(1-t)e^{x(1-t)}}{1-te^{x(1-t)}}$. Hence, $\sum_{d\geq 0}h(\widehat{\mathcal{P}}(K_d), t)\frac{x^d}{d!} = (1+xt)\frac{(1-t)e^{x(1-t)}}{1-te^{x(1-t)}}$.

Finally, we give two theorems which provide a recursive algorithm for computing $h(\widehat{\mathcal{P}}, t)$.

Definition 30 *Let G be a graph with vertex set $[d]$ and $\mathcal{P}$ defined as usual in terms of G. Let $S \subset [d]$. Then $\mathcal{P}_S := \{\mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathcal{P} \mid x_i = 0 \text{ if } i \in S\}$.*

That is, $\mathcal{P}_S$ is isomorphic to $\mathcal{P}(G_S)$, where G_S is the subgraph of G induced by $[d] \setminus S$. We also define $\widehat{\mathcal{P}}_S$ similarly, i.e. $\widehat{\mathcal{P}}_S := \widehat{\mathcal{P}} \cap \mathcal{P}_S$.

Theorem 31 *Let G be a graph with vertex set $[d]$ and no isolated vertices. Then $h(\widehat{\mathcal{P}}, t) = \sum_S h(\widehat{\mathcal{P}}_S, t)(t-1)^{\#S-1}$, where S ranges over all nonempty subsets of $[d]$.* ■

Theorem 32 *Let G be a graph with $d-1$ vertices and denote by G' the graph obtained by adding to G an isolated vertex. Suppose $h(\widehat{\mathcal{P}}(G), t) = a_0 + a_1 t + \cdots + a_{d-1} t^d$. Then $h(\widehat{\mathcal{P}}(G'), t) = b_0 + b_1 t + \cdots + b_d t^d$, where $b_k = (2k+1)a_k + (2d-2k+1)a_{k-1}$.* ■

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