Height preserving minimal interval extensions

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Abstract

We consider minimal interval extensions of a partial order which preserve the height of each vertex. We show that minimal interval extensions having this property bijectively correspond to the maximal chains of a sublattice of the lattice of maximal antichains of the given order. We show that they also correspond to the set of minimal interval extensions of a certain extension of this order.

Résumé

Nous considérons les extensions intervallaires minimales d'un ordre qui préservent la hauteur de chaque sommet. Nous montrons que les extensions intervallaires minimales ayant cette propriété sont en correspondance bijective avec les chaînes maximales d'un sous-treillis du treillis des antichaînes maximales de l'ordre de départ. Nous montrons qu'elles correspondent aussi à l'ensemble des extensions intervallaires minimales d'une certaine extension de ce même ordre.

Keywords: Partially ordered sets, lattices, interval orders, minimal interval extensions, height, lattice of maximal antichains.

1 Introduction

Minimal interval extensions have been introduced in [5] in the context of modelling distributed executions. They have been characterized as follows in [3, 4]. The set of minimal

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interval extensions of a partial order is in a one-to-one correspondence (in the finite case) with the set of maximal chains of the lattice of maximal antichains of the given order. This fact can be seen in close relationship with the one-to-one correspondence between the set of linear extensions of an ordered set and the set of maximal chains of its lattice of (not necessarily maximal) antichains [1].

In this paper we present some results on the set of height preserving minimal interval extensions. Surprisingly, these extensions correspond to an interesting structured object in order and lattice theory: they can be bijectively related to the maximal chains of a certain sublattice of the lattice of maximal antichains.

In Section 2, we present the definitions and notations used in the paper.

In Section 3, we characterize the set of height preserving minimal interval extensions. This can be done by associating to an ordered set P an extension of P denoted by $Sup(P)$ — in some sense it is a "support of P". Then the minimal interval extensions of $Sup(P)$ are exactly the height preserving minimal interval extensions of P. We also show that they are in a one-to-one correspondence with the set of maximal chains of a sublattice of the lattice of maximal antichains of the given order.

2 Definitions and notations

2.1 Ordered sets and related notions

A *partially ordered set* is a pair $P = (X, <_P)$ where X is the *ground set* and $<_P$ an irreflexive and transitive binary relation called *order relation*. We denote by $x \leq_P y$ the fact that $x <_P y$ or $x = y$. We say that two elements x and y are *comparable* if $x \leq_P y$ or $y \leq_P x$. Otherwise x and y are called *incomparable*, which is denoted by $x||_P y$. A set containing pairwise comparable (resp. incomparable) elements is called a *chain* or *total order* (resp. an *antichain*). We say that x *is covered by* y, denoted by $x \prec_P y$, if $x <_P y$ and $\nexists z \; x <_P z <_P y$.

For $x \in X$, we define the set of predecessors (resp. immediate predecessors, successors, immediate successors) $Pred_P(x) = \{y \in X, y <_P x\}$ (resp. $ImPred_P(x) = \{y \in X, y \prec_P x\}$, $Succ_P(x) = \{y \in X, x <_P y\}$, $ImSucc_P(x) = \{y \in X, x \prec_P y\}$). An order is *bipartite* if each vertex either has no predecessor or no successor. For $P = (X, <_P)$ and

$Y \subseteq X$ we denote by $P(Y) = (Y, <_{P(Y)})$ the induced suborder of P on Y. A bipartite order is often denoted by $P = (X, Y, <_P)$ where $X \cup Y$ is the ground set and $<_P \subseteq X \times Y$.

For $P = (X, <_P)$, $Q = (X, <_Q)$ is an *order extension* of P, denoted by $P \ll Q$, if for all $x, y \in X$, $x <_P y$ implies $x <_Q y$.

Let $x \in X$, the *height* of x, denoted by $\mathcal{H}_P(x)$ is the size of the longest path ending at x, in other words $\mathcal{H}_P(x) = \max_{y \in Pred_P(x)}(\mathcal{H}_P(y)) + 1$ if $Pred_P(x) \neq \emptyset$ and $\mathcal{H}_P(x) = 0$ otherwise.

