# Generating Trees and Forbidden Subsequences

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*We discuss an enumerative technique called* generating trees *which was introduced in the study of Baxter permutations. We apply the technique to some other classes of* permutations with forbidden subsequences. *We rederive some known results, e.g.* $S_n(132,231) = 2^n$ *and* $S_n(123,132,213) = F_n$*, and add several new ones:* $S_n(123,3241), S_n(123,3214), S_n(123,2143)$. *Finally, we argue for the broader use of generating trees in combinatorial enumeration.*

*Nous présentons la méthodologie appellée* arbres de génération, *introduite pour étudier les permutations Baxter. Nous utilisons cette méthodologie pour étudier d'autres classes de* permutations à motifs exclus. *Nous retrouvons quelques résultats connus, e.g.* $S_n(132,231) = 2^n$ *et* $S_n(123,132,213) = F_n$*, et ajoutons quelques résultats nouveaux:* $S_n(123,3241)$, $S_n(123,3214)$, $S_n(123,2143)$. *En conclusion, nous suggerons l'application plus générale des arbres de génération dans la combinatoire énumérative.*

A *generating tree* is a rooted, labelled tree having the property that the labels of the set of children of each node $x$ can be determined from the label of $x$ itself. Thus, any particular generating tree may be specified by a recursive definition consisting of

(1) the label of the root

(2) a set of *succession rules* explaining how to derive, given the label of a parent, the quantity of children and their labels (there being exactly one rule conforming to each possible parent-label).

Obviously, (2) corresponds to an induction step and (1) to the basis of the induction.

Since every succession-rule must first state the quantity of children as a function of the label of the parent, at least this much information must be contained in the parent's label. In some sense, the simplest situation to imagine is one in which there the label contains no further information. In this case we can imagine the label simply *to be* a record of the number of children. As a first example:

*Example 1.* **The complete binary tree.**

| | |
|---|---|
| Root: | (2) |
| Rule: | $(2) \to (2)(2)$ |

We are generally interested in recording how many nodes appear on level $n$ of the tree, and occasionally interested in knowing their distribution by label. We will call these the level-numbers and reserve the notation $\Sigma_n$ and $(\text{label})_n$ for them. For the complete binary tree, the level-numbers

$\Sigma_n = (2)_n = 2^n$. (We establish the convention that the root be considered level 0, although for many of our combinatorial applications, level 1 might be more natural.)

With no extra effort we have

*Example 2.* **The complete $k$-ary tree.**

| | |
|---|---|
| Root: | (k) |
| Rule: | (k) $\rightarrow$ $(\text{k})^k$ |

This tree has level-numbers $\Sigma_n = k^n$, a popular combinatorial function. (The most trivial possible example has $k = 1$.)

For a less trivial example, take

*Example 3.* **The Fibonacci tree.**

| | |
|---|---|
| Root: | (1) |
| Rules: | (1) $\rightarrow$ (2) |
| | (2) $\rightarrow$ (1)(2) |

Notice that our decision to use *numbers* as the labels is somewhat arbitrary, being made purely to reflect the basic information which is inherent in them. We could have chosen, for instance, (non-breeding pair) and (breeding pair) in place of (1) and (2).

If we rewrite the succession rules into a *transition matrix* we will find it easy to read off recurrence relations for the level-sums. In this case we can read off

| | | | |
|---|---|---|---|
| | | n-1 | |
| | | 1 | 2 |
| n | 1 | 0 | 1 |
| | 2 | 1 | 1 |

giving us that $(1)_n = (2)_{n-1}$ and $(2)_n = (1)_{n-1} + (2)_{n-1}$.

If the transition matrix is $A$ and the root has label $l_r$, the vector giving the $n$-th level-numbers, $[(l_1)_n, \ldots, (l_q)_n]^T$ is $A^n e_r^T$. For instance if the 0-th level-numbers are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the transition matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix},$$

as in the Fibonacci example 3, the $n$-th level-numbers are

$$\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

the solution to which is $\begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$ where $a_n = xa_{n-1} + a_{n-2}$.

