Finding All q-Hypergeometric Solutions of q-Difference Equations

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Abstract

We present an algorithm for finding all solutions $y(x)$ of a linear homogeneous q-difference equation such that $y(qx)/y(x)$ is a rational function of q and x. The algorithm can also be used to construct q-hypergeometric series solutions of q-difference equations.

Résumé

Nous présentons un algorithme qui trouve toutes les solutions $y(x)$ des équations linéaires homogènes aux q-différences, telles que $y(qx)/y(x)$ est une fonction rationnelle de q et de x. On peut utiliser cet algorithme aussi pour construire les solutions des équations aux q-différences ayant la forme d'une série q-hypergéométrique.

1 Introduction

Let $\mathbb{Q}$ be the rational number field, q transcendental over $\mathbb{Q}$, K a computable extension of $\mathbb{Q}(q)$, and x transcendental over K. Denote by Q the unique automorphism of $K(x)$ which fixes K and satisfies $Qx = qx$. Then $K(x)$ together with Q is an inversive difference field.

Let M be a difference extension ring of $K(x)$. An element $a \in M$ is *q-polynomial* if $a \in K[x]$, and *q-rational* if $a \in K(x)$. An element $a \in M \setminus \{0\}$ is a *q-hypergeometric term* if $Qa = ra$ for some $r \in K(x)$. All these concepts are relative to the field K.

We are interested in q-hypergeometric solutions y of $Ly = 0$ where

$$L = \sum_{i=0}^{\rho} p_i Q^i$$

is a linear q-difference operator of order ρ with coefficients $p_i \in K(x)$, with $p_\rho, p_0 \neq 0$. By clearing denominators in $Ly = 0$ we can restrict our attention to operators L with $p_i \in K[x]$. An algorithm for this problem is presented in Section 4. It is a q-analogue of the algorithm for finding hypergeometric solutions of difference equations described in [6]. In preparation, we show how to find q-polynomial solutions of $Ly = 0$ in Section 2, and give a normal form for q-rational functions in Section 3. Finally, in Section 5, we describe solution of various related problems such as solving nonhomogeneous equations, finding solutions in the form of q-hypergeometric series, and deriving q-hypergeometric identities.

We use $\mathbb{N}$ to denote the set of nonnegative integers. By $(a;q)_n$ we denote the expression $(1-a)(1-aq)\cdots(1-aq^{n-1})$.

In our examples we use two algebraic settings which are special cases of the general framework described above. In one we work with sequences of elements of K, identifying sequences which agree from some point on. More precisely, we take $M = K^{\mathbb{N}}/J$ where $K^{\mathbb{N}}$ is the ring of sequences over K, and J is the ideal of sequences with finitely many nonzero terms. In particular, all equalities among sequences (of the form $a_n = b_n$) are meant to hold for all but finitely many $n \in \mathbb{N}$. Further we take $x = (q^n)_{n=0}^{\infty} + J$ and define Q as the unique automorphism of M satisfying $Q(a + J) = Ea + J$ for all $a \in K^{\mathbb{N}}$. Here E denotes the shift operator acting on $K^{\mathbb{N}}$ by $Ea_n = a_{n+1}$. Obviously K can be embedded in M as the subring of constant sequences. To simplify notation, we will henceforth identify $a + J \in K^{\mathbb{N}}/J$ with its representative $a \in K^{\mathbb{N}}$. Note that in this context a sequence a_n is q-polynomial if $a_n = p(q^n)$ for some $p \in K[x]$, q-rational if $a_n = r(q^n)$ for some $r \in K(x)$, and a q-hypergeometric term if $a_{n+1} = r(q^n)a_n$ for some $r \in K(x)$.