2.2 The Lattice of Maximal Antichains

Among the set of all antichains of P we are going to focus on *maximal antichains*: an antichain is maximal if it is not strictly contained in another one. Let us denote by $\mathcal{MA}(P)$ the set of maximal antichains of P equipped with the following order relation: $\forall A,\ B \in \mathcal{MA}(P)$, $A \leq_{\mathcal{MA}(P)} B$ if $\forall x \in A,\ \exists y \in B,\ x \leq_P y$; this order is a lattice (every pair of elements admits a lower and an upper bound) called *lattice of maximal antichains*. In this lattice, the lower bound (resp. upper bound) of A and B is the maximal antichain denoted by $A \wedge B = \max_P \{x \in X \ s.t. \forall y \in A \cup B,\ y \not<_P x\}$ (resp. $A \vee B = \min_P \{x \in X \ s.t. \forall y \in A \cup B,\ x \not<_P y\}$).

Note that a maximal antichain A is covered by another B iff for all $x \in A \setminus B$ and for all $y \in B \setminus A$, $x \prec_P y$.

2.3 Interval Orders and Interval Extensions

An ordered set $I = (X, <_I)$ is an *interval order* if and only if we can associate to each element $x \in X$ an interval $[x_1, x_2]$ where $(x_1, x_2) \in \mathbb{R}^2$, such that $x <_I y$ if and only if $x_2 < y_1$. We have the following characterization theorem [2]:

Theorem 1 *For any ordered set $I = (X, <_I)$ the following statements are equivalent:*

1. *I is an interval order.*
2. *I does not contain $2 \oplus 2$ as suborder i.e. there are not x, y, z, $t \in X$ with $x <_I y, z <_I t, x \,||_I\, t$ and $z \,||_I\, y$ (see Figure 1).*
3. *The sets of predecessors (resp. successors) in I, are totally ordered by inclusion.*
4. *The lattice of maximal antichains is a total order.*

Figure 1: $2 \oplus 2$

Let $I = (X, <_I)$ and $P = (X, <_P)$. I is an *interval extension* of P if and only if $P \ll I$ and I is an interval order. The set of interval extensions of P is denoted by $\mathcal{I}(P)$. $I = (X, <_I) \in \mathcal{I}(P)$ is a *minimal interval extension* of $P = (X, <_P)$ if $\forall J \in \mathcal{I}(P)$: $P \ll J \ll I \Rightarrow J = I$. The set of minimal interval extensions of P is denoted by $\mathcal{MI}(P)$. Minimal interval extensions have been characterized in [3, 4] as being in a one-to-one correspondance with the maximal chains of $\mathcal{MA}(P)$. More precisely, $I \in \mathcal{MI}(P)$ if and only if $\mathcal{MA}(I)$ is a maximal chain of $\mathcal{MA}(P)$.

We say that $Q = (X, <_Q)$ is a *height preserving extension* of $P = (X, <_P)$ if and only if $P \ll Q$ and $\forall x \in X$: $\mathcal{H}_P(x) = \mathcal{H}_Q(x)$. The set of height preserving interval extensions of P is denoted by $\mathcal{HI}(P)$. The set of height preserving minimal interval extensions of P is denoted by $\mathcal{HMI}(P)$. Note that if $P = (X, <_P) \ll Q = (X, <_Q)$ then $\forall x \in X$, $\mathcal{H}_P(x) \leq \mathcal{H}_Q(x)$. So clearly $\mathcal{HMI}(P) \neq \emptyset$ since the order defined on X by $x \prec y$ if $\mathcal{H}_P(x) < \mathcal{H}_P(y)$ is obviously an interval order which extends P. Figure 2 illustrates the correspondance between the lattice of maximal antichains and minimal interval extansions of an order, in this example, one extension is height preserving and the other not.

