Lattices of Parabolic Subgroups in Connection with Hyperplane Arrangements

Hélène Barcelo and Edwin Ihrig
Department of Mathematics, Arizona State University, Tempe, Arizona 85287-1804
helene@math.la.asu.edu
ihrig@math.la.asu.edu

Abstract

A un groupe W (réel, fini) de réflections correspond le treillis L_W de toutes les intersections d'hyperplans associés à W. On démontre que L_W est isomorphe au treillis de tous les sous-groupes paraboliques de W. Cet isomorphisme est ensuite utilisé pour caractériser les groupes de réflexions (réels et finis) qui possèdent un treillis superrésoluble. Il existe une procédure combinatoire, "la procédure des mains", pour engendrer toutes les bases de circuits non brisés (NBC) d'un treillis superrésoluble. On démontre que le treillis L_W est superrésoluble si et seulement si toutes ses bases NBC peuvent être obtenues grâce a cette méthode.

Let L be the lattice consisting of all intersections of hyperplanes in the arrangement associated with a finite real reflection group W. We show that L is isomorphic to the lattice consisting of all parabolic subgroups of the reflection group. This isomorphism is used to determine all W for which L is supersolvable. Also, there is a well known combinatorial procedure for the generation of all non-broken circuit bases (NBC bases) of a supersolvable lattice. If the NBC bases of a geometric lattice can be obtained by this procedure, we say that the NBC bases are "obtainable by hands." We show that L is supersolvable if and only if all the NBC bases of L are obtainable by hands.

1. Introduction

By an arrangement $\mathcal{A}$, we mean a finite collection of codimension 1 subspaces of a real vector space V. Associated to $\mathcal{A}$ is a lattice which consists of all possible intersections of elements of $\mathcal{A}$, ordered by reverse set inclusion. A rich theory has been developed to study the properties of this lattice (see [7]). However, these lattices are somewhat abstract, and it can sometimes be useful to have a more concrete realization of them. One example of such a realization arises within the class of reflection arrangements. If W is a finite group generated by a set of reflections acting on $\mathbf{R}^n$, the reflection arrangement corresponding to W is the arrangement consisting of the reflecting hyperplanes of all possible reflections in W. We call the intersection lattice corresponding to this arrangement a reflection lattice (with group W). When W is the symmetric group S_n, with its usual action by permutation matrices on $\mathbf{R}^n$, the corresponding reflection lattice is isomorphic to the partition lattice, the lattice consisting of all partitions of the set $\{1, \ldots, n\}$ ordered by refinement.

It is the purpose of this paper to give a generalization of this correspondence which applies to all reflection lattices, and to illustrate its utility by using it to resolve the question of which reflection lattices are supersolvable. Also, we are able to use it to characterize the reflection lattices which admit a certain combinatorial procedure for generating all of the non-broken circuit bases of the lattice.

Primary 05E15, 52B30; Secondary 20D30, 20F55, 06B99

It may not be immediately clear what the generalization of the partition lattice should be. In fact, it was only in retrospect, after our correspondence theorem was proven, that it became evident that this theorem gave the correspondence between the reflection lattice of S_n and the partition lattice as a special case.

The key idea crucial to the discovery of this theorem came from a seemingly unrelated result; namely, the fundamental theorem of Galois theory.

Let E be a field, and K be a subfield of E (such that E is Galois over K). The fundamental theorem of Galois theory states that there is a bijective correspondence between the subfields of E which contain K, and the subgroups of the group of all field automorphisms of E which fix every element of K. For our purposes, we will think of the situation in a slightly different way.

We first define a more general context in which the Galois correspondence belongs. Let K be a field, as before, and let V be a finite dimensional vector space over K. Assume that we are given a sublattice L of the lattice of vector subspaces of V (ordered by reverse set inclusion). Assume also that we are given a subgroup G of $GL(V)$, the group of invertible linear transformations of V. Now, for any $U \in L$, we can define the subgroup $Gal(U)$ of G by

$$Gal(U) = \{g \in G : g(u) = u \text{ for all } u \in U\}.$$

In addition, define

$$Gal(L) = \{Gal(U) : U \in L\}.$$

In this context we can then ask two questions. The first is whether there is some concrete characterization of $Gal(L)$. The second is whether the mapping $Gal : L \rightarrow Gal(L)$ is a bijection. If Gal is a bijection, then Gal will be an isomorphism of partially ordered sets (when $Gal(L)$ is ordered by set inclusion). Hence, $Gal(L)$ will be a lattice, and Gal will be a lattice isomorphism.