In our example $x = 1$ so the solution is

$$\begin{pmatrix} (1)_n \\ (2)_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n-2} \\ F_{n-1} \end{pmatrix},$$

where $F_n$ is the Fibonacci numbers initialized to $F_0 = F_1 = 1$, and the level numbers $\Sigma_n = F_n$.

The transition matrix contains exactly the same information as the competing notation for the succession rules. The former is generally more revealing but the latter is frequently more compact, as may be seen in the following

*Example 4.* **The Catalan tree.**

$$\begin{array}{ll} \text{Root:} & (2) \\ \text{Rule:} & (k) \rightarrow (2)(3)\cdots(k+1) \end{array}$$

In this case, the transition matrix would be infinite in extent. This corresponds to the fact that the recurrence equation for $\Sigma_n$ is not of finite order. Nevertheless, there are standard (even elegant) tricks for solving it, to obtain $\Sigma_{n-1} = \frac{1}{n+1}\binom{2n}{n}$, the $n$th Catalan number.

*Example 5.* **The Schröder tree.**

$$\begin{array}{ll} \text{Root:} & (2) \\ \text{Rule:} & (k) \rightarrow (3)\cdots(k+1)(k+1) \end{array}$$

In this case, $\Sigma_n$ is the $n$-th *Schröder number*. The generating function for these numbers is $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$. For a more detailed discussion of the previous two examples, see [6].

Of course, two trees might have completely different appearances and identically equal level-sum-functions $\Sigma_n$. For a taste:

*Example 6.* **A tree having level-sums $2^n$.**

$$\begin{array}{ll} \text{Root:} & (1) \\ \text{Rule:} & (k) \rightarrow (1)^{k-1}(k+1) \end{array}$$

First note that the number of nodes on level $n$, $\Sigma_n$, is equal to the sum of the labels on level $n-1$. Then note that whatever the label $t$ of a node, the sum of all the labels of its children is $2t$. Hence the sum of labels doubles from one level to the next, hence $\Sigma_n = 2^n$, the same level-sums as we saw in example 1.

Generating trees first arose in the study of permutations with excluded subsequences. See [5] for an excellent introduction to these permutations. We give the basic definitions.

**Definition 1.** *For $\tau \in S_k$, a permutation $\pi \in S_n$ is $\tau$-avoiding iff there is no $1 \le i_{\tau(1)} < i_{\tau(2)} < \cdots < i_{\tau(k)} \le n$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. The subsequence $\{\pi(i_{\tau(j)})\}_{j=1}^{k}$ is said to have* type *$\tau$. We write $S_n(\tau)$ for the $\tau$-avoiding permutations of length $n$.*

In [6], we acknowledged [2] as the origin of the idea of generating trees. This paper deals with the reduced Baxter permutations, which can be put into the general form of definition 1 subject to a further modification.

**Definition 2.** *Let $\bar{\tau} = t_1, t_2, \ldots, \bar{t}_r, \ldots, t_k$ be a permutation $\tau$ from $S_k$ together with a* bar *over one of its elements. We explain what it means for a permutation $\pi \in S_n$ to be said to be $\bar{\tau}$-avoiding. In general, $\pi$* is permitted *to contain arbitrarily many subsequences of type $\tau$. By taking all but the $r$-th element from one of these subsequences, we see that $\pi$ may also contain subsequences of type $\tau^\times := \{t_1, t_2, \ldots, t_{r-1}, t_{r+1}, \ldots, t_k\}$. $\pi$ is $\bar{\tau}$-avoiding* iff *it contains no* further *subsequences of type $\tau^\times$.*

Under this definition, the reduced Baxter permutations are those of $S_n(41\bar{3}52) \cap S_n(25\bar{3}14)$. We will abbreviate multiple restictions of this type on the model of $S_n(41\bar{3}52, 25\bar{3}14)$. In [2], a generating tree appears with the reduced Baxter permutations of length $n+1$ being associated with the nodes on level $n$. This tree is constructed according to the rule that a permutation $\pi = p_1, \ldots, p_n$ is made the child of the permutation $p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n$ if $p_j = n$.

