Incidence Matrices, Combinatorial Bases and Matroids

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Abstract

Let C and D be two finite sets. Consider a relation $(A; C, D)$, where $A \subset C \times D$. With a relation $(A; C, D)$ and an ordering $\bar{C}$ of C we associate a matroid $M(\bar{C})$ on the set D. A relation is called regular if there exists an ordering $\tilde{C}$ of C such that for any ordering $\bar{C} \neq \tilde{C}$ we have $\mathcal{S}(\bar{C}) \leq \mathcal{S}(\tilde{C})$, where $\mathcal{S}(\bar{C})$ is the collection of dependent sets of the matroid $M(\bar{C})$. We consider examples of relations that appear from the point configuration in the affine space and relations constructed for a given matroid. From a given relation we construct new relations. There is a nontrivial example of a regular relation. This work is connected with the papers [AGZ] and [A].

Résumé

Soient C et D deux ensembles finis. Considérons une relation $(A; C, D)$, où $A \subset C \times D$. A une telle relation et à un ordre $\bar{C}$ sur C on associe un matroïde $M(\bar{C})$ sur l'ensemble D. Une relation est dite regulière si il existe un ordre $\tilde{C}$ sur C tel que pour tout ordre $\bar{C} \neq \tilde{C}$ on ait $\mathcal{S}(\bar{C}) \leq \mathcal{S}(\tilde{C})$, où $\mathcal{S}(\bar{C})$ est la collection des ensembles dependents du matroïde $M(\bar{C})$. On considère des exemples de relations qui proviennent de configurations de points dans un espace affine, et des relations construites à partir d'un matroïde donné. A partir d'une relation donnée nous en construisons de nouvelles. On présente un exemple non trivial de relation regulière. Ce travail est relié aux articles [AGZ] et [A].

1 Construction of a matroid for a relation $(A; C, D)$ and an ordering of C.

1. Let C and D be two finite sets. A relation is a subset A of pairs $(c, d) \in (C \times D)$. We will denote a relation as $(A; C, D)$.

A relation $(A; C, D)$ can be represented by its incidence matrix $\tilde{A} = \| a_{c,d} \|$, $c \in C$, $d \in D$, where

$$a_{c,d} = 1, \quad if \quad (c, d) \in A, \quad a_{c,d} = 0, \quad if \quad (c, d) \notin A. \tag{1}$$

Let $(A; C, D)$ be a relation and $\bar{C}$ be some fixed ordering of C. The ordering $\bar{C}$ corresponds to some ordering of rows of the matrix $\tilde{A}$.

Denote by D_k the set of columns of $\hat{A}$ that have "0" in the first $k-1$ rows and "1" in the k-th row. We have obtained a partition $D = D_1 \cup D_2 \cup \ldots \cup D_p$, where $D_i \cap D_j = \emptyset$, $i \neq j$. Of course, the partition $(D_1, D_2, \ldots, D_p)$ depends on the ordering $\bar{C}$. Note, that some of the sets D_i can be empty sets.

Consider a subset $B \subset D$ such that for every k the set B contains all elements from D_k except one, i.e.

$$B = \{(D_1 \setminus d_1) \cup \ldots \cup (D_p \setminus d_p)\}, \; where \; d_1 \in D_1, \ldots, d_p \in D_p \tag{2}$$

For any choice of $(d_1, \ldots, d_p)$, $d_i \in D_i$ we obtain a set B. Let us denote by $\mathcal{B}(\bar{C})$ the set of all these sets B, i.e. $\mathcal{B}(\bar{C}) = \{B\}$.

Theorem 1.1 *Let $(A; C, D)$ be a relation with some ordering $\bar{C}$ of C. Then the pair $(D, \mathcal{B}(\bar{C}))$ is a matroid on the set D with the set of bases $\mathcal{B}(\bar{C}) = \{B\}$.*
The rank r of this matroid M is equal to

$$r = \sum_{D_i \neq \emptyset} (| D_i | - 1)$$

where $| D_i |$ is the cardinality of D_i.

We see that if all the sets $D_i \neq \emptyset$ then $r = | D | - p$, where p is the number of subsets D_i in the partition.

Definition 1.2 *Let $(A; C, D)$ be a relation. An ordering $\bar{C}$ of the set C is called correct ordering if in the corresponding partition $D = D_1 \cup \ldots \cup D_p$ we have $D_i \neq \emptyset$ for any $i = 1, \ldots, p$.*

Definition 1.3 *Let $\bar{C}$ be a correct ordering of C and $M(\bar{C})$ be the matroid from Theorem 1.1. The subsets $B \in \mathcal{B}(\bar{C})$ will be called combinatorial bases of a relation $(A; C, D)$ with the ordering $\bar{C}$.*

In general, matroids constructed for different orderings of C can be different matroids. Matroids corresponding to the correct orderings of C have the same rank but can be nevertheless different matroids. Therefore, we have to take into account that combinatorial bases are constructed for a relation with a given ordering of C.

