Towards a methodology for tree enumeration

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Abstract. In this paper, we illustrate a method for enumerating subclasses of plane trees, called first order classes. By means of the plane trees' traversals, which satisfy two particular conditions, we give some recursive descriptions of these subclasses. We use these descriptions to deduce the functional equations verified by the trees' generating functions. We illustrate two applications of the method to the whole class of plane trees and to right-leafed trees and establish both their generating functions according to the number of their internal nodes, the number of their leaves, their right branch length and their internal path length.

Résumé. Dans cet article nous présentons une méthode pour l'énumération des sous-classes des arbres planaires que nous appelons classes du premier ordre. Nous considérons les algorithmes d'exploration des arbres planaires qui satisfont deux conditions particulières et nous définissons des descriptions récursives pour les classes considérées. En utilisant ces descriptions, nous déduisons des équations fonctionnelles qui sont vérifiées par la fonction génératrice de ces classes. Nous décrivons l'application de cette méthode à la classe de tous les arbres planaires et à celle des arbres feuillus à droite. Pour ces classes nous trouvons la fonction génératrice selon le nombre de sommets, le nombre de feuilles, la longueur de la branche droite et la longueur du chemin intérieur.

1 Introduction

Tree enumeration is widely practised in combinatorics and in the analysis of algorithms. Many different tools and tricks have been used in approaching this problem in the various classes of trees. A survey of the results and methods concerning this subject can be found in books by Comtet [4], and Goulden and Jackson [8]. In this paper, we present a construction for enumerating plane trees' subclasses, called first order classes. These subclasses have the following property: each tree belonging to a subclass and having n nodes can be obtained from at least one tree having $n-1$ nodes and a new leaf added to it. The construction is determined by means of some plane trees' traversals able to satisfy two particular conditions. If $\mathcal{S}$ is a first order class, and H is one of these traversals, we obtain a recursive description of $\mathcal{S}$'s plane trees that allows us to deduce a functional equation verified by $\mathcal{S}$'s generating function. Different equations correspond to different traversals. By solving these functional equations, we obtain the plane tree generating functions according to various parameters. A similar method was used in [3, 6, 7]. Section 2 of this paper contains some definitions regarding plane trees. In Section 3, we determine a partition of the first order classes by means of some traversals. From this partition, we deduce a method for constructing $\mathcal{S}$'s trees having n nodes by starting out from $\mathcal{S}$'s trees with $n-1$ nodes. In Section 4, we examine the set of all the plane trees and we perform the constructions obtained by preorder and level traversal. We establish the plane trees' generating function according to the number of their internal nodes, the number of their leaves, their right branch length and their internal path length. In Section 5, the construction obtained by preorder traversal is applied to a class of plane trees studied by Donaghey

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and Shapiro [5] and related to Motzkin numbers. We call these trees right-leafed trees. We determine their generating function according to the same parameters used in section 4.

2 Notations and Definitions

Let A be a finite set. A *plane tree* is an ordered partition $\{\{a\}; A_1; A_2; \ldots; A_m\}$ of A, such that $a \in A$ and each A_i is a plane tree. A's elements are called *internal nodes* (or *nodes*) and node a is called the *root* of the plane tree. A plane tree is an unlabelled tree (i.e., its nodes are indistinguishable). The sets $A_1, A_2, \ldots, A_m$ are the *subtrees* of the root. Each node of a plane tree can be the root of some subtrees contained in the tree. The number of a node's subtrees is called the *degree* of the node. An internal node of degree zero is called a *leaf*. Each root is said to be the *father* of its subtrees' roots, which, in turn, are called *sons* of their father. The sons of the same father are referred to as *brothers*. The *level* of a node is defined as follows: the root's level is 0, and all the other nodes' level is one unit higher than their father's. A plane tree's *k−th level* is the set of its nodes at level k. Let $\mathcal{S}$ be a subclass of plane trees and let $\mathcal{S}_n$ be the set of the plane trees of $\mathcal{S}$ having n internal nodes. A subclass $\mathcal{S}$ of plane trees is a *first order* class if each $P' \in \mathcal{S}_n$ can be produced from at least one $P \in \mathcal{S}_{n-1}$ by adding a new leaf to P. The set of all the plane trees is a first order class. Let us extend each plane tree of $\mathcal{S}$ by attaching a special son to any of its internal nodes. These new nodes are called *external nodes*. If $P \in \mathcal{S}_{n-1}$, then we obtain a plane tree $P' \in \mathcal{S}_n$ by replacing any of P's external nodes with an internal one. In fig. 1, a 1-2 tree (i.e., a plane tree in which each node has degree 0 or 1 or 2) is illustrated. A *traversal* of a plane tree is a linear arrangement of its internal

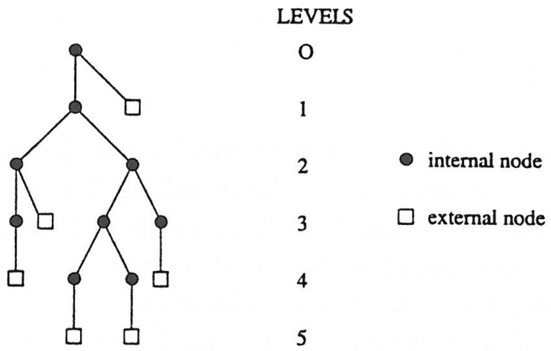

Figure 1: A 1-2 Tree

and external nodes. When each node precedes its sons, a traversal is said to be *hierarchical*. *Preorder traversal* is a hierarchical traversal.