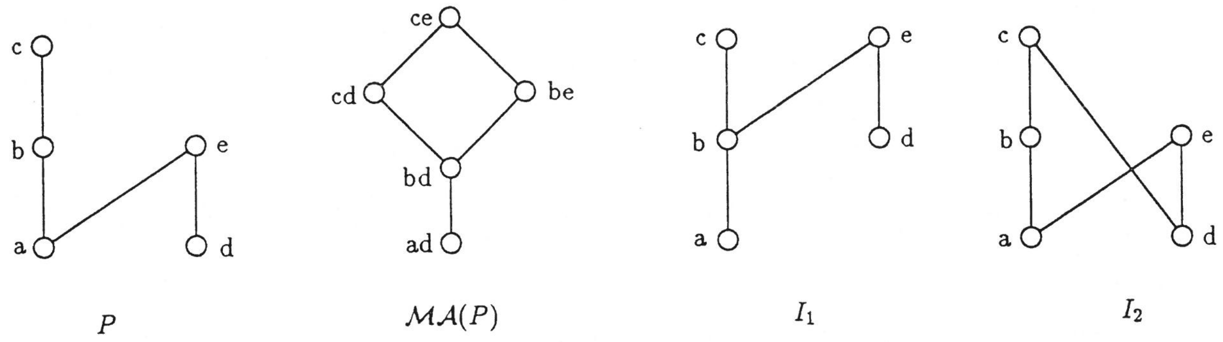

Figure 2: An order P, its lattice of maximal antichains, the minimal interval extensions $I_1 \notin \mathcal{HMI}(P)$ in correspondance with the left chain of $\mathcal{MA}(P)$ and $I_2 \in \mathcal{HMI}(P)$ in correspondance with the right chain of $\mathcal{MA}(P)$

The aim of this paper is to show that this definition of height preserving minimal interval extension in not artificial in the context of order theory. Moreover, we are going to see that these results open new perspectives in the context of lattice theory.

3 Characterization of height preserving minimal interval extensions

Let us consider the added comparabilities in a minimal interval extension I of an order P. Clearly, each arc $u \in I \setminus P$ breaks a $2 \oplus 2$ in $<_I \setminus\{u\}$. Proposition 1 shows that u also breaks a $2 \oplus 2$ in P .

Proposition 1 *Let $P = (X, <_P)$ be an ordered set and let $I \in \mathcal{MI}(P)$. For all $x, y \in X$ we have*

$$x <_I y \text{ and } x \,||_P\, y \Rightarrow \exists z,\ t \in X \text{ with } x <_P z, t \,||_I\, z \text{ and } t <_P y$$

Proof. If $x <_I y$ then the last maximal antichain A containing y is below the first maximal antichain B including y. Furthermore $x \not<_P y$ implies the existence of two other antichains A' and B' (not necessarily distinct) such that $A \prec A' \leq B' \prec B$.

So $\forall z \in A' \setminus A,\ x \prec_P z$. And $\forall t \in B' \setminus B,\ t \prec_P y$. Since $A' \leq B'$, $t \not<_I z$ which achieves the proof. $\square$.

We now state Theorem 2 which characterizes height preserving minimal interval extensions by considering some bipartite orders contained in P. The minimal height of a successor of an element x will be denoted by $\mathcal{MHS}_P(x)$; it is more formally defined as follows: $\mathcal{MHS}_P(x) = \min_{y \in Succ_P(x)} (\mathcal{H}_P(y))$ if $Succ_P(x) \neq \emptyset$ and $\mathcal{MHS}_P(x) = \max_{y \in X} (\mathcal{H}_P(y)) + 1$ otherwise. Before stating Theorem 2, let us show the two following lemmas.

Lemma 1 *Let $P = (X, <_P)$ be an ordered set and let $I = (X, <_I) \in \mathcal{HI}(P)$. For all $x, y \in X$ we have:*

$$\mathcal{H}_P(y) > \mathcal{MHS}_P(x) \Rightarrow x <_I y$$

Proof. $I \in \mathcal{HI}(P)$ then $\forall x, y \in X,\ \mathcal{H}_P(y) > \mathcal{MHS}_P(x) \Rightarrow \begin{cases} Pred_P(y) \neq \emptyset & (1) \\ \quad and & \\ Succ_P(x) \neq \emptyset & (2) \end{cases}$

(1) $\Rightarrow \exists u \in X, u <_P y$ and $\mathcal{H}_P(u) = \mathcal{H}_P(y) - 1$