We have had considerable success using this approach in general, to construct generating trees for various classes of permutations. We remark that not all classes of permutations are hereditary under the operation of inserting a new largest element, as defined in the previous paragraph. A class which is not hereditary in this way will not give rise to a tree. However, there are remedies. Most notably, in the case of permutations, the entire class can be rotated under some global transformation. Equivalently, the generating operation could be replaced by inserting a new ***smallest*** element (adjusting the other elements appropriately), or by inserting a new *first* element or a new *last* element.

Here is a quick survey of some simple generating trees which arise in the context of permutations with forbidden subsequences. All of these enumerative results have previously appeared, though the binary and Fibonacci cases (both due to [5]) have not been put into this form.

$S_n(12)$: the complete *unary* tree of example 2; $\Sigma_n = 1$.

$S_n(123)$: the *Catalan* tree of example 4; $\Sigma_{n+1} = \frac{1}{n+1}\binom{2n}{n}$.

$S_n(132)$: the *Catalan* tree of example 4.

$S_n(132,231)$: the complete *binary* tree of example 1; $\Sigma_n = 2^n$.

$S_n(123,132,213)$: the *Fibonacci* tree of example 3, with root labelled (2).

$S_n(3142,2413)$: the *Schröder* tree of example 5.

$S_n(4132,4231)$: the *Schröder* tree of example 5.

The proof of any of these assertions requires a combinatorial argument. For instance, to see that $S_n(132,231)$ is generated by the complete binary tree, notice that given any permutation avoiding both 132 and 231, a new largest element can be added to either end without creating a forbidden subsequence, but to no point *between any two elements*. We say that the first and last *sites* of the permutation are *active*. A similar, but generally more complicated, argument applies in all the other cases.

For the Fibonacci tree corresponding to $S_n(123,132,213)$, the argument is as follows: having simultaneously to avoid 123 and 213 guarantees that only the first two sites of a permutation can be active. On the other hand, the first site will always be active, as inserting in the first site could only introduce a *new* 3-subsequence of type 312 or 321, both of which are allowed. The second site in any $\pi \in S_n(123,132,213)$ will be active only if the first two elements are descending rather than ascending. Since this can only happen if the previous insertion was into the first site, we know that the permutation begins with $n$. Therefore inserting $n+1$ into the second site cannot lead to a new 132 and so the second site is active *iff* the first two elements are descending.

This amounts to a new proof of an enumerative result in [5]. The examples having to do with the Catalan and Schröder numbers have already appeared in [6].

The remaining examples all produce enumerative results which, to the best of our knowledge, were not previously known:

*Example 7.* **The (123,3241)-avoiding permutations**

Here (for the first time in this paper) the number of children of a node will not be sufficient information to provide its label. We use letters instead. Another phenomenon which we see for the first time is that the normal behaviour which persists throughout the tree after level 2 collapses on the first few levels, simply because the permutations are so short as to be degenerate. Therefore we use a special symbol for the root to stress that this type of permutation never recurs in the tree; it is on the right-hand-side of none of the arrows.

$$\begin{array}{ll} \text{Root:} & (*) \\ \text{Rules:} & (*) \rightarrow (A)(X) \\ & (X) \rightarrow (B)(X)(Y) \\ & (Y) \rightarrow (B)(Y)(Z) \\ & (Z) \rightarrow (B)(B)(Z) \\ & (A) \rightarrow (A)(Z) \\ & (B) \rightarrow (B)(B) \end{array}$$

The combinatorial proof of these rules is somewhat involved and follows an *ad hoc* argument. Let a given permutation belong to $S_n(123, 3241)$, $(n > 1)$ and have an initial descending subsequence of length $k$. We note immediately that the only possible active sites are the first, second, and $k+1$-th. For insertion anywhere to the right of the first increase forms a forbiden 123, and insertion elsewhere in the initial descending subsequence forms a 3241.