3 The Construction

Let $\mathcal{S}$ be a first order class and let H be a hierarchical traversal. Let $x_l(P)$ be P's last internal node in H, where $P \in \mathcal{S}_n$. Then $x_l(P)$ is a leaf of P since H is hierarchical. Let us denote by $\phi(P)$ the set obtained from P by deleting P's subtree having $x_l(P)$ as its root. The tree $\phi(P)$ is a plane tree having $n-1$ internal nodes, but, in general, $\phi(P) \notin \mathcal{S}_{n-1}$. We denote:

- $\mathcal{F}(P)$ as the set of P's external nodes that follow $x_l(P)$ in H;

- $\mathcal{G}(P)$ as the set of the plane trees obtained from P by replacing a node of $\mathcal{F}(P)$ with an internal node.

For example, if H is the preorder traversal, we obtain the set $\mathcal{G}(P)$ shown in fig.2.

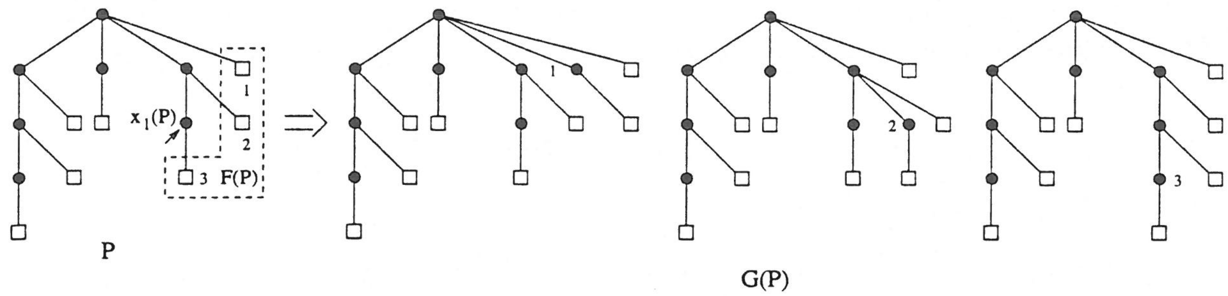

Figure 2: The set $\mathcal{G}(P)$ of a plane tree obtained by means of preorder traversal

Theorem 1 *Let $\mathcal{S}$ be a first order class and let H be a hierarchical traversal that satisfies the following conditions:*

1. *if $P \in \mathcal{S}_n$, then $\phi(P) \in \mathcal{S}_{n-1}$ and $P \in \mathcal{G}(\phi(P))$,*
2. *if $P' \in \mathcal{G}(P)$, then the internal node x added to P in order to obtain P' is the last internal node of P' in H (i.e., $x = x_l(P')$).*

Then the following family of sets: $\mathcal{F}_n = \{\mathcal{E} \subseteq \mathcal{S}_n : \mathcal{E} = \mathcal{G}(P)\ \forall P \in \mathcal{S}_{n-1}\}$, is a partition of $\mathcal{S}_n$.

Proof. Let $\mathcal{B} = \bigcup_{P \in \mathcal{S}_{n-1}} \mathcal{G}(P)$. From the definitions of internal node and $\mathcal{G}(P)$, it follows that if $P \in \mathcal{S}_{n-1}$, then $\mathcal{G}(P) \subset \mathcal{S}_n$. Hence $\mathcal{B} \subseteq \mathcal{S}_n$. Vice versa, from condition 1, it follows that if $P \in \mathcal{S}_n$, then $\phi(P) \in \mathcal{S}_{n-1}$ and $P \in \mathcal{G}(\phi(P))$, and so $\mathcal{S}_n \subseteq \mathcal{B}$. We assume that $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{F}_n$ and $\mathcal{E}_1 \cap \mathcal{E}_2 \neq \emptyset$. Therefore, $P_1, P_2 \in \mathcal{S}_{n-1}$ exist such that $\mathcal{E}_1 = \mathcal{G}(P_1)$, $\mathcal{E}_2 = \mathcal{G}(P_2)$ and $\mathcal{G}(P_1) \cap \mathcal{G}(P_2) \neq \emptyset$. If $P \in \mathcal{G}(P_1) \cap \mathcal{G}(P_2)$, then $P = P_1 \cup \{x_1\}$ and $P = P_2 \cup \{x_2\}$, where x_1 and x_2 are the internal nodes added to P_1 and P_2, respectively, in order to obtain P. But, condition 2 implies that x_1 (resp. x_2) is the last internal node of P in H, and so $x_1 = x_2$. Consequently, $P_1 = P_2$. Hence $\mathcal{F}_n$ is a partition of $\mathcal{S}_n$. □

