TWO VARIABLE PFAFFIAN IDENTITIES AND SYMMETRIC FUNCTIONS

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SUMMARY. We give sign reversing involution proofs of a pair of two variable Pfaffian identities. Applications to symmetric function theory are given, including identities relating Pfaffians and Schur functions. As a corollary we are able to compute the plethysm $p_2 \circ s_{k^n}$. Finally, we discuss some connections to root systems.

SOMMAIRE. On donne des preuves de deux identités de Pfaff, par le biais d'involutions changeant le signe. Ces identités admettent des applications dans le contexte de la théorie des fonctions symétriques en permettant d'obtenir des identités qui relient les fonctions de Pfaff à celles de Schur. Comme corollaire, on en déduit un calcul du plethysme $p_2 \circ s_{k^n}$. On conclu par une discussion des liens entre ce travail et les systèmes de racines.

1. INTRODUCTION

The main result (Theorem 2.1) is a two variable generalization of the following pair of identities:

$$\text{Pf}\left(\frac{x_i - x_j}{x_i + x_j}\right) = \prod_{1 \leq i < j \leq 2n} \left(\frac{x_i - x_j}{x_i + x_j}\right) \tag{1.1}$$

$$\text{Pf}\left(\frac{x_i - x_j}{1 + x_i x_j}\right) = \prod_{1 \leq i < j \leq 2n} \left(\frac{x_i - x_j}{1 + x_i x_j}\right) \tag{1.2}$$

These identities are interesting in that they are related to the Weyl identities for the classical root systems. The proof of Theorem 2.1 uses an adaptation of Gessel's sign reversing involution proof of the Vandermonde identity and also shows how the Weyl identity for D_n plays a role in (1.2); in Section 5 we generalize this connection to root systems of types B_n and C_n.

In Sections 3 and 4 we discuss some applications to symmetric function theory, including several identities which express Schur functions in terms of Pfaffians. In particular, we obtain a Pfaffian expression for the plethysm $p_2 \circ s_{k^n}$ (Corollary 3.1) for which we are able to give an explicit expansion into Schur functions (Theorem 4.3).

2. TWO PFAFFIAN IDENTITIES

In this section we state two-variable generalizations of (1.1) and (1.2) (Theorem 2.1). Much of our notation is taken from [7].

Definitions. For n an integer let $[2n] = \{1, 2, \ldots, 2n\}$. Let $\mathcal{F}_{2n}$ denote the set of perfect matchings in the complete graph on vertices $[2n]$. We refer to elements of $\mathcal{F}_{2n}$ as matchings and represent them as sets of ordered pairs of integers; an ordered pair corresponds to an edge in the matching and by convention the first element of the pair is the smaller vertex. For example

$$\pi_0 = \{(1,2),(3,4),\ldots,(2n-1,2n)\}$$
$$\pi_1 = \{(1,2n),(2,2n-1),\ldots,(n,n+1)\}$$

are matchings in $\mathcal{F}_{2n}$. Given $\pi \in \mathcal{F}_{2n}$ let $\epsilon(\pi) = (-1)^{\text{cross}(\pi)}$, where cross($\pi$) is the crossing number of π, which we can take to be the number of intersections when edges of π are drawn in the upper half-plane as semicircular arcs between integer points of the x-axis.

If $A = (a_{i,j})$ is a $2n \times 2n$ skew-symmetric matrix then we define the Pfaffian of A to be

$$\operatorname{Pf}(A) = \sum_{\pi \in \mathcal{F}_{2n}} \epsilon(\pi) \prod_{(i,j)\in\pi} a_{i,j}.$$

In this way we view the Pfaffian as a weighted generating function for matchings. We often write $\operatorname{Pf}(a_{i,j})$ for $\operatorname{Pf}(A)$.