The salient features of a permutation now depend on exactly three elements: the first two elements, $p_1$ and $p_2$, and the smallest element *outside the initial descending subsequence*, say $p_r$. We now distinguish 5 types of permutations in $S_n(123, 3241)$ according to these elements:

$$\begin{array}{lll} (Z) & : & p_1 > p_r > p_2 \\ (Y) & : & p_r > p_1 > p_2 \\ (X) & : & \text{The all-descending permutation (so that } p_r \text{ does not exist)} \\ (A) & : & \text{Permutations with an initial ascent (so that } p_2 \text{ does not exist)} \\ & & \text{in which } p_r > p_1 \\ (B) & : & \text{Permutations with an initial ascent in which } p_r < p_1, \\ & & \textit{and} \text{ those for which } p_1 > p_2 > p_r \end{array}$$

A methodical attempt to insert a new element $n+1$ into each of the three possible active sites (two of which may coincide) yields the succession-rules quoted above.

The resulting system of linear equations is particularly easy to solve because it can be triangularised. The initial conditions are $(X)_1 = (A)_1 = 1$, $(Y)_1 = (Z)_1 = (B)_1 = 0$ and the system is:

$$\begin{array}{lcl} (X)_n & = & (X)_{n-1} \\ (Y)_n & = & (X)_{n-1} + (Y)_{n-1} \\ (A)_n & = & (A)_{n-1} \\ (Z)_n & = & (Y)_{n-1} + (A)_{n-1} + (Z)_{n-1} \\ (B)_n & = & (X)_{n-1} + (Y)_{n-1} + 2(Z)_{n-1} + 2(B)_{n-1} \end{array}$$

We thus have the luxury of solving the first recurrence first and then plugging its solution into the second one, and so forth. We obtain the solution:

$$\begin{array}{lcl} (X)_n & = & 1 \\ (Y)_n & = & n-1 \\ (A)_n & = & 1 \\ (Z)_n & = & \binom{n}{2} \\ (B)_n & = & 3 \cdot 2^n - (n^2 + 2n + 3) \end{array}$$

and so the number of these permutations of length $n+1$ is $\Sigma_n = 3 \cdot 2^n - n^2/2 - 3n/2 - 2$.

This is the only example examined in detail in this paper in which the number of children does not suffice as a label. Other examples recorded elsewhere are $S_n(2143)$ (the *vexillary permutations* of algebraic geometry) which has succession-rules involving a two-parameter label [6], and $S_n(3\bar{5}241)$, which has a very complex succession rule involving a variable number of parameters [4].

*Example 8.* **The (123,3214)-avoiding permutations**

Consider a permutation of length $n$ avoiding both 123 and 3214. The first three elements will either contain an ascent or be all-decreasing; in either case insertion in any site to the right of the third element will always be *forbidden*. Likewise, insertion in the first two sites will always be *permitted*: a new element can only create a subsequence of type 123 if it is inserted in the third site and the first two elements are ascending; it cannot create a subsequence of type 3214 at all, if inserted in the first 3 sites. So a permutation beginning with an ascent will have two children, one beginning with an ascent and one with a descent. A permutation beginning with a descent will have three children, only one beginning with an ascent.

We summarize these rules (after making sure that they hold in the first few levels of the tree, where degeneracies may occur) as:

$$\begin{array}{ll} \text{Root:} & (2) \\ \text{Rules:} & (2) \rightarrow (2)(3) \\ & (3) \rightarrow (2)(3)(3) \end{array}$$

Here, indexing the transition matrix by the labels [(2),(3)], we have

$$\begin{pmatrix} (2)_n \\ (3)_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for the $n$-th level numbers. Since this transition matrix is just $A^2$ for our $A$ of example 3, the result is again a consecutive pair of Fibonacci numbers $\begin{pmatrix} F_{2n-2} \\ F_{2n-1} \end{pmatrix}$, giving $\Sigma_n = F_{2n}$. Alternatively, we could find these succession-rules by skipping a generation and sending inheritances directly to grandchildren in example 3.

There is an entertaining bijection with another class counted by the same function, the column-convex directed animals on $n$ cells studied by Delest and Dulucq in [3].

An column-convex directed animal is a set $A$ of lattice points in the first quadrant such that

(1) $(0,0) \in A$

(2) if $(x,a),(x,b) \in A$ and $a < y < b$ then $(x,y) \in A$

(3) if $(x,y) \in A$ then $(x,y)$ may be reached from $(0,0)$ by a sequence of steps to the north and east, always staying in $A$.

Essentially, condition (2) is the column-convexity condition and condition (3) states that the animal is directed.