(2) $\Rightarrow \exists z \in X, x <_P z$ and $\mathcal{H}_P(z) = \mathcal{MHS}_P(x)$

As I is an interval order we have $u <_I z$ or $x <_I y$. $u <_I z \Rightarrow \mathcal{H}_I(u) < \mathcal{H}_I(z)$ which leads to a contradiction because $\mathcal{H}_P(y) > \mathcal{MHS}_P(x) \Rightarrow \mathcal{H}_I(u) \geq \mathcal{H}_I(z)$. Therefore $x <_I y$ $\square$.

Lemma 2 *Let $P = (X,\ <_P)$ be an ordered set and let $I \in \mathcal{HMI}(P)$. For all $x, y \in X$ we have:*

$$x <_I y \Rightarrow \mathcal{H}_P(y) \geq \mathcal{MHS}_P(x)$$

Proof. Suppose there exist some x, y of X such that $\mathcal{H}_P(y) < \mathcal{MHS}_P(x)$ and $x <_I y$. $\mathcal{H}_P(y) < \mathcal{MHS}_P(x) \Rightarrow x \,||_P\, y$, and thus from proposition 1, we deduce that there exist z and t in X such that $x <_P z$, $t <_P y$ and $t \,||_I\, z$. So $\mathcal{MHS}_P(t) \leq \mathcal{H}_P(y) < \mathcal{MHS}_P(x) \leq \mathcal{H}_P(z)$, since $I \in \mathcal{HMI}(P)$ and Lemma 1, we have $t <_I z$ which leads to a contradiction. $\square$.

We are now able to characterize the height preserving minimal extensions of P among its minimal interval extensions. For all possible value k of the height of a vertex in P, let us define the induced bipartite order $Bip_k(P) = (X_k,\ Y_k,\ <_{P(X_k \cup Y_k)})$ where $X_k = \{x \in X$ such that $\mathcal{MHS}_P(x) = k\}$ and $Y_k = \{y \in X$ such that $\mathcal{H}_P(y) = k\}$ (see Figure 3).

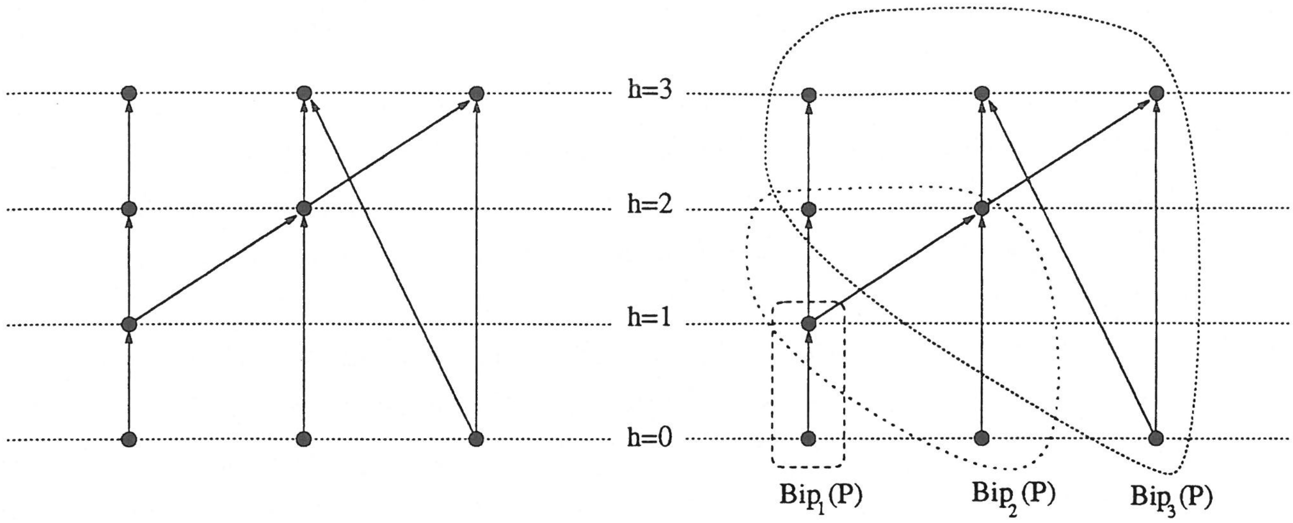

Figure 3: An ordered set P and the associated $Bip_i(P)$ for $i = 1,\ 2,\ 3$.