The fundamental theorem of Galois theory can be viewed as a resolution of these two questions for a specific choice of the lattice L and the group G. As before, we start with E a field containing K (such that E is finite dimensional and Galois over K). It is convenient to define a kind of "forgetful functor" here. Given a field F containing K, define F' to be a vector space over K as follows. The underlying set of F' is the same as F. Moreover, the addition in F' is the same as the field addition in F, and scalar multiplication in F' is the natural restriction of field multiplication in F. Now, let V be E', where E is the field mentioned above, and let

$$L = \{F' : F \text{ is a field with } K \subset F \subset E\}.$$

Also, define G to be the set of$g \in GL(V)$ such that g is a field automorphism when considered as a function from E to E. With this choice of L and G, the fundamental theorem of Galois theory is exactly the statement that $Gal(L)$ is the lattice of all subgroups of G, and that Gal is a lattice isomorphism.

In this paper we concentrate a different choice of L and G. We start with a reflection group W acting on a vector space V over $\mathbf{R}$. We let K be $\mathbf{R}$, and we let L be the reflection lattice corresponding to W defined above. Also let $G = W$. Our main result, theorem 3.1, is that in this case $Gal(L)$ is the collection of all parabolic subgroups of W, and that the correspondence Gal is indeed a lattice isomorphism. Section 3 is devoted to developing the machinery needed to prove this result. In section 4, we show that when W is S_n , there is an explicit natural isomorphism between the lattice of parabolic subgroups and the partition lattice. In section 5, we use Theorem 3.1 to characterize supersolvable reflection lattices. This result is Theorem 5.1. Finally, in section 6, we give a relationship between the parabolic subgroups of a reflection group and the non-broken circuit bases of the corresponding reflection lattice when the atoms of this lattice are given a certain natural total order. This constitutes Theorem 6.1.

2. Preliminaries.

First we review a few facts about reflection groups that can be found, for example in [6]. We are borrowing his notation. Let $\mathbf{R}^n$ be the $n-$dimensional Euclidean space endowed with a certain positive definite symmetric bilinear form (v, u) (for $v, u \in \mathbf{R}^n$). A *reflection* $r_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ sends the nonzero vector α to its negative while fixing pointwise the hyperplane H_α orthogonal to α. It is easy to check that

$$r_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$$

and that r_α is an orthogonal transformation of order 2. Consider a finite set Φ of non-zero vectors in $\mathbf{R}^n$ satisfying the following two conditions, for all $\alpha \in \Phi$:

$$\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}, \tag{$R1$}$$

where $\mathbf{R}\alpha$ is the line spanned by α, and

$$r_\alpha \Phi = \Phi. \tag{$R2$}$$

Define W to be the group generated by all reflections r_α, $\alpha \in \Phi$. Φ is said to be a *root system* of W, and the elements of Φ are called *roots*. In general roots need not be of unit length, but hereafter we will always choose root systems with roots of length one. It happens that the reflections r_α are *all* the reflections in W, and W is said to be a *real (finite) reflection group*.

Given a total ordering (compatible with the vector space structure) of $\mathbf{R}^n$, a subset Π of Φ is called a *positive system* if it consists of all the roots which are positive with respect to the given order. A subset Δ of Φ is said to be a *simple system* if Δ is a vector basis for the $\mathbf{R}-$span of Φ in $\mathbf{R}^n$, and if each $\alpha \in \Phi$ is a linear combination of the elements of Δ with coefficients all of the same sign. To a positive system Π there corresponds a unique simple system Δ, with $\Delta \subset \Pi$. Moreover simple (resp. positive) systems are all conjugate to one another in W ([6, p.10]). Thus it makes sense to define the rank $r(W)$ of a reflection group W to be the number of elements in a simple system Δ; that is, $r(W) = |\Delta|$.

Each element $w \in W$ can be expressed in the form:

$$w = r_{\alpha_1} r_{\alpha_2} \dots r_{\alpha_k},$$

where $\alpha_i \in \Pi$, for $i = 1, \ldots, k$. The smallest value of k in any such expression for w is denoted $al(w)$, and is called the *absolute length* of w. An expression $r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_k}$ is said to be *totally reduced* if $k = al(r_{\alpha_1} \cdots r_{\alpha_k})$

Given a simple system of roots Δ for W, the subgroups of W generated by subsets $I \subseteq \Delta$ are of fundamental importance to our work.