We show that the tree of example 8 is a generating tree for this class of objects as follows. Identify the root of the tree with the animal $(0,0)$. We will grow the animal square by square, according to one of the following three operations:

(A) add a square to the top of the leftmost column

(B) shift the animal up one square and add a square to the bottom of the leftmost column

(C) shift the animal one square to the right and add a new square at $(0,0)$.

Evidently operations (A) and (B) commute, so to avoid generating the same animal twice we will insist that between occurrences of operation (C) all occurrences of (A) precede all occurrences of (B). Hence every (C) may be followed by any letter, every (A) may be followed by any letter, but every (B) must be followed by a (B) or a (C). When the animal has only one column, (A)s and (B)s have the same effect, so we deem that the root square was added by a (B) operation, so that only (B)s will follow until we move to the next column.

Our succession-rules are thus seen to be identical to those for (123,3214)-avoiding permutations:

| | |
|---|---|
| Root: | (B) |
| Rules: | (B) $\rightarrow$ (B)(C) |
| | (A) $\rightarrow$ (A)(B)(C) |
| | (C) $\rightarrow$ (A)(B)(C) |

*Example 9.* **The (123,2143)-avoiding permutations**

We derive the succession rules: first note that the active sites of a 123,2143-avoiding permutation with $k$ children must be the leftmost $k$ sites. If a site is inactive because it would lead to a 123, all sites to the right are immediately inactive as well. If a site is inactive not for this reason but because it would lead to a 2143, then all the elements preceding it must be decreasing. Either they continue to decrease, in which case the sites continue to be made inactive by the same $21-3$, or there is an increase in which case the sites are inactive because of a $12-$, as before.

Now consider introducing a new largest element into the first site: it extends the initial decreasing sequence, and disturbs nothing, increasing the number of active sites by 1. On the other hand, consider inserting a new largest element into an active site $k > 1$. It cancels all active sites to its right, by reason of $12-$, and it also cancels all active sites to its left after the second element, by reason of $21-3$. This is because the first two elements are decreasing (else site $k$ would not have been active) and smaller than the new element. Hence the succession-rules are:

| | |
|---|---|
| Root: | (2) |
| Rule: | (k) $\rightarrow$ (k+1)$(2)^{k-1}$ |

This tree can *also* be used to grow the column-convex directed animals seen above, and so $\Sigma_n = F_{2n}$, as above. To see this, consider growing an animal as follows: let $k-1$ be the number of cells in its rightmost column, which might then be $(r,b),(r,b+1),\ldots,(r,b+k-2)$. We add a new cell in any of the $k$ positions $(r,b+k-1)$ (i.e. atop the rightmost column) or $(r+1,b),(r+1,b+1),\ldots,(r+1,b+k-2)$ (i.e. in a newly created column one to the right). Each column-convex directed animal can be grown, column by column, according to these rules in exactly one way.

If an animal has $k$ children, then these children will respectively have $k+1$ children (the rightmost column being one higher), or 2 children (the rightmost column being newly created with one cell). We have rederived the succession-rules as above.

*Table.* **The 3,4-restricted permutations**

For completeness, we offer the following table, which treats permutations suffering two restrictions, one of length 3 and the other of length 4 (as in the three previous cases). Note that if the 4-restriction $\rho$ contains a subsequence of type $\pi$, then $\rho$ does not remove any permutations which pass muster with regard to $\pi$, and so $S_n(\pi,\rho) = S_n(\pi) = c_n = 1,2,5,14,\ldots$. Otherwise the sequence begins $1,2,5,13,\ldots$, one extra permutation from $S_4$ having been removed.