Theorem 2 *Let$P = (X,\ <_P)$ be an ordered set and let $I \in \mathcal{MI}(P)$. We have*

$$I \in \mathcal{HMI}(P) \text{ if and only if } \forall k : I(X_k \cup Y_k) \in \mathcal{MI}(Bip_k(P))$$

Proof.

$\Rightarrow$:

$\forall k$, $\forall x \in X_k$, $\forall y \in Y_k$, since $I \in \mathcal{MI}(P)$,we have from proposition 1: $x <_I y$ and $x \,||_P\, y \Rightarrow \exists (z,\ t) \in X^2$ such that $x <_P z$, $t <_P y$ and $t \,||_I\, z$. Clearly, we have $\mathcal{H}_P(z) \geq k$ and $\mathcal{MHS}_P(t) \leq k$ and from $t \,||_I\, z$ and Lemma 1, we have $\mathcal{H}_P(z) \leq \mathcal{MHS}_P(t)$. Then $\mathcal{H}_P(z) = \mathcal{MHS}_P(t) = k$; so, from proposition 1, $I(X_k \cup Y_k) \in \mathcal{MI}(Bip_k(P))$

$\Leftarrow$:

For proving that $I \in \mathcal{HMI}(P)$, we may show that $x <_I y$ and $x \,||_P\, y \Rightarrow \mathcal{H}_P(y) > \mathcal{H}_P(x)$. This is immediate because if $\mathcal{H}_P(y) \leq \mathcal{H}_P(x)$, there exists some $z \in X$ such that $\mathcal{H}_P(z) = \mathcal{H}_P(y)$ and $z \leq_I x$, so by transitivity $z \leq_I y$, and then, $I(X_{\mathcal{H}_P(y)} \cup Y_{\mathcal{H}_P(y)}) \notin \mathcal{MI}(Bip_{\mathcal{H}_P(y)}(P))$, since a minimal interval extension of a bipartite order is a bipartite order. This leads to a contradiction. $\square$.

This first characterization theorem can be rewritten in an easier way. Using Lemma 1, let us define from P the ordered set $\mathcal{S}up(P) = (X,\ <_{\mathcal{S}up(P)})$ by $x <_{\mathcal{S}up(P)} y \iff x <_P y$ or $\mathcal{H}_P(y) > \mathcal{MHS}_P(x)$. The following corollary is immediate:

Corollary 1 *For any ordered set P, $\mathcal{HMI}(Sup(P)) = \mathcal{MI}(Sup(P)) = \mathcal{HMI}(P)$*

So the height preserving minimal interval extensions of P are exactly the minimal interval extensions of $Sup(P)$. Using the characterization of $\mathcal{MI}(P)$ in term of maximal chains of $\mathcal{MA}(P)$, we deduce from Corollary 1 that the elements of $\mathcal{HMI}(P)$ are in a one-to-one correspondence with the maximal chains of $\mathcal{MA}(Sup(P))$. We are now going to see that $\mathcal{MA}(Sup(P))$ is strongly related to $\mathcal{MA}(P)$.

Let us define $Allowed(P)$ to be the induced suborder of $\mathcal{MA}(P)$ such that $\forall A \in Allowed(P)$, $\forall x,\ y \in A$, $\mathcal{H}_P(x) \leq \mathcal{MHS}_P(y)$. $Allowed(P)$ is clearly the set of allowed maximal antichains in any height preserving minimal interval extension of P. We have the following result describing $Allowed(P)$ (let us recall that a sublattice of a lattice is a suborder of the lattice which is itself a lattice which preserves upper and lower bounds):

Theorem 3 *For any ordered set P, $Allowed(P)$ is isomorphic to $\mathcal{MA}(Sup(P))$ and is a sublattice of $\mathcal{MA}(P)$*