If $\mathcal{S}$ is a first order subclass and H is a hierarchical traversal which satisfies conditions 1 and 2, then Theorem 1 allows as to construct $\mathcal{S}_n$ from $\mathcal{S}_{n-1}$. For every $P \in \mathcal{S}_{n-1}$, we construct $\mathcal{G}(P)$ and obtain all of $\mathcal{S}_n$'s plane trees. Moreover, every $P' \in \mathcal{S}_n$ is obtained by one and only one $P \in \mathcal{S}_{n-1}$. Therefore, this construction allows us to obtain $\mathcal{S}_n$ by starting from $\mathcal{S}_{n-1}$; since this construction is determined by means of the hierarchical traversal H, it is denoted as C_H. The construction C_H is a recursive description of $\mathcal{S}$ that allows us to deduce a functional equation verified by $\mathcal{S}$'s generating function. The same approach has been recently used in [3] to enumerate various classes of column-convex polyominoes. In the following sections, we use this construction for enumerating two classes of plane trees according to various parameters.

4 Plane trees

The set of all the plane trees is a first order class. Let us denote this set by $\mathcal{P}$. The rightmost son of each node of a plane tree $P \in \mathcal{P}$ is an external node (see fig.4). Let us now examine *preorder* *traversal* and *level traversal* (i.e., visiting the root and then the other nodes on increasing levels from left to right). It is easy to prove the following:

Proposition 2 *Preorder traversal and level traversal on the class $\mathcal{P}$ satisfy conditions 1 and 2 of Theorem 1.*

By using one of these traversals, we can construct $\mathcal{P}_n$ from $\mathcal{P}_{n-1}$. For example, if H is the preorder traversal, we can construct $\mathcal{P}_5$ from $\mathcal{P}_4$, as shown in fig. 3. The two traversals yield

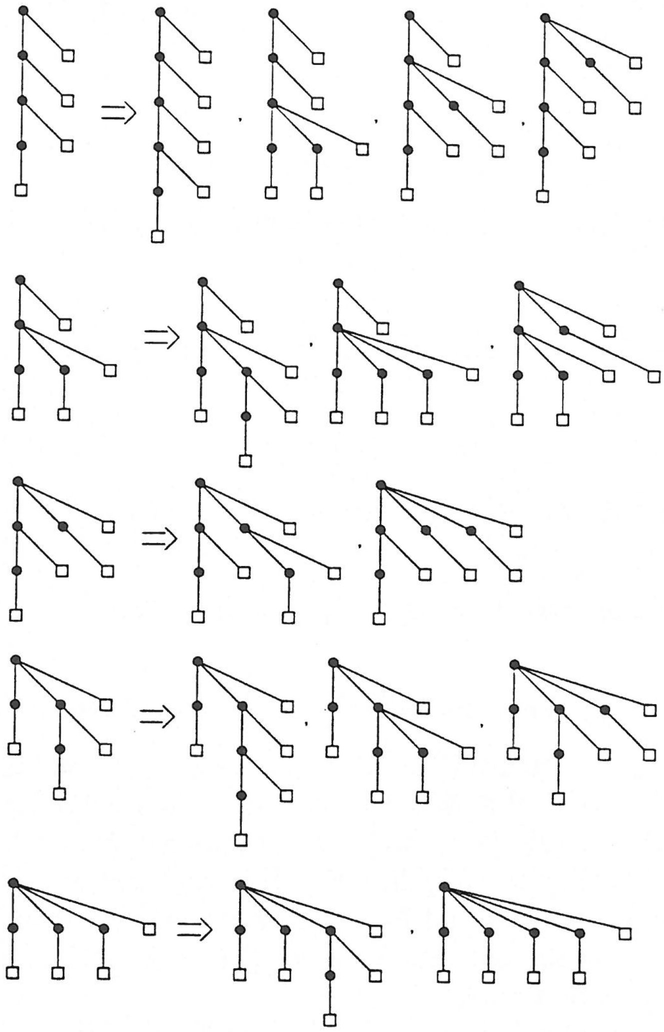

Figure 3: Construction C_H

two different recursive descriptions of plane trees. Each of them allows us to deduce a functional equation verified by the plane trees' generating function.

4.1 Constructing by means of preorder traversal

Let H be the preorder traversal. We begin by translating construction C_H into an equation. Let $P \in \mathcal{P}$. If a and a' are nodes of P, we say that $\{a_1, a_2, \ldots, a_n\}$ is a *path* of length n from a to a' if $a = a_1$, $a' = a_n$ and a_k is the father of a_{k+1} for $0 < k < n$. The *internal path length* of P is the sum (evaluated over all the internal nodes) of the lengths of the paths from the root to each internal node. The *right branch length* of P is the length of the path from its root to its rightmost internal node (the last one in the preorder traversal). We denote (see fig. 4):

- $r(P)$, P's right branch length,
- $n(P)$, P's internal nodes number,
- $l(P)$, P's leaves number,
- $i(P)$, P's internal path length.