In our main result, we express Pfaffians in terms of skew-symmetrizations of certain monomials. Let $x = \{x_1, x_2, \ldots, x_{2n}\}$ and $y = \{y_1, y_2, \ldots, y_{2n}\}$ be two sets of variables and let S_{2n} act on each by permuting indices. For α and β compositions of length n define

$$a_{\alpha,\beta}(x,y) = \sum_{\sigma\in S_{2n}} \epsilon(\sigma)\,\sigma\!\left(x_1^{\alpha_1} y_1 \ldots x_n^{\alpha_n} y_n\; x_{n+1}^{\beta_1} \ldots x_{2n}^{\beta_n}\right).$$

For example, $a_{\delta_n,\delta_n}(x,x^n) = \det(x_i^{2n-j})$ is the usual alternant in x, where $\delta_n = (n-1,\ldots,1,0)$.

We are now ready to state the main result; (2.1) is also due to Proctor.

Theorem 2.1.

(2.1) $$\operatorname{Pf}\left(\frac{y_i - y_j}{x_i + x_j}\right) \prod_{1\le i<j\le 2n} (x_i + x_j) = (-1)^{\binom{n}{2}}\, a_{2\delta_n,2\delta_n}(x,y)$$

(2.2) $$\operatorname{Pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right) \prod_{1\le i<j\le 2n} (1 + x_i x_j) = \sum_{\lambda,\mu} a_{\lambda+\delta_n,\mu+\delta_n}(x,y)$$

where the sum is over pairs of partitions $\lambda = (\alpha_1,\ldots,\alpha_p \mid \alpha_1+1,\ldots,\alpha_p+1)$ *and* $\mu = (\beta_1,\ldots,\beta_q \mid \beta_1+1,\ldots,\beta_q+1)$ *in Frobenius notation, with* $\alpha_1, \beta_1 < n-1$.

Remark 2.1. Equations (2.1) and (2.2) are generalizations of (1.1) and (1.2) respectively. Setting $y = x$ in (2.1) gives

$$\operatorname{Pf}\left(\frac{x_i - x_j}{x_i + x_j}\right) \prod_{1\le i<j\le 2n} (x_i + x_j) = (-1)^{\binom{n}{2}} \sum_{\sigma} \epsilon(\sigma)\,\sigma(x_1^{2n-1} \ldots x_n^1\, x_{n+1}^{2n-2} \ldots x_{2n}^0)$$
$$= \prod_{i<j} (x_i - x_j).$$

Setting $y = x$ in (2.2) we see that

$$\operatorname{Pf}\left(\frac{x_i - x_j}{1 + x_i x_j}\right) \prod_{1\le i<j\le 2n} (1 + x_i x_j) = \sum_{\lambda,\mu}\sum_{\sigma} \epsilon(\sigma)\,\sigma\!\left(x_1^{\lambda_1+n} \ldots x_n^{\lambda_n+1}\, x_{n+1}^{\mu_1+n-1} \ldots x_{2n}^{\mu_n}\right).$$

The inner sum vanishes unless the exponents $\lambda_1 + n, \ldots, \mu_n$ are distinct. Since $\lambda_1, \mu_1 < n$, the exponents must be a permutation of $\{0,1,\ldots,2n-1\}$. We see immediately that $\lambda_1 = n-1$ and $\mu_n = 0$. By induction we get $\lambda = (n-1)^n$ and $\mu = \emptyset$, as desired.

Remark 2.2. The shapes λ and μ which occur in the last sum are those which occur in the expansion of $\prod(1+x_ix_j)$ into Schur functions. In fact (see [4, pages 46–47])

$$\prod_{1\le i<j\le n}(1+x_ix_j)=\sum_\lambda s_\lambda(x)$$

where the sum is over all partitions $\lambda=(\alpha_1,\dots,\alpha_p\mid\alpha_1+1,\dots,\alpha_p+1)$ with $\alpha_1<n-1$. Moreover, the right hand side of (2.1) has a similar interpretation as a sum of terms $a_{\lambda+\delta,\mu+\delta}(x,y)$ where λ and μ range over all shapes in the expansion of $\prod(x_i+x_j)$, namely the single shape δ. The proofs make this connection clear.

3. Pfaffians and Schur functions

In this section we obtain identities expressing Schur functions in terms of certain Pfaffians.