Proof. Obviously $Allowed(P) \subseteq \mathcal{MA}(Sup(P))$. Now suppose there exists

$$A \in \mathcal{MA}(Sup(P)) \setminus Allowed(P)$$

then $A \notin \mathcal{MA}(P)$ so $\exists x \in X \setminus A$ *s.t.* $A \cup \{x\} \in A(P)$. Let $k = \min_{y\in A}(\mathcal{MHS}_P(y))$ and let $l = \max_{y\in A}(\mathcal{H}_P(y))$. by construction of $Sup(P)$, $k \geq l$ which implies since $x \notin A$ *(i)* $(\mathcal{H}_P(x) < \mathcal{MHS}_P(x) < l \leq k)$ or *(ii)* $(\mathcal{MHS}_P(x) > \mathcal{H}_P(x) > k \geq l)$. *(i)* implies $\exists y \in X$ *s.t.* $x <_P y$ and $\mathcal{MHS}_P(x) = \mathcal{H}_P(y)$; furthermore $A\cup\{x\} \in A(P) \Longrightarrow A\cup\{y\} \in A(P)$ and $y \notin A$. By iterating this operation we obtain: $\exists z \in X$ *s.t.* $A\cup\{z\} \in A(P)$ with $x <_P z$, $\mathcal{H}_P(z) < l \leq \mathcal{MHS}_P(z)$ and $z \notin A$ which is impossible. The case *(ii)* is similar.

Let us now prove the second part of the theorem. Suppose that $Allowed(P)$ is not a sublattice of $\mathcal{MA}(P)$, this implies that, $\exists A,\ B \in Allowed(P)$, s.t., one of the two following assertion is right:

- $A \wedge B \notin Allowed(P)$ then $\exists x,\ y \in A \wedge B$ *s.t.* $\mathcal{H}_P(x) > \mathcal{MHS}_P(y)$. Since $x \in A \wedge B$, $\exists t \in A$ and $\exists u \in B$ *s.t.* $x \leq_P t$ and $x \leq_P u$, so $\mathcal{H}_P(x) \leq \mathcal{H}_P(t)$ and $\mathcal{H}_P(x) \leq \mathcal{H}_P(u)$. But $\forall z \in A \cup B$, $\mathcal{MHS}_P(z) \geq \mathcal{H}_P(t)$ or $\mathcal{MHS}_P(z) \geq \mathcal{H}_P(u)$ so $\mathcal{MHS}_P(z) \geq \mathcal{H}_P(x)$. Let $w \in Succ_P(y)$ *s.t.* $\mathcal{H}_P(w) = \mathcal{MHS}_P(y)$ (w exists because $\mathcal{H}_P(x) > \mathcal{MHS}_P(y)$), clearly since $\mathcal{H}_P(x) > \mathcal{MHS}_P(y) = \mathcal{H}_P(w)$, we have $\mathcal{H}_P(w) < \mathcal{MHS}_P(z), \forall z \in A \cup B$ and $z \not<_P w$ which leads to a contradiction with $y \in A \wedge B$. Since y was supposed to be maximal.
- $A \vee B \notin Allowed(P)$ then $\exists x,\ y \in A \vee B$ *s.t.* $\mathcal{H}_P(x) > \mathcal{MHS}_P(y)$. Furthermore, $\exists t \in A$ and $u \in B$ *s.t.* $t \leq_P y$ and $u \leq_P y$ then $\mathcal{MHS}_P(t) \leq \mathcal{MHS}_P(y) < \mathcal{H}_P(x)$ and $\mathcal{MHS}_P(u) \leq \mathcal{MHS}_P(y) < \mathcal{H}_P(x)$. So $x \notin A$ and $x \notin B$ then $\exists z \in X$ *s.t.* $z <_P x$ and $\mathcal{H}_P(z) = \mathcal{H}_P(x) - 1$. But $z \notin A \vee B$ and thus $\exists w \in A \cup B$ *s.t.* $z <_P w$ which implies $\mathcal{H}_P(w) \geq \mathcal{H}_P(x)$ but $\mathcal{MHS}_P(t) < \mathcal{H}_P(w) \Longrightarrow w \notin A$ and $\mathcal{MHS}_P(u) < \mathcal{H}_P(w) \Longrightarrow w \notin B$: contradiction. □.