In another setting we take $M = K[[x]]$ (or $M = K((x))$), the ring of formal power series (resp. the field of formal Laurent series) over K. Again, K, $K[x]$, and $K(x)$ are embedded in M in a natural way. We distinguish between series that are q-hypergeometric terms, and series whose coefficients form a q-hypergeometric sequence. More precisely, a series $f(x) = \sum_{j=0}^{\infty} \alpha_j x^j$ is a *q-hypergeometric term* if $f(qx) = r(x)f(x)$ for some $r(x) \in K(x)$, and a *q-hypergeometric series* if $\alpha_{j+1} = r(q^j)\alpha_j$ for some $r(x) \in K(x)$ and for all large enough $j \in \mathbb{N}$.

Several times we will need to find the largest $n \in \mathbb{N}$ (if any) such that q^n is a root of a given polynomial with coefficients in K. Therefore we assume that K is a *q-suitable field*, meaning that there exists an algorithm which given $p \in K[x]$ finds all $n \in \mathbb{N}$ such that $p(q^n) = 0$. For instance, if $K = k(q)$ where q is transcendental over k we can proceed as follows: Let $p(x) = \sum_{i=0}^{d} c_i x^i$ where $c_i \in k[q]$. Compute $s = \min\{i;\ c_i \neq 0\}$ and $t = \max\{j;\ q^j \,|\, c_s\}$. Then $p(q^n) = 0$ only if $n \leq t$, and the set of all such n can be found by testing the values $n = t, t-1, \ldots, 0$.

2 q–polynomial solutions

First we show how to find solutions $y \in K[x]$ of $Ly = 0$. Let $p_i = \sum_{k=0}^{d} c_{ik} x^k$ where $c_{ik} \in k[q]$ and not all c_{id} are zero. Assume that $y = \sum_{j=0}^{N} \alpha_j x^j$ where $\alpha_N \neq 0$. Substituting these expressions into $Ly = 0$ and replacing k by $l = j + k$ yields

$$\sum_{i,l,j} c_{i,l-j}\alpha_j q^{ij} x^l = 0$$

which implies that

$$\sum_{j=\max\{l-d,0\}}^{\min\{l,N\}} \sum_{i=0}^{\rho} c_{i,l-j}\alpha_j q^{ij} = 0, \quad \text{for } 0 \le l \le N+d. \tag{1}$$

In particular, for $l = N + d$,

$$\sum_{i=0}^{\rho} c_{id} q^{iN} = 0, \tag{2}$$

and for $l = 0$,

$$\alpha_0 \sum_{i=0}^{\rho} c_{i0} = 0. \tag{3}$$

From (2) it follows that q^N is a root of the polynomial $P(x) = \sum_{i=0}^{\rho} c_{id}x^i$. Let N_0 be the largest $n \in \mathbb{N}$ such that $P(q^n) = 0$ (see the last paragraph of Introduction). All q-polynomial solutions y of $Ly = 0$ can now be found by the method of undetermined coefficients. Ultimately, the problem is reduced to a system of linear algebraic equations over K with $N_0 + 1$ unknowns. – A more efficient method leading to a system with at most $\min\{2d, N_0 + 1\}$ unknowns is described in [2].

3 A normal form for q-rational functions

Theorem 1 *Let $r \in K(x) \setminus \{0\}$. Then there are $z \in K$ and monic polynomials $a, b, c \in K[x]$ such that*

$$r(x) = z\frac{a(x)}{b(x)}\frac{c(qx)}{c(x)}, \tag{4}$$

$$\gcd(a(x), b(q^n x)) = 1 \quad \text{for all } n \in \mathbb{N}, \tag{5}$$

$$\gcd(a(x), c(x)) = 1, \tag{6}$$

$$\gcd(b(x), c(qx)) = 1, \tag{7}$$

$$c(0) \neq 0. \tag{8}$$

Proof: Write $r(x) = \frac{f(x)}{g(x)}$ where f, g are relatively prime polynomials. We start by finding the set $\mathcal{S}$ of all $n \in \mathbb{N}$ such that $f(x)$ and $g(q^n x)$ have a nonconstant common factor. To this end consider the polynomial $R(h) = \mathrm{Resultant}_x(f(x), g(hx))$. By the well-known properties of polynomial resultants, $\mathcal{S} = \{n \in \mathbb{N};\ R(q^n) = 0\}$.