Definition 1.4 *A relation $(A; C, D)$ is called regular if there exists an ordering $\tilde{C}$ of C such that for any ordering $\bar{C} \neq \tilde{C}$ we have $\mathcal{S}(\bar{C}) \leq \mathcal{S}(\tilde{C})$, where $\mathcal{S}(\bar{C})$ is the collection of dependent sets of the matroid $M(\bar{C})$.*

In section 2 we will consider an example of a regular relation.

Let $B \in \mathcal{B}(\bar{C})$ be a combinatorial basis of a relation $(A; C, D)$ with the ordering $\bar{C}$ of C . Consider the set $R = D \setminus B$. From formula (2) we have $R = (d_1, d_2, \ldots, d_p)$. Denote by $\mathcal{R}(\bar{C})$ the set of all the sets R corresponding to $B \in \mathcal{B}(\bar{C})$.

It is easy to see that the following theorem holds.

Theorem 1.5 *The pair $(D, \mathcal{R}(\bar{C}))$ is a matroid on D with the set of bases $\mathcal{R}(\bar{C})$. This matroid M^* is dual to the matroid $M = (D, \mathcal{B}(\bar{C}))$ and has the rank $r(M^*) = p$, where p is the number of nonempty sets D_i in the partition corresponding to the ordering $\bar{C}$.*

2. A useful technique in the study of matroids $M(\bar{C})$ constructed for a relation $(A; C, D)$ is the notion of a nill-matrix. The connection between the construction of matroids $M(\bar{C})$ and nill-matrices is established in Propositions 1.7 and 1.8.

Definition 1.6 *A rectangular $m \times l$, $m \geq l$ incidence matrix $\|a_{i,k}\|$ is called a nill-matrix if by permutations of its rows and columns it can be transformed to a matrix such that $a_{i,i} = 1$ and $a_{i,k} = 0$ for $i < k$, $i = 1, \ldots, m$.*

Let $(A; C, D)$ be a relation with some ordering $\bar{C}$ of C and $R \in \mathcal{R}(\bar{C})$ be a set defined in **1.** We have $R = (d_1, d_2, \ldots, d_p)$, $d_i \in D$. To each d_i there corresponds a column of the matrix $\hat{A}$ defined by the formula (1). Let us denote by $\hat{R}$ the submatrix of the matrix $\hat{A}$ which consists of the columns enumerated by $(d_1, d_2, \ldots, d_p)$.

It is easy to see that if $\bar{C}$ is a correct ordering then the matrix $\hat{R}$ has the following property:

$$a_{c_k,d_k} = 1, \; a_{c_i,d_k} = 0, \; for \; i < k.$$

This implies the following proposition.

Proposition 1.7 *Let $(A; C, D)$ be a relation with a correct ordering $\bar{C}$ of C and let $R \in \mathcal{R}(\bar{C})$. Then the matrix $\hat{R}$ is a nill-matrix.*

Let $\hat{A}$ be an incidence matrix of order $m \times n$. Let us introduce some notations:

C is the set of all rows of $\hat{A}$;

$\hat{N}$ is a submatrix of $\hat{A}$ of order $m \times l$, $1 \leq l \leq n$ such that $\hat{N}$ is a nill-matrix;

$\mathcal{N}(A) = \{\hat{N}\}$ is the set of all nill-matrices of the matrix $\hat{A}$;

N is the set of columns of the matrix $\hat{N}$;

$p = max \mid N \mid$, where $\hat{N} \in \mathcal{N}(A)$.

Proposition 1.8 *Let $\hat{N} \in \mathcal{N}(A)$ be a nill-matrix of order $m \times p$ (where p is defined above). Then there exists some ordering $\bar{C}$ of the rows of $\hat{A}$ such that there exists $R \in \mathcal{R}(\bar{C})$ for which $N = R$. This ordering $\bar{C}$ is correct.*

The following Propositions 1.9 and 1.10 describe some useful properties of nill-matrices.