Let us finish with the description of the structure of *Allowed*(P) and then of $\mathcal{MA}(Sup(P))$. Let us first define for any $k \leq \min_{x\in X}(\mathcal{H}_P(x))$ the antichain $A_k = \{x \in X \text{ s.t. } \mathcal{H}_P(x) \leq k \text{ and } \mathcal{MHS}_P(x) > k\}$.

Proposition 2 *$\forall k \leq \min_{x\in X}(\mathcal{H}_P(x))$ we have the three following properties (see Figure 5):*

1. *$A_k \in Allowed(P)$*
2. *$\forall A \in Allowed(P)$, $A \leq_{\mathcal{MA}(P)} A_k$ or $A_k \leq_{\mathcal{MA}(P)} A$*
3. *Let $Allowed_k(P)$ be the suborder of $Allowed(P)$ defined on the antichains A of $Allowed(P)$ such that $A_{k-1} \leq_{\mathcal{MA}(P)} A \leq_{\mathcal{MA}(P)} A_k$. is isomorphic to $\mathcal{MA}(Bip_k(P))$.*

Proof. 1. can be proved by a same way as the proof of the first part of theorem 3

2. $\forall B \in \mathcal{MA}(P)$ *s.t.* $A_k||_{\mathcal{MA}(P)}B$, we have $\exists x,\ y \in A_k$, $\exists u, t \in B$ *s.t.* $x <_P y$ which implies $\mathcal{H}_P(u) > k$ and $t <_P y$ which implies $\mathcal{MHS}_P(t) \leq k$. Then $B \notin Allowed(P)$ since $\mathcal{MHS}_P(t) < \mathcal{H}_P(u)$.

3. Obvious since $A_k \setminus A_{k-1} = Y_k$ and $A_{k-1} \setminus A_k = X_k$ □.

We illustrate this property with a concrete example in Figure 4, and with the general structure of *Allowed*(P) in Figure 5.