Assume that $\mathcal{S} = \{n_1, n_2, \ldots, n_t\}$ where $t \ge 0$ and $n_1 < n_2 < \cdots < n_t$. In addition, let $n_{t+1} = +\infty$. Define polynomials f_i and g_i inductively by setting

$$f_0(x) = f(x), \quad g_0(x) = g(x),$$

and for $i = 1, 2, \ldots, t$,

$$\begin{aligned} s_i(x) &= \gcd(f_{i-1}(x), g_{i-1}(q^{n_i}x)), \\ f_i(x) &= f_{i-1}(x)/s_i(x), \\ g_i(x) &= g_{i-1}(x)/s_i(q^{-n_i}x). \end{aligned}$$

Now take

$$
\begin{aligned}
z &= \alpha/\beta, \\
a(x) &= f_t(x)/\alpha, \\
b(x) &= g_t(x)/\beta, \\
c(x) &= \prod_{i=1}^{t}\prod_{j=1}^{n_i} s_i(q^{-j}x),
\end{aligned}
$$

where α and β denote the leading coefficients of $f_t(x)$ and $g_t(x)$, respectively. Before proving (4) – (8) we state a lemma.

Lemma 1 *Let $n \in \mathbb{N}$. If $0 \le l \le i, j \le t$ and $n < n_{l+1}$, then* $\gcd(f_i(x), g_j(q^n x)) = 1$.

Proof: Assume first that $n \notin \mathcal{S}$. Then $R(q^n) \neq 0$, hence $\gcd(f(x), g(q^n x)) = 1$. Since $f_i(x) \mid f(x)$ and $g_j(x) \mid g(x)$ it follows that $\gcd(f_i(x), g_j(q^n x)) = 1$, too.

To prove the lemma for $n \in \mathcal{S}$ we use induction on l.

$l = 0$: In this case there is nothing to prove since there is no $n \in \mathcal{S}$ such that $n < n_1$.

$l > 0$: Assume that the lemma holds for all $n < n_l$. It remains to show that it also holds for $n = n_l$. Since $f_i(x) \mid f_l(x)$ and $g_j(x) \mid g_l(x)$ it follows that $\gcd(f_i(x), g_j(q^{n_l}x))$ divides $\gcd(f_l(x), g_l(q^{n_l}x)) = \gcd(f_{l-1}(x)/s_l(x), g_{l-1}(q^{n_l}x)/s_l(x))$. By the definition of $s_l(x)$, the latter gcd is 1, completing the proof. $\square$

Now we proceed to verify properties (4) – (8).

(4):

$$
\begin{aligned}
z\frac{a(x)}{b(x)}\frac{c(qx)}{c(x)} &= \frac{f_t(x)}{g_t(x)}\prod_{i=1}^{t}\prod_{j=1}^{n_i}\frac{s_i(q^{1-j}x)}{s_i(q^{-j}x)} \\
&= \frac{f_0(x)}{\prod_{i=1}^{t} s_i(x)}\frac{\prod_{i=1}^{t} s_i(q^{-n_i}x)}{g_0(x)}\prod_{i=1}^{t}\frac{s_i(x)}{s_i(q^{-n_i}x)} = \frac{f(x)}{g(x)} = r(x).
\end{aligned}
$$

(5): Let $i = j = l = t$ in Lemma 1. Then $\gcd(f_t(x), g_t(q^n x)) = 1$ for all $n < n_{t+1} = +\infty$. In other words, $\gcd(a(x), b(q^n x)) = 1$ for all $n \in \mathbb{N}$.

(6): If $a(x)$ and $c(x)$ have a non-constant common factor then so do $f_t(x)$ and $s_i(q^{-j}x)$, for some i and j such that $1 \le i \le t$ and $1 \le j \le n_i$. Since $g_{i-1}(q^{n_i-j}x) = g_i(q^{n_i-j}x)s_i(q^{-j}x)$, it follows that $g_{i-1}(q^{n_i-j}x)$ contains this factor as well. As $n_i - j < n_i$, this contradicts Lemma 1. Hence $a(x)$ and $c(x)$ are relatively prime.