For α a composition of length $2n$ let

$$a_\alpha(x)=\sum_{\sigma\in S_{2n}}\epsilon(\sigma)\sigma(x_1^{\alpha_1}\dots x_{2n}^{\alpha_{2n}}).$$

The Schur function of shape λ is $s_\lambda(x)=a_{\lambda+\delta}(x)/a_\delta(x)$. In (2.1) or (2.2), if we replace x and y by powers of x and divide both sides by $a_\delta(x)$ the right hand side is easily expressed in terms of Schur functions. One case of interest is

Proposition 3.1.

$$\frac{1}{a_\delta(x)}\operatorname{Pf}\left(\frac{x_i^N-x_j^N}{x_i^M+x_j^M}\right)\prod_{1\le i<j\le 2n}(x_i^M+x_j^M)=\pm s_{\Lambda_N^M}(x_1,\dots,x_{2n}),$$

where

$$\Lambda_N^M=((2M\delta_n+N)\cup(2M\delta_n))-\delta_{2n},$$

the union is the shuffle union, and the sign is $(-1)^{\binom{n}{2}}$ *times the sign of the shuffle permutation.*

Proof. In (2.1) replace x by x^M, replace y by x^N, and divide both sides by $a_\delta(x)$. □

The next corollary, originally due to Proctor [6], expresses the plethysm $p_2\circ s_{k^n}$ in terms of a Pfaffian.

Corollary 3.1.

$$\frac{1}{a_\delta(x)}\operatorname{Pf}\left(\frac{x_i^{2(n+k)}-x_j^{2(n+k)}}{x_i+x_j}\right)=(-1)^{\binom{n}{2}}s_{k^n}(x_1^2,\dots,x_{2n}^2).$$

Proof. Set $N=(n+k)$ and $M=1/2$ in Proposition 3.1 and replace x by x^2. Then $\Lambda_{2(n+k)}^{1/2}=k^n$ and the shuffle permutation is the identity. On the left hand side, the factors $\prod(x_i^M+x_j^M)/a_\delta(x)$ become $1/a_\delta(x)$ as desired. □

There is another way to express Schur functions in terms of Pfaffians. More generally any determinant can be written as a Pfaffian [2],[5]. Given an even order matrix A, choose J skew-symmetric with determinant 1 and set $B=AJA^t$. Then

$$\text{(3.1)}\qquad \operatorname{Pf}^2(B)=\det(B)=\det{}^2(A)\quad\text{so}\quad\operatorname{Pf}(B)=\det(A).$$

For A of odd order let $\bar A=A\oplus(1)$ (matrix direct sum) so that $\bar A$ has even order and $\det(A)=\det(\bar A)$.

Thus any Schur function can be written as a quotient of a Pfaffian by $a_\delta(x)$ in many ways. One such way is

Proposition 3.2.

$$s_\lambda(x) = \frac{1}{a_\delta(x)} \operatorname{Pf}\left(f_\lambda(x_i, x_j)\right),$$

where

$$f_\lambda(x,y) = \sum_{i=0}^{2n-1} (-1)^i \, x^{(\delta+\lambda)_{i+1}} y^{(\delta+\lambda)_{2n-i}}.$$

Proof. Apply (3.1) to $a_{\lambda+\delta}(x)$ where J has entries $(-1)^{i+1}$ on the antidiagonal. □

Similarly, we can express a Schur function as a single Pfaffian by applying our method to the Jacobi–Trudi identity:

Proposition 3.3. *Let λ be a partition of length at most $2n$. For $1 \le i < j \le 2n$ let*

$$s_{i,j} = s_{\lambda_i - i+1, \lambda_j - j+2} + s_{\lambda_i - i+3, \lambda_j - j+4} + \cdots + s_{\lambda_i - i+2n-1, \lambda_j - j+2n},$$

(sums of Schur functions with two parts), and for $i > j$ let $s_{i,j} = -s_{j,i}$. Then $s_\lambda(x) = \operatorname{Pf}(s_{i,j})$.