The table omits cases yielding $c_n$, but represents all other cases. Many cases have the same enumerative result because of obvious symmetry arguments (e.g. $S_n(\pi,\rho) = S_n(\pi^{-1},\rho^{-1})$; see [5] or [6]) and usually only one representative of each symmetry-class has been given. The exception is in the classes counted by $F_{2n}$, here each case conforming to the listed succession-rules is given.

| restrictions | succession-rules | formula | first terms (1,2,5,13,...) |
|---|---|---|---|
| $S_n(123,4321)$ | | $0\ (n \geq 7)$ | 25,25,0,... |
| $S_n(123,3421)$ | (S) $\rightarrow$ (S)<br>(A) $\rightarrow$ (S)(A)<br>(X) $\rightarrow$ (S)(A)(X)<br>(Y) $\rightarrow$ (A)(X)(Y)<br>(Z) $\rightarrow$ (A)(Y)(Z)<br>$(A)_2 = (Y)_2 = 2$, $(Z)_2 = 1$ | $\binom{n}{4} + 2\binom{n}{3} + n$ | 30,61,112,190,303,460.... |
| $S_n(132,4321)$ | (S) $\rightarrow$ (S)<br>(A) $\rightarrow$ (S)(A)<br>(B) $\rightarrow$ (A)(B)<br>(C) $\rightarrow$ (C)(Y)<br>(X) $\rightarrow$ (S)(A)(X)<br>(Y) $\rightarrow$ (B)(X)(Y)<br>$(C)_0 = 1$ | $\binom{n}{4} + \binom{n+1}{4} + \binom{n}{2} + 1$ | 31,66,127,225,373,586... |
| $S_n(123,4231)$ | see below | $\binom{n}{5} + 2\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + 1$ | 32,72,148,281,499,838... |
| $S_n(123,3241)$ | (C) $\rightarrow$ (A)(X)<br>(A) $\rightarrow$ (A)(Z)<br>(B) $\rightarrow$ (B)(B)<br>(X) $\rightarrow$ (B)(X)(Y)<br>(Y) $\rightarrow$ (B)(Y)(Z)<br>(Z) $\rightarrow$ (B)(B)(Z)<br>$(C)_0 = 1$ | $3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$ | 32,74,163,347,722,1480... |
| $S_n(123,3412)$ | R.P.Stanley | $2^{n+1} - \binom{n+1}{3} - 2n - 1$ | 33,80,185,411,885,1862... |
| $S_n(132,4231)$ | O.Guibert | $1 + (n-1)2^{n-2}$ | 33,81,193,449,1025,2305... |
| $S_n(132,3421)$ | (A) $\rightarrow$ (A)(X)<br>(B) $\rightarrow$ (B)(B)<br>(X) $\rightarrow$ (B)(X)(X)<br>$(A)_0 = 1$ | $1 + (n-1)2^{n-2}$ | 33,81,193,449,1025,2305... |
| $S_n(132,3214)$ | (A) $\rightarrow$ (A)(X)<br>(B) $\rightarrow$ (B)(X)<br>(C) $\rightarrow$ (B)(C)<br>(X) $\rightarrow$ (C)(X)(X)<br>$(A)_0 = 1$ | g.f. $= \frac{(1-x)^3}{1-4x+5x^2-3x^3}$ | 33,82,202,497,1224,3017... |
| $S_n(123,2143)$<br>$S_n(123,2413)$<br>$S_n(132,2314)$<br>$S_n(132,2341)$<br>$S_n(312,2314)$ | (k) $\rightarrow$ $(2)^{k-1}$(k+1)<br>$(2)_0 = 1$ | $F_{2n}$ | 34,89,233,610,1597,4181... |
| $S_n(312,3241)$<br>$S_n(312,3214)$<br>$S_n(123,3214)$<br>$S_n(312,4321)$<br>$S_n(312,3421)$<br>$S_n(132,3241)$ | (2) $\rightarrow$ (2)(3)<br>(3) $\rightarrow$ (2)(3)(3)<br>$(2)_0 = 1$ | $F_{2n}$ | 34,89,233,610,1597,4181... |
| $S_n(132,3412)$<br>$S_n(312,1432)$<br>$S_n(312,1342)$ | $(k^*)$ $\rightarrow$ (2)(3)...(k)$(k+1^*)$<br>(k) $\rightarrow$ (2)(3)...(k)(k)<br>$(2^*)_0 = 1$ | $F_{2n}$ | 34,89,233,610,1597,4181... |

A few comments are necessary. Firstly, two results have been ascribed to R.Stanley [1] and O.Guibert (unpublished). Neither of these proofs involved generating trees; we omit them here. Secondly, the case marked "see below" both requires a two-parameter label and so falls a little beyond the main purpose of the present paper. Indeed, there are numerous cases requiring more complicated labelling schemes which we have omitted; except $S_n(123, 4231)$, however, each is equivalent to one of the listed cases under the "obvious" symmetries. Thirdly, note the appearance of a third set of succession-rules generating $F_{2n}$. These are not terribly complicated; after remarking that a permutation labelled ($k^*$) is the all-decreasing permutation on $k-1$ elements we leave the further details to the reader.