(7): If $b(x)$ and $c(qx)$ have a non-constant common factor then so do $g_t(x)$ and $s_i(q^{-j}x)$, for some i and j such that $1 \le i \le t$ and $1 \le j+1 \le n_i$. Since $f_{i-1}(q^{-j}x) = f_i(q^{-j}x)s_i(q^{-j}x)$, it follows that $f_{i-1}(x)$ and $g_t(q^j x)$ contain this factor as well. As $j < n_i$, this contradicts Lemma 1. Hence $b(x)$ and $c(qx)$ are relatively prime.

(8): It is easy to see that $s_i(x)$ divides both $f(x)$ and $g(q^{n_i}x)$. Hence $s_i(0) = 0$ would imply that $f(0) = g(0) = 0$, contrary to the assumption that f and g are relatively prime. It follows that $s_i(0) \neq 0$ for all i, and consequently $c(0) \neq 0$. $\square$

Theorem 2 *Let a, b, c, A, B, $C \in K[x]$ be polynomials such that $c(0) \neq 0$ and $\gcd(a(x), c(x)) = \gcd(b(x), c(qx)) = \gcd(A(x), B(q^n x)) = 1$, for all $n \in \mathbb{N}$. If*

$$\frac{a(x)}{b(x)} \frac{c(qx)}{c(x)} = \frac{A(x)}{B(x)} \frac{C(qx)}{C(x)}, \tag{9}$$

then $c(x)$ divides $C(x)$.

Proof: Let

$$\begin{aligned} g(x) &= \gcd(c(x), C(x)), \\ d(x) &= c(x)/g(x), \\ D(x) &= C(x)/g(x). \end{aligned}$$

Then $\gcd(d(x), D(x)) = \gcd(a(x), d(x)) = \gcd(b(x), d(qx)) = 1$ and $d(0) \neq 0$. Clear denominators in (9) and cancel $g(x)g(qx)$ on both sides. The result $A(x)b(x)d(x)D(qx) = a(x)B(x)D(x)d(qx)$ shows that

$$\begin{aligned} d(x) &\mid B(x)d(qx), \\ d(qx) &\mid A(x)d(x). \end{aligned}$$

Using these two relations repeatedly we find that

$$\begin{aligned} d(x) &\mid B(x)B(qx)\cdots B(q^{n-1}x)d(q^n x), \\ d(x) &\mid A(q^{-1}x)A(q^{-2}x)\cdots A(q^{-n}x)d(q^{-n}x), \end{aligned}$$

for all $n \in \mathbb{N}$. It is easy to see that since $d(0) \neq 0$ and q is not a root of unity, $d(x)$ and $d(q^n x)$ are relatively prime for all large enough n. It follows that $d(x)$ divides both $B(x)B(qx)\cdots B(q^{n-1}x)$ and $A(q^{-1}x)A(q^{-2}x)\cdots A(q^{-n}x)$ for all large enough n. But these polynomials are relatively prime by assumption, so $d(x)$ is constant. Hence $c(x) \mid C(x)$. □

Corollary 1 *The factorization of $r(x)$ described in Theorem 1 is unique.*

Proof: If

$$z\frac{a(x)}{b(x)} \frac{c(qx)}{c(x)} = Z\frac{A(x)}{B(x)} \frac{C(qx)}{C(x)}$$

are two such factorizations then $c(x) \mid C(x)$ and $C(x) \mid c(x)$, by Theorem 2. Since these polynomials are monic, $c = C$. It follows that $z = Z$ and $aB = Ab$. Hence $a \mid A$ and $A \mid a$, so $a = A$ and $b = B$. □

Corollary 2 *Among all factorizations of $r(x)$ satisfying (4) and (5) of Theorem 1, the one satisfying (4) - (8) has $c(x)$ of least degree.*