Proposition 1.9 *Let $\hat{N}$ be a nill-matrix and N be its set of columns. Consider $N' \subset N$. Then the matrix $\hat{N}'$ consisting of the columns N' is a nill-matrix.*

Proposition 1.10 *Let $\hat{N}$ be a nill-matrix of order $m \times l$ and d be an arbitrary vector-column of length m consisting of "0" and "1". Then there exists a column $d' \in N$ such that the matrix consisting of the columns $(N \setminus d') \cup d$ is a nill-matrix.*

Warning: Proposition 1.10 gives an illusion that if $\hat{A}$ is an incidence matrix and $\mathcal{N}_p(A) \subset \mathcal{N}(A)$ is the set of all its nill-matrices of order $m \times p$, (where $p = max \mid N \mid$, $\hat{N} \in \mathcal{N}(A)$) then $\mathcal{N}_p(A)$ satisfies the exchange axiom for bases of a matroid. However, Proposition 1.10 differs from the exchange axiom for bases of a matroid in the following way. Indeed, let $\hat{N}, \hat{N}' \in \mathcal{N}(A)$ and $d \in N' \setminus N$. Then by Proposition 1.10 there exists a column $d' \in N$ such that the matrix consisting of the columns $(N \setminus d') \cup d$ is a nill-matrix. We have not required that $d' \in N \setminus N'$.

2 Incidence matrices for a point configuration

Let $E = (e_1, e_2, \ldots, e_N)$, $N > n$ be a finite set of points in the n-dimensional affine space. Let $P = conv(E)$ be the convex hull of E. Let us denote by σ an n-dimensional simplex spanned by some $n+1$ points from E that are in general position. Denote by $\Sigma = \{\sigma\}$ the set of all such simplices. All simplices σ (as a rule overlapping) cover the polytope P. Simplices σ divide the polytope P into a finite number of chambers γ. Denote by Γ the set of all chambers in P.

One can naturally associate with the obtained two sets (the set Σ of simplices and the set Γ of chambers) the following incidence matrix $A = \| \; a_{\sigma,\gamma} \; \|$, $\sigma \in \Sigma, \quad \gamma \in \Gamma$, where

$$a_{\sigma,\gamma} = 1, \quad if \quad \gamma \subset \sigma, \quad a_{\sigma,\gamma} = 0, \quad if \quad \gamma \not\subset \sigma \tag{3}$$

We can now define two linear spaces : the linear space generated by the rows of A ("the linear space of simplices") and the linear space generated by the columns of A ("the linear space of chambers"). The study of bases in these linear spaces see in [AGZ], [A], and [B]. However, such bases are not quite combinatorial objects by the two following reasons: 1) in general case such a basis, for example, a basis in V_Σ, consists not only of objects (i.e. simplices) but of their linear combinations. The notion of a linear combination is not quite combinatorial; 2) in order to construct such a basis one has to use linear independency of objects which is again not quite a combinatorial notion.

Remark. We want to mention that in [A] some class of bases of chambers (i.e. class of bases in V_Γ) is introduced; these bases are called there "combinatorial bases of chambers". In order not to make confusion with our definition we will refer to these bases from [A] as to "geometrical bases". Thus, a geometrical basis (of chambers) is a basis in V_Γ that has some additional property (see [A] for details).

In section 1 we have defined combinatorial bases for a relation $(A; C, D)$ and some ordering of C. However, combinatorial bases constructed for the incidence matrix $\hat{A}$ defined by the formula (3) do not give us bases in V_Σ or in V_Γ, (i.e. bases of chambers or bases of simplices).

We will associate with a point configuration other incidence matrices, see formulae (4) and (5). Connections between the combinatorial bases constructed for these incidence matrices and bases in V_Γ and in V_Σ are established in Theorem 2.1 and Theorem 2.3.

Combinatorial bases of chambers. Consider again a finite set of points $E = (e_1, \dots, e_N)$ in the n-dimensional affine space. Some of the vertices of chambers $\gamma \in \Gamma$ are points from E and some are not. A vertex w of a chamber is called a *new point* if $w \notin E$. Let $W = \{w\}$ be a set of all new points that appear in the point configuration.

Consider the following incidence matrix $\hat{A} = \| a_{w,\gamma} \|, \quad w \in W, \quad \gamma \in \Gamma$, where

$$a_{w,\gamma} = 1, \quad if \quad w \in \gamma, \quad a_{w,\gamma} = 0, \quad if \quad w \notin \gamma \tag{4}$$

Let B be a geometrical basis of chambers defined in [A] (see Remark above). The existence of geometrical bases of chambers is established in [A] for $n = 2$ by an explicit construction of such bases.

Theorem 2.1 *1. Let B be a geometrical basis of chambers. There exists a correct ordering $\bar{W}$ of W such that B is a combinatorial basis for this ordering, i.e. $B \in \mathcal{B}(\bar{W})$, where $\mathcal{B}(\bar{W})$ is the set of bases of the matroid $M(\bar{W})$ constructed for the matrix $\hat{A}$ and the ordering $\bar{W}$. (see Theorem 1.1.)*

2. For this ordering $\bar{W}$ any $B \in \mathcal{B}(\bar{W})$ defines a geometrical basis of chambers.

It seems that from this theorem it is possible to obtain that the relation defined by formula (4) is regular.