Proof. Actually, this is a special case of a theorem of Stembridge [7, Theorem 3.1], but we can also obtain it by applying (3.1) to the Jacobi–Trudi identity with J equal to a block diagonal matrix with blocks $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. □

Using the matrix J with entries -1 on the upper half of the antidiagonal and 1 on the lower half of the antidiagonal we can show

$$a_{2\delta_n, 2\delta_n}(x,y) = \operatorname{Pf}\left(\frac{(y_i - y_j)(x_i^{2n} - x_j^{2n})}{x_i^2 - x_j^2}\right). \tag{3.2}$$

Then we have

Proposition 3.4.

$$\operatorname{Pf}\left(\frac{y_i - y_j}{x_i + x_j}\right) \operatorname{Pf}\left(\frac{x_i^{4n} - x_j^{4n}}{x_i^2 + x_j^2}\right) = (-1)^{\binom{n}{2}} \operatorname{Pf}\left(\frac{x_i^{2n} - x_j^{2n}}{x_i + x_j}\right) \operatorname{Pf}\left(\frac{(y_i - y_j)(x_i^{2n} - x_j^{2n})}{x_i^2 - x_j^2}\right).$$

Proof. Multiply both sides of (2.1) by $a_{\delta_{2n}}(x) = (-1)^{\binom{n}{2}} a_{2\delta_n, 2\delta_n}(x,x)$. Now use (3.2) to convert determinants to Pfaffians. □

4. Symmetric function expansions

In this section we study how Pfaffians give rise to alternating functions and give a technique for expanding such Pfaffians.

We say that the formal power series $f(x_1, \ldots, x_{2n})$ is *alternating* if $\sigma f(x) = \epsilon(\sigma) f(x)$ for all permutations $\sigma \in S_{2n}$. We say that the formal power series $f(u,v)$ (in two variables) is *skew symmetric* if $f(u,v) = -f(v,u)$. The following lemma follows immediately from properties of Pfaffians.

Lemma 4.1. *Let $f(u,v)$ be skew symmetric and define $a_{i,j} = f(x_i, x_j)$. Then $\operatorname{Pf}(a_{i,j})$ is alternating.*

Consequently we have

Theorem 4.1. *Let f be a skew symmetric formal power series in two variables with expansion $f(u,v)=\sum_{r,s} c_{r,s}\, x^r y^s$. Then*

$$\frac{1}{a_\delta(x)}\operatorname{Pf}(f(x_i,x_j))=\sum_\lambda \operatorname{Pf}(C_{\lambda+\delta})\, s_\lambda(x),$$

where C_μ is the skew symmetric matrix with entries c_{μ_i,μ_j}.

For certain alternating functions, the coefficients $\operatorname{Pf}(C_{\lambda+\delta})$ can be determined explicitly. We give two such applications of Theorem 4.1.

Theorem 4.2.

$$\frac{1}{a_\delta(x)}\operatorname{Pf}\left(\frac{x_i-x_j}{1-x_i^2x_j^2}\right)=\sum_\lambda s_\lambda(x),$$

where the sum is over all shapes with even length rows and even length columns.

We can use (2.2) to get a different expansion of the previous Pfaffian involving plethysms.

Corollary 4.1.

$$\sum_\pi s_\pi(x)=\sum_\nu (-1)^{|\nu|/2} s_\nu(x^2)\sum_{\lambda,\mu}\epsilon(\lambda,\mu)(-1)^{\binom{n}{2}+|\lambda|/2+|\mu|/2} s_{\Lambda(\lambda,\mu)}(x),$$

where π has even rows and columns, ν has Frobenius type $(\alpha_1,\dots|\alpha_1+1\dots)$ with $\alpha_1<2n-1$, λ and μ have Frobenius type $(\alpha_1,\dots|\alpha_1+1\dots)$ with $\alpha_1<n-1$, $\Lambda(\lambda,\mu)=((\lambda+2\delta_n+1)\cup(\mu+2\delta_n))-\delta_{2n}$, and $\epsilon(\lambda,\mu)$ is the sign of the shuffle permutation that defines $\Lambda(\lambda,\mu)$.

As a second application, we can expand the Pfaffian in Corollary 3.1 to get an explicit expansion of the plethysm $p_2\circ s_{k^n}$ into Schur functions. This is also in [6].