Definition 2.1. $W_I \subset W$ is a parabolic subgroup of W if there exists a simple system of roots Δ for W with a subset $I \subset \Delta$ such that W_I is generated by the set $\{r_{\alpha_i} \mid \alpha_i \in I\}$.

There is a very nice presentation for W in terms of the simple roots of W that is also of importance to us. For any roots $\alpha, \beta \in \Phi$, let $m(\alpha, \beta)$ denote the order of the product $r_\alpha r_\beta$ in W.

Proposition 2.1 [6, p.16]). *Fix a simple system Δ in Φ. Then W is generated by the set $S = \{s_\alpha | \alpha \in \Delta\}$, subject only to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \quad (\alpha, \beta \in \Delta).$$

This presentation of W shows that W is determined up to isomorphism by the set of integers $m(\alpha, \beta)$, (for $\alpha, \beta \in \Delta$). Coxeter (see [5]) encoded this information in a labelled graph Γ constructed as follows: Let Γ be a graph whose vertex set is indexed by the elements of Δ; two distinct vertices α, β are joined by an edge, labelled $m(\alpha, \beta)$, whenever $m(\alpha, \beta) \geq 3$. A pair of vertices not joined by an edge implicitly means that $m(\alpha, \beta) = 2$. This graph is called the Coxeter graph of W and uniquely determines (up to isomorphism) W. Note that since simple systems are conjugate, Γ does not depend on the choice of Δ.

Later on we will need the notion of exponent together with that of degree for finite reflection groups. Let Δ be a simple system for W with corresponding simple reflections $r_{\alpha_1}, r_{\alpha_2}, \ldots r_{\alpha_n}$; the product $R = r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_n}$ is said to be a *Coxeter element.* The fact that all Coxeter elements are conjugate in W ([6, p.74]) insures that all Coxeter elements have the same order as well as the same characteristic polynomial and eigenvalues. Thus if h is the order of R and if we let $\varepsilon = exp(2\pi i/h)$, then the eigenvalues of R are of the form

$$\varepsilon^{m_1}, \ \varepsilon^{m_2}, \ \ldots \ , \varepsilon^{m_n}.$$

The m_i so defined are called the *exponents* of W, and we see that they do not depend on the particular product R. Next we turn to the definition of degrees of W. Let $\mathbf{R}[x_1, \ldots, x_n]$, be the polynomial ring in n variables over the field of real numbers (where n is the rank of W). Since W is a group of orthogonal transformations of $\mathbf{R}^n$ it acts naturally on $\mathbf{R}[x_1, \ldots, x_n]$. The set of polynomials fixed under the action of W forms a (invariant) ring $I(W)$ which is generated by n algebraically independent homogenous polynomials: $f_1, \ f_2, \ \ldots, f_n$. Such an independent set is not unique, but the corresponding sequence $d_1, \ d_2, \ \ldots, d_n$ of degrees is unique. Moreover the d_i's are related to the exponents by the following identity

$$m_i = d_i - 1.$$

We now turn to the definition of the lattice L_W. Let the orthogonal complement of any subset $X \in \mathbf{R}^n$ be denoted by

$$X^{\perp} = \{v \in \mathbf{R}^n | (v, x) = 0, \forall x \in X\}.$$

Let $\mathcal{A}$ be the set of all *reflecting hyperplanes* associated with W; that is,

$$\mathcal{A} = \{\alpha^{\perp} | \alpha \in \Phi\} = \{H_{\alpha} \mid \alpha \in \Phi\},$$

and let L_W denote the poset of all possible intersections of hyperplanes in $\mathcal{A}$ ordered by *reverse set inclusion.* Denote the partial order of L_W by $\leq$ ($X \leq Y$ if and only if $Y \subseteq X$). It is a known fact [1, p. 23] that L_W is a *geometric lattice*, with rank function given by $r(X) = codim(X)$ for any $X \in L_W$. Thus, all the reflecting hyperplanes H_{α} have rank one and are called the *atoms* of L_W. Moreover, for any two elements X and Y of L_W the *meet* of X and Y is given by:

$$X \wedge Y = \bigcap \{Z \in L_W | X \cup Y \subseteq Z\},$$

while if $X \cap Y \neq 0$, the *join* of X and Y is defined to be:

$$X \vee Y = X \cap Y.$$

We also need to review the notions of independent set and basis for geometric lattices. Let L be a geometric lattice. The elements of L of rank one are called the *atoms* of L, and the set of all atoms will be denoted by A. A subset $B = \{b_1, \ldots, b_m\} \subseteq A$ is said to be *independent* if the rank of the join of its elements $\vee B = b_1 \vee \cdots \vee b_m$ satisfies, $r(\vee B) = |B|$. Otherwise, B is said to be *dependent.* A subset $B \subseteq A$ is said to be a *base* for an element $X \in L$ if and only if B is independent and if $\vee B = X$. A *circuit* is a dependent set $B \subseteq A$ such that all its proper subsets $C \subset B$ are independent. Given a total order $\prec$ on the set of atoms A, we say that $B = \{b_1, \ldots, b_k\} \subseteq A$ is a *broken circuit*, denoted BC, if there is an atom $a \in A$ such that $a \prec b_i$ for all $i = 1, \ldots, k$ and $B \cup \{a\}$ is a circuit. In other words, the broken circuits are obtained from the circuits by removing the smallest atom. A *non-broken-circuit*, NBC, is a set of atoms that does not contain any broken circuit. It can be shown that NBC sets are independent sets of atoms. There is a fundamental link between the NBC bases of L_W and the elements of W. Indeed the first author together with A. Goupil and A. Garsia established in [2] the following correspondence. Let $\{H_{\alpha_1}, \ldots, H_{\alpha_k}\}$ be an NBC base where $\alpha_i < \alpha_j$ if $i < j$. Let this NBC base correspond to w defined by

$$w = r_{\alpha_1} \ldots r_{\alpha_k}. \tag{2.1}$$

It turns out that equation (2.1) is a totally reduced expression for w, and this correspondence is a bijection between W and the set of all NBC basis of L_W. Moreover, the enumerating polynomial for all the NBC bases of L_W

$$\sum_{S \in NBC(W)} t^{|S|} \tag{2.2}$$

has a factorization that involves the exponents of W (see [2]):

$$\sum_{S \in NBC(W)} t^{|S|} = \prod_i (1 + m_i t).$$

We shall return to this factorization in section 4.

3. The lattice $\mathcal{P}_W$ of parabolic subgroups of W.

Definition 3.1. Let $\mathcal{P}_W$ be the poset of all the parabolic subgroups of W, ordered by set inclusion.

Fixing a simple system and taking all the parabolic subgroups with respect to this simple system gives rise to a Boolean lattice [6, p.24]. But, here we do *not* fix the simple system a priori, we consider the set of all parabolic subgroups with respect to any simple system. A word of caution is needed here. For the time being $\mathcal{P}_W$ is simply a poset. Thus there is no extra structure (like a rank function) nor property (such as being a geometric lattice, etc...) associated with $\mathcal{P}_W$. These structures and properties will be obtained gratuitously once we show there is an isomorphism between L_W and $\mathcal{P}_W$. Hence, when speaking of the rank of a parabolic subgroup W_I (with simple system of roots given by I) we are simply referring to the cardinality of I. We will see later on, that indeed this notion of rank is an appropriated rank function on $\mathcal{P}_W$. We now define two correspondences that will enable us to go from the lattice L_W to the lattice $\mathcal{P}_W$ and vice-versa. Let W be a finite reflection group acting on $\mathbf{R}^n$, and for any given subset S of W let $\langle S \rangle$ be the subgroup of W generated by the elements of S.

Definition 3.2

(i) Let $G \subseteq W$be any subgroup of W. The fixpoint space of G is

$$Fix(G) = \{v \in \mathbf{R}^n | gv = v, \text{ for all } g \in G\},$$

(ii) Let $X \subseteq \mathbf{R}^n$. Define $Gal(X)$ by

$$Gal(X) = \{w \in W | wx = x \text{ for all } x \in X\}.$$

For example, consider the subgroup of W generated by a single reflection r_α. Then clearly $Fix(\langle r_\alpha \rangle) = H_\alpha$, which indeed belongs to L_W. It turns out that if G is a reflection subgroup of W then the fixpoint space of G belongs to L_W. More precisely, if G is generated by a given set $\{r_{\alpha_1}, \ldots, r_{\alpha_k}\}$ of reflections then its fixpoint space $Fix(G)$ corresponds to the intersection of the reflecting hyperplanes $\{H_{\alpha_1}, \ldots, H_{\alpha_k}\}$; that is,