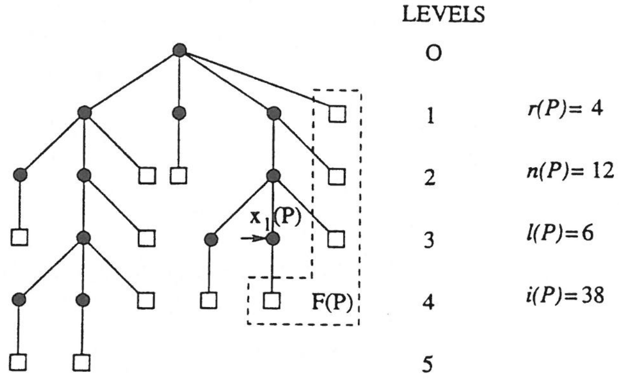

Figure 4: Parameters $r(P)$, $n(P)$, $l(P)$ and $i(P)$ of a plane tree

$\mathcal{P}$'s generating function according to the above-listed four parameters is the following:

$$A(s,x,y,q) = \sum_{P\in\mathcal{P}} s^{r(P)}x^{n(P)}y^{l(P)}q^{i(P)}.$$

We often denote $A(s,x,y,q)$ by $A(s)$. P's right branch length is equal to the number of $\mathcal{F}(P)$'s nodes (i.e., $|\mathcal{F}(P)| = r(P)$). Furthermore, for each $k \in [1..r(P)]$, there is $e \in \mathcal{F}(P)$, such that e's level is k (see fig. 4). Let $\mathcal{F}(P) = \{e_1, e_2, \ldots, e_{r(P)}\}$, where e_k is the node at level k. We now perform the construction C_H on P; that is, we determine $\mathcal{G}(P)$. Each tree of $\mathcal{G}(P)$ is obtained from P by replacing a node of $\mathcal{F}(P)$ with an internal node (see fig. 5).

- If we replace e_k, with $k \in [1..r(P)-1]$, then we obtain $P' \in \mathcal{G}(P)$ such that: $r(P') = k+1$, $n(P') = n(P)+1$, $l(P') = l(P)+1$ and $i(P') = i(P)+k+1$,
- if we replace $e_{r(P)}$, then we obtain $P' \in \mathcal{G}(P)$ such that: $r(P') = r(P)+1$, $n(P') = n(P)+1$, $l(P') = l(P)$ and $i(P') = i(P)+r(P)+1$.

Therefore, the translation of this construction into the generating function $A(s,x,y,q)$, gives us:

$$\sum_{P\in\mathcal{P}} \sum_{k=1}^{r(P)-1} s^{k+1}x^{n(P)+1}y^{l(P)+1}q^{i(P)+k+1} + \sum_{P\in\mathcal{P}} s^{r(P)+1}x^{n(P)+1}y^{l(P)}q^{i(P)+r(P)+1} =$$
$$= \frac{sxyq}{1-sq}(sqA(1,x,y,q) - A(sq,x,y,q)) + sxqA(sq,x,y,q).$$

Construction C_H, applied to the set P gives all the plane trees P such that $n(P) > 1$, and so:

Proposition 3 *The plane trees' generating function $A(s,x,y,q)$ verifies the following functional equation:*

$$A(s) = sxyq + \frac{s^2xyq^2}{1-sq}A(1) + \frac{sxq(1-y-sq)}{1-sq}A(sq). \qquad (1)$$

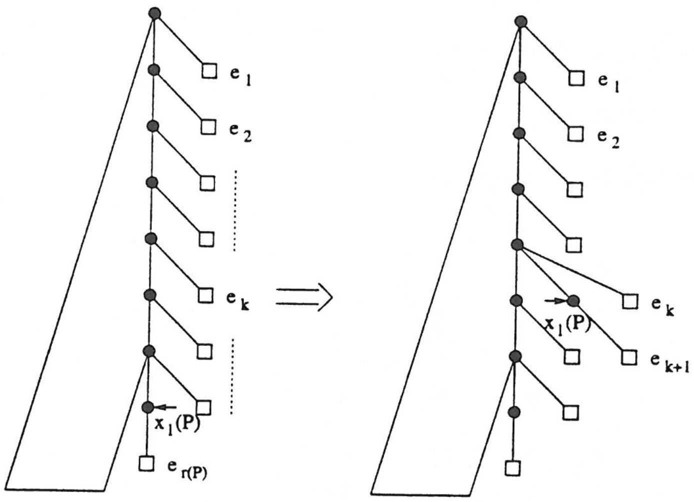

Figure 5: The construction according to preorder traversal

4.1.1 Solving equations

Let us now solve the functional equation 1 by using the following Lemma [3]:

Lemma 4 *Let $\mathcal{R} = \mathbb{R}[[s,x,q]]$ be the algebra of the formal power series in variables s, x and q with real coefficients and let $\mathcal{A}$ be a sub-algebra of $\mathcal{R}$ such that the series converge for $s = 1$. Let $X(s,x,q)$ be a formal power series in $\mathcal{A}$. Let assume that:*

$$X(s) = xe(s) + xf(s)X(1) + xg(s)X(sq),$$

where $e(s), f(s)$ and $g(s)$ are some given power series in $\mathcal{A}$. Then:

$$X(s) = \frac{E(s) + E(1)F(s) - E(s)F(1)}{1 - F(1)},$$

where

$$E(s) = \sum_{n \geq 0} x^{n+1} g(s)g(sq)\ldots g(sq^{n-1})e(sq^n),$$

and

$$F(s) = \sum_{n \geq 0} x^{n+1} g(s)g(sq)\ldots g(sq^{n-1})f(sq^n).$$

By means of Lemma 4 and Proposition 3, we get the following:

Theorem 5 *The generating function $A(s,x,y,q)$ is given by:*

$$A(s,x,y,q) = \frac{J_1(s)J_0(1) - J_1(1)J_0(s) + J_1(1)}{J_0(1)},$$

with

$$J_0(s) = 1 - s^2xyq^2 \sum_{n \geq 0} \frac{x^n s^n q^{\frac{n(n+5)}{2}}}{(sq;q)_{n+1}} \prod_{k=0}^{n-1} (1 - y - sq^{k+1}),$$

and

$$J_1(s) = sxyq \sum_{n \geq 0} \frac{x^n s^n q^{\frac{n(n+3)}{2}}}{(sq;q)_n} \prod_{k=0}^{n-1} (1 - y - sq^{k+1}).$$

where we denote $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$.

Let us now take generating function $A(1,x,y,q)$ into consideration. We denote the functions $A(1,x,y,q)$, $J_1(1,x,y,q)$ and $J_0(1,x,y,q)$ as $A(x,y,q)$,$J_1(x,y,q)$ and $J_0(x,y,q)$ for brevity's sake. By means of some computations, we obtain:

Lemma 6 *The functions* $J_0(x,y,q)$ *and* $J_1(x,y,q)$ *satisfy the following equations:*

$$xq\,J_0(xq,y,q) = xq(1-y)\,J_0(x,y,q) + J_1(x,y,q),$$

$$xq\,J_1(xq,y,q) = J_1(x,y,q) - xyq\,J_0(x,y,q).$$

Let us now use this lemma to prove that:

Proposition 7 *The plane trees' generating function* $A(x,y,q)$ *satisfies:*

$$A(x,y,q) = \frac{J_1(x,y,q)}{J_0(x,y,q)} = \cfrac{xq}{1+xq^2(1-y)-\cfrac{xq^2}{1+xq^3(1-y)-\cfrac{xq^3}{1+xq^4(1-y)-\cfrac{xq^4}{\cdots}}}} - xq(1-y).$$

Proof. From Lemma 6, we deduce that:

$$A(xq,y,q) = \frac{xq\,J_1(xq,y,q)}{xq\,J_0(xq,y,q)} = \frac{J_1(x,y,q) - xyq\,J_0(x,y,q)}{xq(1-y)\,J_0(x,y,q) + J_1(x,y,q)},$$

and so we obtain:

$$A(x,y,q) = xyq + xq(1-y)A(xq,y,q) + A(x,y,q)A(xq,y,q). \tag{2}$$

If $\bar{A}(x,y,q) = xq(1-y) + A(x,y,q)$, then

$$\bar{A}(x,y,q) = \frac{xq}{1+xq^2(1-y) - \bar{A}(xq,y,q)},$$

so the proposition follows. □

Proposition 3 and equation 2 give us:

Corollary 8 *The generating function* $A(s,x,y,1)$ *satisfies:*

$$A(s,x,y,1) = \frac{sxy}{1-s-sx+sxy+s^2x}\left(1-s+s\,A(1,x,y,1)\right),$$

where:

$$A(1,x,y,1) = \frac{1-x+xy-\sqrt{1-2x+2xy+x^2-2x^2y+x^2y^2}}{2}.$$

Remark 9 The generating function $A(1,x,y,1)$ is given in [8, pag. 385]. Note that if $y=1$, we find the same results as in [1]; that is, the generating function $A(x,1,q)$ is Ramanujan's continued fraction and the average internal path length of the plane trees with n internal nodes is:

$$I_n = \frac{\sqrt{\pi}}{2}n^{\frac{3}{2}} + o(n).$$

4.2 Constructing by means of level traversal

In this section, we use construction C_H obtained by level traversal. Given a plane tree P, let $x_l(P)$ be P's last internal node in H. From the definition of $\mathcal{F}(P)$, it follows that $\mathcal{F}(P)$ is the set of P's external nodes following $x_l(P)$ in H. In this case, if $x \in \mathcal{F}(P)$, then x belongs either to the highest level or to the highest minus one level of P and it is on the right of $x_l(P)$ (see fig. 6). We call *right fringe length* of P the number of $\mathcal{F}(P)$'s nodes (i.e., $|\mathcal{F}(P)|$). We denote:

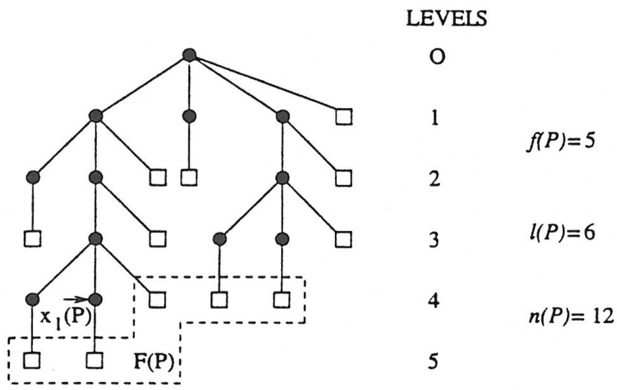

Figure 6: The set $\mathcal{F}(P)$ of a plane tree P obtained by level traversal

- $f(P)$, P's right fringe length,
- $l(P)$, P's leaves number,
- $n(P)$, P's internal nodes number.