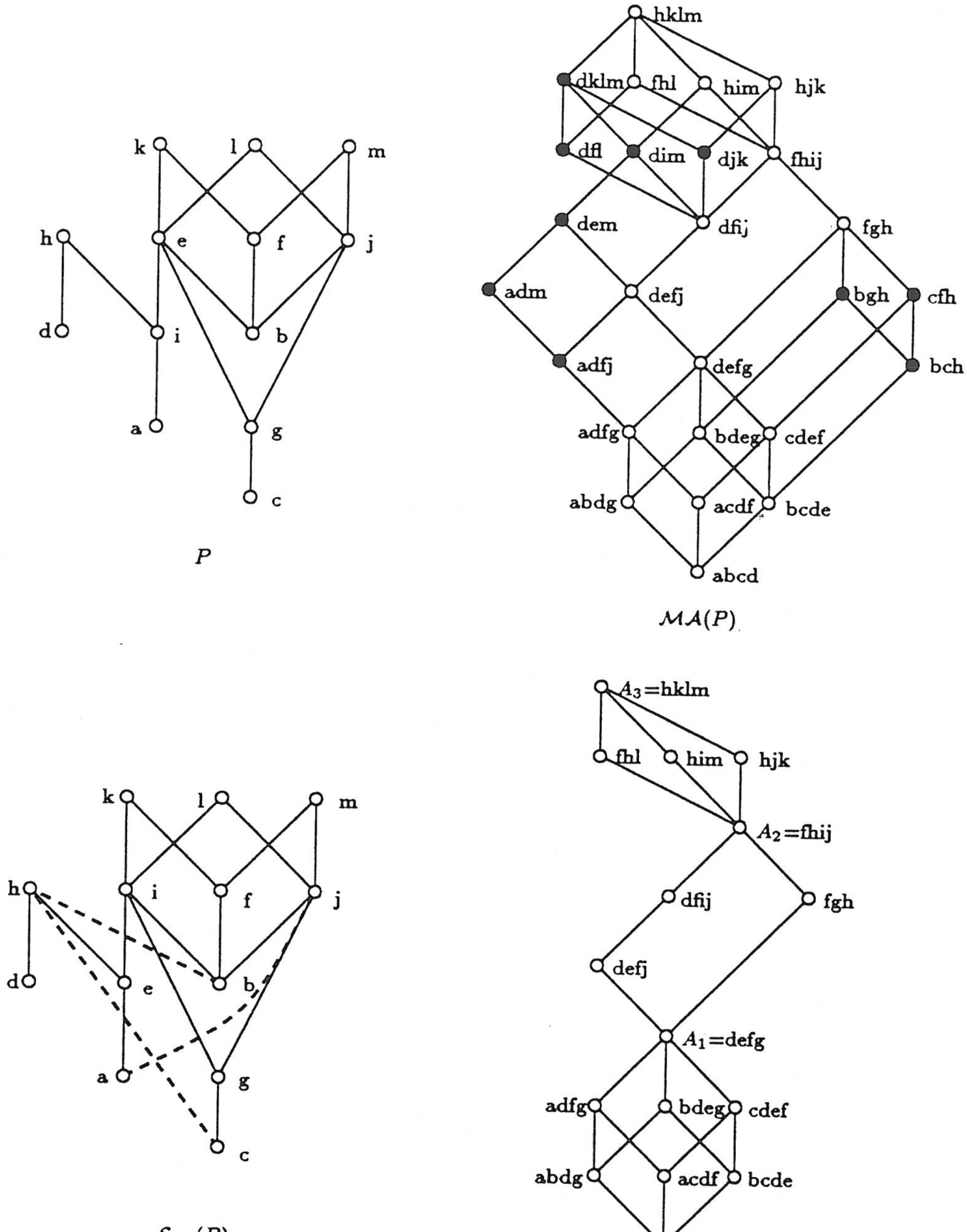

Figure 4: An order P, $\mathcal{MA}(P)$, $\mathcal{S}up(P)$, $\mathcal{A}llowed(P)$

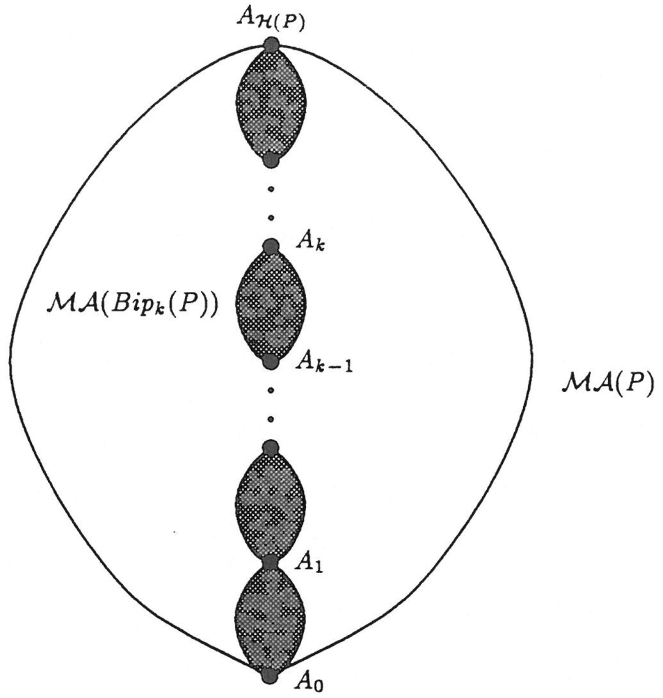

Figure 5: Structure of *Allowed*(P) (grey regions represent *Allowed*(P))

4 Perspectives

More than the particular results stated in this paper, this work opens promising new ways for future research. It was already known that the maximal chains of the lattice of maximal antichains of a partial order — which are some of its sublattices — represent interesting objects — the minimal interval extensions of the order. This paper shows that some other sublattices of such a lattice can represent important objects. So it would be natural to study the correspondences between other sublattices of this lattice of maximal antichains and the given order. For example, a natural question could be to characterize sublattices which induce an extension or a restriction of the original order.

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