Careful inspection of the first 5 sets of succession rules in the table reveals that each appears to be a "mutated" or "deficient" variant of the rules we derived for $S_n(123, 3214)$, exactly as though some further pruning of the generating-tree has taken place in a controlled fashion. The rules for $S_n(123, 3214)$ are in like manner "mutated" versions of the succession-rules for the Catalan tree, which send $(2) \rightarrow (2)(3)$ and $(3) \rightarrow (2)(3)(4)$. (Other rules such as $(4) \rightarrow (2)(3)(4)(5)$ are then irrelevant as a (4) is never generated in the mutated version.) The rules which generate $S_n(123, 2143)$ and $S_n(132, 3412)$ are also mutated versions of the $S_n(123)$ rules in a very similar way. (Although we have not presented them here, each of these rules seems to undergo similar further mutations to produce those more complicated schemes which we left off the chart.)

The apparently controlled manner of these mutations seems to be evidence that some sort of general approach could be made to these problems, rather than solving each (as here) in an *ad hoc* fashion. (It remains, however, perplexing that no *single* restriction of length 4, e.g. $S_n(1234)$, appears to be susceptible to such successful treatment. The two-parameter succession-rules for that case are more complicated than anything we have seen here [6].)

*Conclusion.* We have had considerable success considering classes of permutations as generated by these trees. This is consistent with the advice of [2], which suggested adapting the technique to various problems involving permutations. Although in some sense permutations may be the "natural" objects to grow on trees, we believe the concept is more general still, and introduce in evidence the directed animals considered above. In [6] we generated some minimal semiorders with the Catalan tree, and a great many combinatorial objects counted by the Catalan numbers can be produced in this way. In [4] we use a very complicated set of succession-rules to generate non-separable planar maps.

Objects we suspect might be amenable to this treatment are those which are in some way **linear** or can be made linear. For instance, permutations have their elements ordered from 1 to $n$ in a straight-forward one-dimensional way, and the maps of [4] while themselves two-dimensional, are coded using the one-dimensional code of Lehmann and Walsh. We suspect this is necessary for the succession-rules to apply themselves in a controlled manner: it will be necessary to distinguish which is the *first* child created by the operation (e.g. insertion of a new largest element), which the *second*, and so forth.

Furthermore, the criterion for membership in the class to be enumerated should be according to some **local** criteria, for instance possession of an excluded subsequence, existence of a cut-vertex. Thus, insertion of a new largest element into a permutation does not disturb any of the existing subsequences (these can then be regarded as distant from the new element). Furthermore the new element itself participates (therefore, is local to) each of the new subsequences it creates.

Examples of a less-tractable **global** criterion might be the cycle index of a permutation or chromatic number of a graph. Insertion of a new large element into a permutation can drastically alter the cycle-index in hard-to-predict ways. For our purposes it is important that the relevant statistic on each child can be obtained from knowing the statistic on the parent, together with what has changed between generations.

Finally, it is important for this genealogical technique that the class under consideration be **hereditary**. This condition is better defined, though perhaps less stringent, than the other conditions (linear, local). Heredity means precisely that the generating tree must genuinely be a tree; that is, that there are no sudden appearances of foundlings with no ancestry. However, fixes are possible; if foundlings appear in a controlled fashion (a given number per level, with known properties) then they can be factored in (for instance, treated as children of imaginary parents who do not themselves contribute weight to the tree).

Much work remains to be done. In particular, a general explanation of how succession-rules mutate as new conditions are applied is eagerly desired. There may be enough data in the $3,4$-restricted permutations tabulated (though surely not adequately) here to formulate this explanation. Alternatively, we propose $3,5$-restrictions as next on the agenda.

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