Proposition 3.1. *If $G \subset W$ is a reflection group, then $Fix(G)$ belongs to L_W. In particular, if $G \in \mathcal{P}_W$ then $Fix(G) \in L_W$.*

As an immediate consequence of this lemma we see that:

Corollary 3.1. *If both $T = \{\alpha_1, \ldots, \alpha_k\} \subset \Phi$, and $S = \{\beta_1, \ldots \beta_m\} \subset \Phi$ generate $G \subseteq W$ then*

$$H_{\alpha_1} \vee \ldots \vee H_{\alpha_k} = H_{\beta_1} \vee \ldots \vee H_{\beta_m}.$$

We now claim that for any element V of L_W, $Gal(V)$ is indeed a parabolic subgroup of W. Being an element of L_W means that V is the intersection of certain reflecting hyperplanes of W: $V = \bigcap_{i=1}^{k} H_{\alpha_i}$. It will not be difficult to prove that indeed $Gal(V)$ is a reflection subgroup of W. Unfortunately, in general there are reflection subgroups of W that are not necessarily parabolic subgroups of W. For example the dihedral group $\mathbf{D}_a$ of order a is a reflection subgroup of any dihedral group $\mathbf{D}_b$ of order b provided that a divides b. On the other hand $\mathbf{D}_a$ is certainly not a parabolic subgroup of $\mathbf{D}_b$, since each of these groups have rank 2. Thus we will also have to show that there exists a simple system of roots for W that generates $Gal(V)$.

Proposition 3.2. *If $V \in L_W$ then $Gal(V)$ belongs to $\mathcal{P}_W$. Moreover, the root system Φ' for $Gal(V)$ is:*

$$\Phi' = \Phi \cap V^{\perp}.$$

Before we can show that the posets L_W and $\mathcal{P}_W$ are isomorphic we need to show one more property.

Proposition 3.3. *Let $Fix : \mathcal{P}_W \mapsto L_W$ and $Gal : L_W \mapsto \mathcal{P}_W$ be the two correspondences defined earlier. Both Fix and Gal are order and rank preserving correspondences.*

We are finally ready to state our main theorem.

Theorem 3.1. *The (ranked) partially ordered sets L_W and $\mathcal{P}_W$ are isomorphic. Gal is an isomorphism with inverse Fix.*

Corollary 3.2. *The poset $\mathcal{P}_W$ is a geometric lattice. Moreover, its rank function is given by: $r(W_I) = |I|$.*

Before showing the importance of this isomorphism we give an example.

4. Example: $\mathcal{P}_{S_n}$.

Let S_n be the symmetric group. It can be thought of as a subgroup of the group $O(n, \mathbf{R})$ of $n \times n$ orthogonal matrices; that is, we can think of $\sigma \in S_n$ as a permutation matrix. A permutation σ acts on $\mathbf{R}^n$ by permuting the elements e_i (for $i = 1, \ldots, n$) of the standard basis: $\sigma e_i = e_{\sigma(i)}$. A transposition (ij) acts as a reflection, by sending the vector $e_i - e_j$ to its negative and by fixing pointwise the orthogonal complement (i.e.: all vectors in $\mathbf{R}^n$ whose ith and jth coordinates are equal). Since S_n is generated by transpositions, it is a finite reflection group. Indeed, the set $\Phi = \{e_i - e_j\}_{1 \leq i < j \leq n}$ is a root system, and the usual lexicographical order on $\mathbf{R}^n$ yields $\Delta = \{e_i - e_{i+1}\}_{1 \leq i \leq n-1}$ as its corresponding simple root system. Since all simple systems are conjugate to one another we have that any other simple system is of the form

$$\Delta' = \sigma\Delta, \text{ for some permutation } \sigma \in S_n.$$

This means that any system of simple reflections for S_n will be of the form

$$\sigma\{(12),(23),\ldots(n-1n)\}\sigma^{-1} = \{(\sigma_1\sigma_2),(\sigma_2\sigma_3),\ldots(\sigma_{n-1}\sigma_n)\} \tag{3.1}$$