4 q–hypergeometric solutions

After this preparation we turn to the algorithm for finding q-hypergeometric solutions y of $Ly = 0$. Let $Qy = ry$ where $r \in K(x)$, then $Q^i y = \prod_{j=0}^{i-1} r(q^j x) y$. We look for $r(x)$ in the normal form described in Theorem 1. After inserting (4) into $Ly = 0$, clearing denominators and cancelling y we obtain

$$\sum_{i=0}^{\rho} z^i f_i(x) c(q^i x) = 0 \tag{10}$$

where

$$f_i(x) = p_i(x) \prod_{j=0}^{i-1} a(q^j x) \prod_{j=i}^{\rho-1} b(q^j x).$$

Since all terms in (10) except for $i = 0$ are divisible by $a(x)$ it follows that $a(x)$ divides $p_0(x) \prod_{j=0}^{\rho-1} b(q^j x) c(x)$. Because of (5) and (6), $a(x)$ divides $p_0(x)$. Similarly, all terms in (10) except for $i = \rho$ are divisible by $b(q^{\rho-1}x)$, therefore $b(q^{\rho-1}x)$ divides $z^\rho p_\rho(x) \prod_{j=0}^{\rho-1} a(q^j x) c(q^\rho x)$. Because of (5) and (7), $b(q^{\rho-1}x)$ divides $p_\rho(x)$. Thus we have a finite choice for $a(x)$ and $b(x)$.

For each choice of $a(x)$ and $b(x)$, equation (10) is a q-difference equation for the unknown polynomial $c(x)$. However, $z \in K$ is also not known yet. Let u_{ik} denote the coefficient of x^k in f_i. Since $c(0) \neq 0$, we have $\alpha_0 \neq 0$ in (3), hence applying (3) to (10) we obtain

$$\sum_{i=0}^{\rho} u_{i0} z^i = 0. \tag{11}$$

We may assume that not all u_{i0} are zero, or else we start by first cancelling a power of x from the coefficients of (10). Thus z is a nonzero root of $f(z) = \sum_{i=0}^{\rho} u_{i0} z^i$, and is algebraic over K.

If $N = \deg c(x)$ then by (2),

$$\sum_{i=0}^{\rho} u_{id} z^i q^{iN} = 0, \tag{12}$$

hence $w = zq^N$ is a nonzero root of $g(w) = \sum_{i=0}^{\rho} u_{id} w^i$. It follows that q^N is a root of $p(x) = \mathrm{Resultant}_w(f(w), g(wx))$, thus to obtain an upper bound on N computation in algebraic extensions of K is not necessary.

In summary, we find the factors of $r(x)$ as follows:

1. $a(x)$ is a monic factor of $p_0(x)$,
2. $b(x)$ is a monic factor of $p_\rho(q^{1-\rho}x)$,
3. z is a root of Eqn. (11),
4. $c(x)$ is a nonzero q-polynomial solution of (10).

Then $r = z(a/b)(Qc/c)$ and $Qy = ry$.

Example 1 Let us find a q-hypergeometric solution y of $Ly = 0$ where

$$L = xQ^3 - q^3x^2Q^2 - (x^2+q)Q + qx(x^2+q).$$

The candidates for $a(x)$ are

$$1, x, x^2+q, x(x^2+q),$$

and the candidates for $b(x)$ are

$$1, x.$$

Here we explore only the choice $a(x) = x$ and $b(x) = 1$. The corresponding equation (10) is, after cancelling one x,

$$z^3q^3x^3c(q^3x) - z^2q^4x^3c(q^2x) - z(x^2+q)c(qx) + q(x^2+q)c(x) = 0, \tag{13}$$

whence $f(z) = -qz + q^2$ with unique root $z = q$, and $g(w) = q^3w^3 - q^4w^2$ with unique nonzero root $w = q = zq^N = q^{N+1}$. It follows that $N = 0$ is the only possible degree for c. Equation (13) is satisfied by $c = 1$. Thus we have found $r = z(a/b)(Qc/c) = qx$, and the corresponding q-hypergeometric solution of $Ly = 0$ satisfies $Qy = qxy$. We can take, for instance, $y_n = x(x/q)(x/q^2)\cdots(x/q^n) = q^{\binom{n+1}{2}}$.

