Plethysm, the partition lattice and Euler numbers

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This talk will focus on formal methods for computing homology representations. The main examples will be the lattice of set partitions, and various subposets.

We shall give a brief introduction to plethysm and wreath products, another brief introduction to homology representations, and then tie the two together in the setting of the partition lattice. In this case the formal methods give rise to a large family of plethystic identities for the Frobenius characteristic of the homology representation. These methods were introduced in [3].

Interesting identities arise from the partition lattice itself, the d-divisible sublattice, and rank-selected subposets. In the case of sublattices which are intersection lattices of subspace arrangements, these identities can be seen to encode the fact that the cohomology of the arrangement is essentially that of an exterior algebra. These ideas are pursued for the k-equal lattice in the joint paper [4].

In theory these formal methods can be applied to non-Cohen-Macaulay posets as well. For instance, this approach gives an interesting plethystic identity for the alternating sum of homology modules of the k-equal lattice. However unlike the Cohen-Macaulay case, in order to determine completely the representation on each graded component of the homology, one needs additional topological information about the poset (see [4]).

Invariants such as the multiplicity of the trivial representation were first shown to be interesting from the enumerative point of view by Stanley. This collection of numbers, computed for the rank-selected subposets of the partition lattice, form a refinement of the Euler number (see [2]). The Euler numbers seem to appear with curious frequency in this context. As another example, the Möbius number (up to sign) of the lattice of partitions of $2n$ elements with all block-sizes even, is the tangent number E_{2n-1}.

Of all the proper subposets of the partition lattice studied in [3], one stands out for the remarkable properties of its homology representation. This subposet, obtained by selecting alternate ranks, is a good example of the success of the formal methods described in this talk (details will appear in [5]). In particular we conjecture a decomposition of the homology module which would imply the existence of new refinements, into sums of powers of 2, of the tangent number, the Genocchi number and the number of alternating André or simsun permutations. These results and conjectures also suggest connections with the cd-index (see [6]).

References

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