The plane trees' generating function according to the above-listed three parameters is the following:

$$A_1(t,x,y) = \sum_{P\in\mathcal{P}} t^{f(P)} y^{l(P)} x^{n(P)}.$$

Let $\mathcal{F}(P) = \{e_1, e_2, \ldots, e_{f(P)}\}$, where for each $k \in [1..f(p)]$, e_k follows e_{k+1} in H. If we perform the construction C_H on P (i.e., we determine $\mathcal{G}(P)$), then each tree of $\mathcal{G}(P)$ is obtained from P by replacing a node of $\mathcal{F}(P)$ with an internal node (see fig. 7).

- If we replace e_k with $1 \le k \le f(P)-1$, then we obtain $P' \in \mathcal{G}(P)$ such that: $f(P') = k+1$, $n(P') = n(P)+1$, $l(P') = l(P)$;
- if we replace $e_{f(P)}$, then we obtain $P' \in \mathcal{G}(P)$ such that: $f(P') = f(P)+1$, $n(P') = n(P)+1$, $l(P') = l(P)+1$.

By proceeding as in the previous section, we obtain:

Proposition 10 *The generating function $A_1(t,x,y,1)$ satisfies:*

$$A_1(t,x,y,1) = txy - \frac{t^2x}{1-t+tx-txy+t^2xy}(txy - A(1,x,y,1))).$$

From Corollary 8, $A(s,x,1,1) = A_1(t,x,1,1)$, and so:

Proposition 11 *The number $p_{n,k}$ of plane trees with n internal nodes and right branch length k is equal to the number of plane trees with n internal nodes and right fringe length k. Moreover,*

$$p_{n,k} = \binom{2n-k-1}{n-1}\frac{k-1}{2n-k-1}.$$

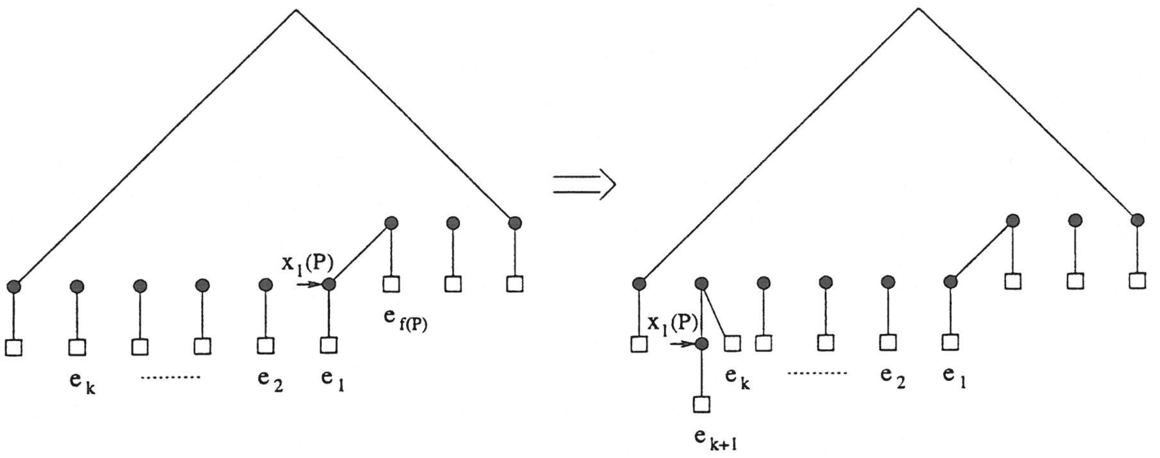

Figure 7: The construction according to level traversal

5 Right-leafed trees

In this section, we examine a particular subclass of plane trees studied by Donaghey and Shapiro [5] and related to Motzkin numbers.
A *right-leafed tree* is a plane tree in which each node is the father of one leaf at most and the leaf must be its rightmost son. Each internal node of degree 1 of a right-leafed tree is not the father of any external node (see fig.8(a)). The set of the right-leafed trees is a first order class. Let us denote this set by $\mathcal{L}$. It is easy to prove that preorder traversal on the class $\mathcal{L}$ satisfies conditions 1 and 2 of Theorem 1, while *level traversal* does not satisfy condition 1 (see fig.8).

Remark 12 By using level traversal on $\mathcal{L}$, it is possible to obtain a construction of $\mathcal{L}_n$ from $\mathcal{L}_{n-1}$ and $\mathcal{L}_{n-2}$ (not illustrated in this paper).