To find other q-hypergeometric solutions (if any), the remaining combinations for $a(x)$ and $b(x)$ could be tried; or even better, the order of the equation could be reduced using the obtained solution, and the algorithm used recursively on the reduced equation. Our *Mathematica* implementation of this algorithm (which we call `qHyper`) shows that up to a constant factor, there are in fact no other q-hypergeometric solutions:

```
In[1]:= qHyper[x y[q^3 x] - q^3 x^2 y[q^2 x] -
                    (x^2 + q) y[q x] + q x (x^2 + q) y[x] == 0, y[x]]
Out[1]= {q x}
```

Note that `qHyper` returns a list of quotients Qy/y rather than solutions y themselves. □

Example 2 Consider the equation $Ly = 0$ where $L = Q^2 - (1+q)Q + q(1-qx^2)$. As shown by `qHyper`,

```
In[2]:= qHyper[y[q^2 x] - (1 + q) y[q x] + q (1 - q x^2) y[x], y[x]]

Out[2]= {1 - Sqrt[q] x, 1 + Sqrt[q] x}
```

this equation has two linearly independent q-hypergeometric solutions, $(\sqrt{q}; q)_n$ and $(-\sqrt{q}; q)_n$. Here K is the splitting field of $1 - qx^2$. □

5 Some related problems

5.1 Nonhomogeneous equations

Consider the problem of finding q-hypergeometric solutions y of the nonhomogeneous equation $Ly = b$ where $b \neq 0$. Let $Qy = ry$ where $r \in K(x)$. Then $Ly = fy$ where $f = \sum_{i=0}^{\rho} p_i \prod_{j=0}^{i-1} Q^j r \in K(x)$. This simple fact has two important consequences:

1. $b = fy$ is q-hypergeometric,

2. $y = b/f$ is a q-rational multiple of b.

Let $Qb = sb$ where $s \in K(x)$ is given. We look for y in the form $y = fb$ where $f \in K(x)$ is an unknown q-rational function. Substituting this into $Ly = b$ gives

$$\sum_{i=0}^{\rho} p_i \left(\prod_{j=0}^{i-1} Q^j s \right) Q^i f = 1.$$

Now q-rational solutions of this equation can be found using the algorithm given in [1].

In particular, this gives an algorithm for the problem of *indefinite q-hypergeometric summation*: Given a q-hypergeometric sequence b_n, decide if $y_n = \sum_{j=0}^{n-1} b_j$ is q-hypergeometric, and if so, express it in closed form. Obviously y_n satisfies $y_{n+1} - y_n = b_n$. Since we are interested in q-hypergeometric solutions, we can rewrite this as $Qy - y = b$ and use the technique described above.

Example 3 Let $y_n = \sum_{j=0}^{n-1} b_n$ where $b_n = q^n(q;q)_n$. Then y satisfies the equation

$$Qy - y = b \tag{14}$$

where $s = Qb/b = q(1 - qx)$. The equation for f is

$$q(1 - qx)Qf - f = 1,$$

with unique q-rational solution $f = -1/(qx)$. Hence $y_n = C - (q;q)_n/q$ where C is a constant. Since $y_0 = 0$ it follows that $C = 1/q$ and $y_n = (1 - (q;q)_n)/q$. □

The same technique for solving nonhomogeneous equations also works when we look for q-hypergeometric term solutions in $M = K[[x]]$.