Theorem 4.3.

$$s_{k^n}(x_1^2,\dots,x_{2n}^2)=\sum_\lambda(-1)^{\lambda_1+\cdots+\lambda_n}s_\lambda(x_1,\dots,x_{2n}),$$

where the sum is over all self complementary partitions inside the $2n\times 2k$ rectangle, i.e. partitions satisfying $\lambda_i+\lambda_{2n+1-i}=2k$ for $i=1,\dots,n$.

5. Remarks

Remark 5.1. Identity (2.2) corresponds to the root system D_n in the sense that the shapes which occur in the expansion on the right hand side are those which appear in $\prod(1-x_ix_j)$, the product half of Weyl's identity for the root system D_n [4, p. 46]. Other identities corresponding to root systems B_n and C_n can easily be developed. More generally we have

$$\operatorname{Pf}\left(\frac{(y_i-y_j)(1-x_i^p)(1-x_j^p)}{1-x_ix_j}\right)\prod_{1\le i<j\le 2n}(1-x_ix_j)=\sum_{\lambda,\mu}c_\lambda^{(p)}c_\mu^{(p)}\,a_{\lambda+\delta,\mu+\delta}(x,y),\tag{5.1}$$

where the coefficients $c_\lambda^{(p)}$ are determined by

$$\prod_{1\le i<j\le n}(1-x_ix_j)\prod_{i=1}^n(1-x_i^p)=\sum_\lambda c_\lambda^{(p)}\,s_\lambda(x).$$

The cases $p = 1$, $p = 2$, and $p = \infty$ (where $p = \infty$ is interpreted in a limiting formal power series sense, that is $1 - x_i^\infty$ means 1) correspond to root systems B_n, C_n, and D_n respectively.

The factors $(1 - x_i^p)$ and $(1 - x_j^p)$ can be factored out of the Pfaffian in (5.1) as $\prod(1 - x_i^p)$. This leads to the identity

$$\prod_{i=1}^{2n}(1 - x_i^p)\sum_{\lambda,\mu} c_\lambda^{(\infty)} c_\mu^{(\infty)} a_{\lambda+\delta,\mu+\delta}(x,y) = \sum_{\lambda,\mu} c_\lambda^{(p)} c_\mu^{(p)} a_{\lambda+\delta,\mu+\delta}(x,y). \tag{5.2}$$

We can similarly modify (2.1) to get

$$\mathrm{Pf}\left(\frac{(y_i - y_j)(1 - x_i^p)(1 - x_j^p)}{x_i + x_j}\right) \prod_{1\le i<j\le 2n} (x_i + x_j) = \sum_{\lambda,\mu} d_\lambda^{(p)} d_\mu^{(p)} a_{\lambda+\delta,\mu+\delta}(x,y), \tag{5.3}$$

where the coefficients $d_\lambda^{(p)}$ are determined by

$$\prod_{1\le i<j\le n} (x_i + x_j)\prod_{i=1}^{n}(1 - x_i^p) = \sum_{\lambda} d_\lambda^{(p)} s_\lambda(x).$$

Remark 5.2. It is easy to modify Theorem 4.2 to obtain other symmetric function expansions. For example, it is known how to find the coefficient of $s_\lambda(x)$ in

$$\frac{1}{a_\delta(x)} \mathrm{Pf}\left(\frac{x_i^N - x_j^N}{1 - x_i^M x_j^M}\right),$$

and all coefficients are -1, 0, or 1 (see [8]). In some cases, the Pfaffian can be computed from (1.2), resulting in a Littlewood formula.

Remark 5.3. The two-variable identities may have three-variable generalizations. For instance, it is known that

$$\mathrm{Pf}\left(\frac{(y_i - y_j)(z_i - z_j)}{x_i - x_j}\right) \prod_{1\le i<j\le 2n} (x_i - x_j) = a_{\delta,\delta}(x,y)\, a_{\delta,\delta}(x,z).$$

This generalizes (2.1) since the change of variables $x \mapsto x^2$, $z \mapsto x$ yields (2.1).

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