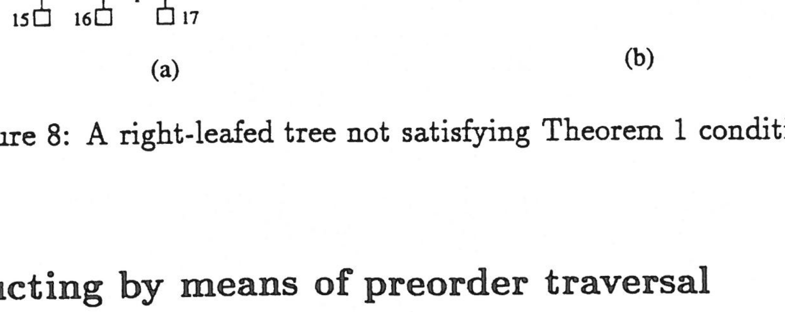

Figure 8: A right-leafed tree not satisfying Theorem 1 conditions

5.1 Constructing by means of preorder traversal

Let H be the preorder traversal. The right-leafed trees' generating function according to the number of their internal nodes, the number of their leaves, their right branch length and their internal path length is the following formal power series:

$$L(s,x,y,q)=\sum_{P\in\mathcal{L}} s^{r(P)}x^{n(P)}y^{l(P)}q^{i(P)}.$$

The right branch length of $P\in\mathcal{L}$ is equal to the number of $\mathcal{F}(P)$'s nodes plus one (i.e., $r(P)=|\mathcal{F}(P)|+1$). Furthermore, for each $k\in\{1,2,\ldots,r(P)-2,r(P)\}$, there is $e\in\mathcal{F}(P)$, such that e's level is k. Let $\mathcal{F}(P)=\{e_1,e_2,\ldots,e_{r(P)-2},e_{r(P)}\}$, where e_k is the node at level k. We now perform the construction C_H on P (see fig. 9). Each tree $P\in\mathcal{L}$ is such that

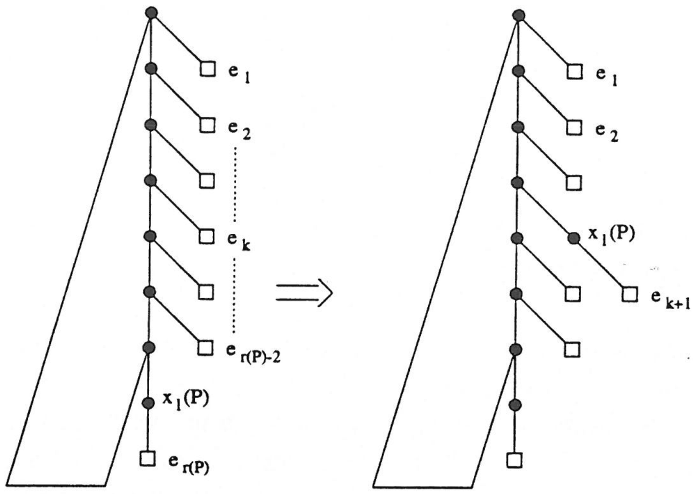

Figure 9: The construction according to preorder traversal

$n(P)\geq 2$, therefore the construction C_H applied to the set $\mathcal{L}$ gives all plane trees $P\in\mathcal{L}$ such that $n(P)>2$. By proceeding as for class $\mathcal{P}$, we have:

Proposition 13 *The right-leafed trees' generating function $L(s,x,y,q)$ satisfies:*

$$L(s)=s^2x^2yq^3+\frac{s^2xyq^2}{1-sq}L(1)+\frac{x(sq-s^2q^2-y)}{1-sq}L(sq). \tag{3}$$

5.1.1 Solving equations

By means of Lemma 4 and Proposition 13, we obtain the following:

Theorem 14 *The generating function $L(s,x,y,q)$ is given by:*

$$L(s,x,y,q)=\frac{J_1(s)J_0(1)-J_1(1)J_0(s)+J_1(1)}{J_0(1)},$$

with

$$J_0(s)=1-s^2xyq^2\sum_{n\geq 0}\frac{x^nq^{2n}}{(sq;q)_{n+1}}\prod_{k=0}^{n-1}(sq^{k+1}-s^2q^{2(k+1)}-y),$$

and

$$J_1(s)=s^2x^2yq^3\sum_{n\geq 0}\frac{x^nq^{2n}}{(sq;q)_n}\prod_{k=0}^{n-1}(sq^{k+1}-s^2q^{2(k+1)}-y).$$

We denote the functions $L(1,x,y,q)$, $J_1(1,x,y,q)$ and $J_0(1,x,y,q)$ as $L(x,y,q)$,$J_1(x,y,q)$ and $J_0(x,y,q)$ for brevity's sake. After some computations, we prove that:

Lemma 15 *The functions $J_0(x,y,q)$ and $J_1(x,y,q)$ satisfy the equations:*

$$xq\, J_0(xq,y,q) = xq\, J_0(x,y,q) + J_1(x,y,q),$$

$$xq\, J_1(xq,y,q) = -x^2yq^3\, J_0(x,y,q) + J_1(x,y,q).$$

By using this lemma, we obtain:

Proposition 16 *The generating function $L(x,y,q)$ satisfies:*

$$L(x,y,q) = \frac{J_1(x,y,q)}{J_0(x,y,q)} = \cfrac{xq(1+xyq^2)}{1+xq^2 - \cfrac{xq^2(1+xyq^3)}{1+xq^3 - \cfrac{xq^3(1+xyq^4)}{1+xq^4 - \cfrac{xq^4(1+xyq^5)}{\cdots}}}} - xq.$$

Proof. From Lemma 16, we deduce that:

$$L(xq,y,q) = \frac{xq\, J_1(xq,y,q)}{xq\, J_0(xq,y,q)} = \frac{J_1(x,y,q) - x^2yq^3\, J_0(xq,y,q)}{xq\, J_0(x,y,q) + J_1(x,y,q)},$$

and so obtain:

$$L(x,y,q) = x^2yq^3 + xqL(xq,y,q) + L(x,y,q)L(xq,y,q). \tag{4}$$

If $\bar{L}(x,y,q) = xq + L(x,y,q)$, then

$$\bar{L}(x,y,q) = \frac{xq(1+xyq^2)}{1+xq^2 - \bar{L}(xq,y,q)}.$$

Hence, the proposition is proven. □

From Proposition 13 and equation 4, it follows that:

Corollary 17 *The generating function $L(s,x,y,1)$ satisfies:*

$$L(s,x,y,1) = \frac{s^2xy}{1+xy-s(1+x)+s^2x}\left(x - xs + L(1,x,y,1)\right),$$

where:

$$L(1,x,y,1) = \frac{1-x-\sqrt{1-2x+x^2-4x^2y}}{2}.$$

Remark 18 From corollary 17, we have:

$$L(x,1,1) = \frac{1-x-\sqrt{1-2x-3x^2}}{2}.$$

Hence, we obtain the same result as Donaghey and Shapiro [5]. That is, the number of right-leafed trees with n internal nodes is equal to $(n-2)-th$ Motzkin number:

$$[x^n]L(x,1,1) = m_{n-2} = \sum_{k\geq 0} \frac{1}{k+1}\binom{n-2}{2k}\binom{2k}{k}.$$

Remark 19 If $y = 1$, we have:

$$L(x,1,q) = \frac{J_1(x,1,q)}{J_0(x,1,q)},$$

where:

$$J_0(x,1,q) = \sum_{n\geq 0}(-1)^n \frac{x^n q^{2n}}{1-q^n+q^{2n}} \frac{(-q^3;q^3)_n}{(q^2;q^2)_n},$$

and

$$J_1(x,1,q) = \sum_{n\geq 0}(-1)^n x^{n+2} q^{2n+3} \frac{(-q^3;q^3)_n}{(q^2;q^2)_n}.$$

Let us now determine the average internal path length I_n of a right-leafed tree with n internal nodes:

$$I_n = \frac{[x^n]\frac{\partial}{\partial q}L(x,1,q)_{|q=1}}{[x^n]L(x,1,1)}.$$

From equation 4, we obtain:

$$\frac{\partial L(x,1,q)}{\partial q} = 3x^2q^2 + L(xq,1,q)\left(x + \frac{\partial L(x,1,q)}{\partial q}\right) + (xq + L(x,1,q))\left(x\frac{\partial L(xq,1,q)}{\partial(xq)} + \frac{\partial L(xq,1,q)}{\partial q}\right)$$

By setting $W(x) = \frac{\partial}{\partial q}L(x,1,q)_{|q=1}$, we find:

$$W(x) = \frac{3x^2}{2\sqrt{1-2x-3x^2}} + \frac{3x^2}{2(1-3x)}.$$

Hence:

$$I_n = \frac{3\beta_{n-2}}{2m_{n-2}} + \frac{3^{n-1}}{2m_{n-2}},$$

where β_n is the central trinomial coefficient (i.e., $\beta_n = [x^n](1-2x-3x^2)^{-1/2}$). In [2], it is shown that:

$$m_{n-2} \approx \frac{3^{n-1}}{2n-1}\sqrt{\frac{3}{\pi n}}\left(1+\frac{1}{16(n-2)}\right) \qquad \frac{\beta_{n-2}}{m_{n-2}} \approx \frac{2n-1}{6}\left(1+\frac{3}{4(n-2)}\right),$$

thus:

Theorem 20 *The average internal path length of right-leafed trees with n internal nodes is:*

$$I_n = \sqrt{\frac{\pi}{3}}n^{\frac{3}{2}} + o(n).$$

6 Conclusions

We illustrated a construction for enumerating classes of plane trees. We described two applications of this method, namely to the whole class of plane trees and to the class of right-leafed trees, but the method can be applied to many other classes such as 1-2 trees, binary and m-ary trees, and so on. For instance, the 1-2 trees' generating function defined according to the number of their internal nodes, the number of their leaves, their right fringe length and obtained by means of level traversal, is:

$$B(t,x,y) = \frac{2txy(1+tx)(1-t) + t^2(1+xy)\left(1-x-\sqrt{1-2x+x^2-4x^2y}\right)}{2(1-t+tx+t^2yx^2)}.$$

Moreover, this method can be extended to other classes of combinatorial objects, namely lattice paths and directed animals.

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