Example 4 Let

$$Q^2y(x) - (1 - qx)Qy(x) + qy(x) = b(x) \tag{15}$$

where

$$b(x) = \sum_{i=0}^{\infty} \frac{x^i}{(q;q)_i}.$$

Here $b(qx) = (1 - x)b(x)$, as can be easily verified. Thus $s = 1 - x$ and the equation for f is

$$(1 - qx)(1 - x)Q^2f - (1 - qx)(1 - x)Qf + qf = 1$$

with q-rational solution $f = 1/q$. Hence $y(x) = b(x)/q$ solves (15). □

5.2 q–hypergeometric series solutions

Assume that $y = \sum_{j=0}^{\infty} \alpha_j x^j$ and $Ly = b$ where $b = \sum_{j=0}^{\infty} \beta_j x^j$. As in (1), we obtain

$$\sum_{j=\max\{l-d,0\}}^{l} \sum_{i=0}^{\rho} c_{i,l-j} \alpha_j q^{ij} = \beta_l, \quad \text{for } l \geq 0. \tag{16}$$

We separate the cases $0 \leq l < d$ and $l \geq d$. In the former case, (16) yields initial conditions

$$\sum_{j=0}^{l} \alpha_j \sum_{i=0}^{\rho} c_{i,l-j} q^{ij} = \beta_l, \quad \text{for } 0 \leq l < d, \tag{17}$$

while in the latter, substitutions $m = l - d$, $s = j - m$, and $X = q^m$ transform (16) into the *associated q-difference equation*

$$\sum_{s=0}^{d} \alpha_{m+s} \sum_{i=0}^{\rho} c_{i,d-s} q^{is} X^i = \beta_{m+d}, \quad \text{for } m \geq 0, \tag{18}$$

for the unknown sequence $(\alpha_m)_{m=0}^{\infty}$. We use the algorithms of Sections 4 and 5.1 to find all solutions of (18) which are linear combinations of q-hypergeometric terms, then select the constants in these combinations so that conditions (17) are satisfied (if possible).

Example 5 Let us find q-hypergeometric series solutions y of

$$q^2 x^2 Q^3 y + (1+q) x Q^2 y + (1-x) Q y - y = 0. \tag{19}$$

The associated equation (18) in this case is

$$(q^2 X - 1)\alpha_{m+2} + (q^2(q+1)X^2 - qX)\alpha_{m+1} + q^2 X^3 \alpha_m = 0 \tag{20}$$

and qHyper finds two solutions:

```
In[3]:= qHyper[(q^2 X - 1) y[q^2 X] + (q^2 (1 + q) X^2 - q X) y[q X] +
                                    q^2 X^3 y[X] == 0, y[X]]


                  2
                q X
Out[3]= {-X, -------}
              1 - q X
```

Thus the general solution of (20) is $\alpha_m = Cq^{m^2}/(q;q)_m + D(-1)^m q^{\binom{m}{2}}$ where C and D are arbitrary constants. Equations (17) imply that $D = 0$. Hence $y^{(1)} = \sum_{m=0}^{\infty} q^{m^2} x^m/(q;q)_m$ is a q-hypergeometric series solution of (19).

Note that running qHyper on equation (19) itself we obtain another solution $y^{(2)} = (-1)^n/q^{\binom{n}{2}}$. □

Example 6 The right-hand side of the equation (15) is both a q-hypergeometric term and a q-hypergeometric series. The associated nonhomogeneous equation

$$(qX^2 - X + 1)\alpha_{m+1} + X\alpha_m = \frac{1}{q(q;q)_{m+1}}$$

can be solved as described in Section 5.1. Here $s = 1/(1-q^2X)$ and the equation for f

$$\frac{1 - X + qX^2}{1 - q^2X}Qf + Xf = 1$$

is satisfied by the q-rational function $f = 1 - qX$. Thus $\alpha_m = (1 - qX)/(q(q;q)_{m+1}) = 1/(q(q;q)_m)$, and we find the same solution $y(x) = b(x)/q$ as in Example 4. □

5.3 Deriving q-hypergeometric identities

Another important application is definite q-hypergeometric summation. The corresponding algorithm of [7] will produce a q-difference equation for the sum, but in general it will not be of minimal order. Thus it can happen that the equation will be of order 2 or more while the sum can actually be expressed in closed form. In this case one can use our algorithm to find the q-hypergeometric solutions of the equation, and then test them to see which linear combination – if any – gives the initial sum.

In analogy with the ordinary hypergeometric case [4], we also expect our algorithm to play an important role in the factorization algorithm for linear q-difference operators.

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