Combinatorial bases of simplices. Let E be a finite set of points in the n-dimensional affine space and let $\Sigma = \{\sigma\}$ be the set of all n-dimensional simplices with the vertices in E.

Definition 2.2 *An extended circuit s is a subset of $n+2$ points from E such that at least $n+1$ points from s are in general position.*

Let us denote by S the set of all extended circuits in the considered point configuration, i.e. $S = \{s\}$.

We use the terminology of "an extended circuit" for a point configuration since in the next section we will define similarly an extended circuit of a matroid.

Consider a simplex $\sigma \in \Sigma$. Denote by $\bar{\sigma}$ the set of its vertices. Let us define the incidence matrix $\hat{A} = \|a_{s,\bar{\sigma}}\|, \quad s \in S, \ \sigma \in \Sigma$ as follows

$$a_{s,\bar{\sigma}} = 1, \ if \ \bar{\sigma} \subset s, \ a_{s,\bar{\sigma}} = 0, \ if \ \bar{\sigma} \not\subset s \tag{5}$$

Remark. Instead of the incidence between s and $\bar{\sigma}$ one can also consider the incidence between $conv(s)$ and σ. It is easy to see that even for the same point configuration the incidence matrices that will arise in each case might be different.

Similarly to the definition of a geometrical basis of chambers (given in [A]) we can define a geometrical basis of simplices, i.e. some basis in V_Σ. We will not give this definition here since it will require some additional explanations. However, we will formulate the theorem.

Let $\hat{A}$ be the matrix defined by the formula (5).

Theorem 2.3 *1. Let B' be a geometrical basis of simplices. Then there exists an ordering $\bar{S}$ of S such that B' is a combinatorial basis constructed for the matrix $\hat{A}$ with this ordering, i.e. $B \in \mathcal{B}(\bar{S})$, where $\mathcal{B}(\bar{S})$ is the set of bases of the matroid $M(\bar{S})$ constructed for the matrix $\hat{A}$ with the ordering $\bar{S}$.*

2. For the ordering $\bar{S}$ any basis $B \in \mathcal{B}(\bar{S})$ is a geometrical basis of simplices.

3 Incidence matrix of a matroid

For a matroid M one can consider different incidence matrices, for example, we can consider the incidence between the elements of M and circuits of M, the incidence between the elements of M and the bases of M, etc.

We will define some other incidence matrix.

Let $M = (E, B)$ be a matroid on the set E, where B is the set of its bases $\{b\}$. Denote by $C = \{c\}$ the set of all circuits of M.

Definition 3.1 *A subset $x \subset E$ is an extended circuit of a matroid M if there exists a circuit $c \in C$ such that $x \supseteq c$ and for any $e \in c$, $(x \setminus e) \in B$.*

Denote by X the set of all extended circuits of a matroid M, i.e. $X = \{x\}$. It is clear that $X \supseteq C$ and that for any $x \in X$ we have $|\, x \,| = r + 1$, where r is the rank of the matroid M.

Let us define the incidence matrix $\hat{A} = \|a_{x,b}\|$, where $x \in X$ and $b \in B$ as follows

$$a_{x,b} = 1, \text{ if } b \subset x, \text{ and } a_{x,b} = 0, \text{ if } b \not\subset x \tag{6}$$

Conjecture: the relation $(A; X, B)$ is regular.

Some properties of the incidence matrix $\hat{A} = \|a_{x,b}\|$.

Definition 3.2 *We will say that an incidence matrix $\hat{A} = \|a_{i,k}\|$ has T-property if it does not have a second order minor consisting only of "1".*

Proposition 3.3 *Let M be a matroid on E with the set B of bases and let X be the set of all extended circuits of M. Let $\hat{A} = \|a_{x,b}\|$, $x \in X$, $b \in B$ be the incidence matrix for the matroid M (i.e. defined by the formula (6)).*

1. *The matrix $\hat{A}$ has T-property.*
2. *$\sum_b a_{x,b} > 0$, for any $x \in X$*
3. *$\sum_x a_{x,b} > 1$, for any $b \in B$.*

We will also consider a related geometrical notion to the notion of a T-matrix that will be called a T-graph.

Definition 3.4 *Let $V = \{v\}$ be a finite set (a set of vertices) and $\mathcal{F} = \{F\}$ be a set of subsets $F \subset V$. A pair $(V, \mathcal{F})$ is called a T-graph if the following conditions are satisfied:*

1) $\mid F \cap F' \mid \leq 1$ *for any* $F,\ F' \in \mathcal{F}$ *(T-property)*

2) $\bigcup\limits_{F \in \mathcal{F}} F = V$

3) $\mid F \mid > 1$.

The notion of a T-graph generalizes the notion of a graph. Indeed, if $\mid F \mid = 2$ for any $F \in \mathcal{F}$, then $(V, \mathcal{F})$ is a graph.

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