The lattice of parabolic subgroups of S_n is naturally isomorphic to the partition lattice Π_n; that is, the lattice whose elements are partitions of the set $[n] = \{1,2,\ldots,n\}$ ordered by refinement. In order to avoid cumbersome notation, we will not give a formal proof of this isomorphism. Rather we will illustrate how the isomorphism works in a special case. This should enable the reader to produce the general isomorphism without difficulty. Indeed, given any subset S_I of a simple system of reflections $S = \{(\sigma_1\sigma_2),(\sigma_2\sigma_3),\ldots(\sigma_{n-1}\sigma_n)\}$ we associate a partition of $[n]$ in the following manner. $\{1,\ldots,n\}$ is partitioned by the classes of the equivalence relation generated by the set of ordered pairs S_I. For example let $n = 7$ and let $S_I = \{(12),(23),(45)\}$ then the equivalence classes are

$$\{1,2,3\},\ \{4,5\},\ \{6\},\ \{7\}.$$

Conversely given a partition π of $\{1,\ldots,7\}$ how can we recover a simple system of roots whose equivalence classes would be the blocks of π. For example let

$$\pi = \{\{1,3,4,7\},\ \{2,5\},\ \{6\}\}$$

be a partition of [7]. First we realize that the set of all reflections corresponding to π can only be

$$R_I = \{(13),(14),(17),(34),(37),(47),(25)\},$$

thus yielding the reflection group $W_I = \langle R_I\rangle$. Now, how can we choose a subset of R_I that is a simple system for W_I? Since we know that such a subset must be of the form given by equation 3.1 we realize that certain subsets like $\{(13),(14),(17),(25)\}$ are not good ones. On the other hand $S_I = \{(13),(34),(47),(25)\}$ would be a good choice. In fact we claim that S_I is a simple system for W_I when we take

$$S_\Delta = \{(13),(34),(47),(72),(25),(56)\}$$

for simple system of S_7. We only need to show that there is a permutation $\sigma \in S_7$ such that $S_\Delta = \sigma\{(12),(23),\ldots,(67)\}\sigma^{-1}$. Clearly this permutation (in two lines notation)is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 7 & 2 & 5 & 6 \end{pmatrix}.$$

From this example we see that this correspondence between $\mathcal{P}_{S_n}$ and Π_n is a bijection. A moment's thought reveals that it is also a rank preserving bijection. Thus we have that $\mathcal{P}_{S_n}$ and Π_n are isomorphic lattices. We now use this correspondence to relate the lattice isomorphism Gal to the known correspondence between the intersection lattice of the reflecting hyperplanes of S_n and Π_n ([7]). This bijection is obtained as follows. Let

$V \in L_{S_n}$, and denote the reflecting hyperplane corresponding to a reflection (ij) by $H_{i,j}$. Also let $\mathbf{R}^n = H_{i,i}$ for all $i \in [n]$. Define an equivalence relation $\sim_V$ on $[n]$ by $i \sim_V j$ if and only if $V \subseteq H_{i,j}$. Let π_V be the partition of $[n]$ define by $\sim_V$. The map $\pi : L_{S_n} \to \Pi_n$ given by $\pi(V) = \pi_V$ is a lattice isomorphism. Hence, while Theorem 3.1 ensures us of a direct correspondence between V and $Gal(V)$, the correspondence going through Π_n allows us to explicitly find simple systems for $Gal(V)$.

5. Supersolvable lattices.

One of the main problems concerning the lattices L_W is to determine if they are supersolvable. We will not give details here, but let us say that if L_W is a supersolvable lattice then there is a combinatorial way of determining its characteristic polynomial. For an overview and references of this subject see [3]. As we mentioned in the introduction it is not easy in general to determine if a lattice is supersolvable. When the reflection group is either S_n, B_n (the group of signed permutations), or $\mathbf{D}_n$ (the dihedral group), it is known that the corresponding lattices are supersolvable. But for the other reflection groups there is no complete list in the literature answering this question. Through personal communications with G. Ziegler and H. Terao it was suggested that none of the others were supersolvable. In this section, we give an elegant combinatorial proof (using the lattice of parabolic subgroups), of the fact that the only supersolvable lattices L_W are the ones corresponding to either $\mathbf{D}_n$ or the reflection groups of type A_n and B_n. Let us first recall the definition of supersolvability. Let L be a finite geometric lattice of rank $r(L) = n$. An element $m \in L$ is called *modular* [8] if

$$r(m) + r(m') = r(m \vee m') + r(m \wedge m')$$

for every $m' \in L$. Let $\hat{0}$ be the minimal element of L and $\hat{1}$ be its maximal element. A geometric lattice L is said to be *supersolvable* [9] if it has a maximal chain

$$\hat{0} = m_0 < m_1 < \cdots < m_n = \hat{1}$$

of modular elements, (called an M-chain of L). Let A be the set of atoms of L, and let $\sim$ be an equivalence relation on A.

Definition 5.1.

Define $\wp(\sim)$ to be

$$\wp(\sim) = \{S | S \subseteq A \text{ and } S \text{ contains at most one element from each equivalence class of } \sim\}.$$

Note that when $\sim$ is equality, then $\wp(\sim) = \wp(A)$, the power set of A.

Let $\prec$ be a total order on A. We say that the NBC bases of L, $NBC(L)$, with respect to $\prec$ are obtainable by the *hands* of $\sim$ if

$$NBC(L) = \wp(\sim).$$

In the next theorem we use the classification of all the real finite reflection groups, together with their Coxeter diagrams and lists of degrees. See for example [6, p.32 and p.59].

By analogy to previous work of the first author, the equivalence classes of $\sim$ are called the hands of $\sim$.

Theorem 5.1. *Let W be an irreducible real finite reflection group, A be the collection of all its reflecting hyperplanes, and L_W its corresponding lattice. The following are equivalent:*

(a) L_W is supersolvable.
(b) There is a total order $\prec$ and an equivalence relation $\sim$ on A, so that the $NBC(L_W)$ bases with respect to $\prec$ are obtainable from the hands of $\sim$.
(c) There is a label of the Coxeter diagram of W (other than 2) which is a degree of W.
(d) W is either of type A_n, B_n or is $\mathbf{D}_n$.

6. Non-Broken Circuit Bases.

As we saw earlier on the NBC bases play a fundamental role in many aspects of the theory of reflection groups, and moreover the elements of $NBC(L_W)$ are in one to one correspondence with the elements of W. So now if we consider the lattice of parabolic subgroups $\mathcal{P}_W$ what can be said about its NBC bases? Are they easy to characterize? In this section we identify some of the NBC bases of $\mathcal{P}_W$ and show that when translated into the L_W lattice they remain NBC bases. Unfortunately this characterization does not yield all NBC basis. For this entire section we fix a total order on $\mathbf{R}^n$. Let H_α be an atom of L_W. Then there will be a unique positive root $\alpha \in H_\alpha^\perp$. Thus, the total ordering on $\mathbf{R}^n$, when restricted to the roots of W, gives rise to a total ordering on the atoms of L_W. Using this total ordering one can define NBC bases for L_W. Also, for any reflection subgroup $W_I \subseteq W$, we may use this total ordering of $\mathbf{R}^n$ to induce a total ordering on the root space of W_I. This ordering defines a unique system of simple roots for W_I, denoted by Δ_I.

Theorem 6.1. *Let W_I be a parabolic subgroup of W. Then*

$$NBC_{W_I} \equiv \{\alpha^\perp | \alpha \in \Delta_I\}$$

is an NBC basis for L_W.

References

1. H. Barcelo, *On the action of the symmetric group on the free Lie algebra and the partition lattice,* J. of Combinatorial Theory, Series A **55**, No. 1 (1990), 93–129.
2. H. Barcelo and A. Goupil, *Non-broken circuits of reflection groups and factorization in D_n*, Israel Journal of Mathematics,
3. —, *Combinatorial aspects of the Poincaré polynomial associated with a reflection group*, Contemporary Mathematics, **178** (to appear).
4. R. W. Carter, *Conjugacy classes in the Weyl group*, Compositio Mathematica **25** (1972), 1–59.
5. H. S. M. Coxeter, *Regular complex-polytopes*, Second ed., Cambridge University Press, 1991.
6. James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29** (1990), Cambridge University Press.

7. P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Series of Comprehensive Studies in Math. 300, Springer-Verlag, Berlin, Heidelberg, 1992.
8. R. Stanley, *Modular elements of geometric lattices*, Algebra Universalis **1** (1971), 214–217.
9. —, *Supersolvable lattices*, Algebra Universalis **2** (1972), 197–217.
10. H. Terao, *Modular elements of lattices and topological fibration*, Advances in Mathematics **62** No. 2 (1986), 135–154.