MULTINOMIAL SUMS AS NUMBERS OF STANDARD TABLEAUX

WILLIAM BECKNER*
MATHEMATICS DEPARTMENT
UNIVERSITY OF TEXAS AT AUSTIN
AUSTIN, TX 78712

AND

AMITAI REGEV†
MATHEMATICS DEPARTMENT
THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PA 16802
AND
DEPARTMENT OF THEORETICAL MATHEMATICS
THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT 76100, ISRAEL

ABSTRACT. The Catalan and the Motzkin numbers can be expressed in terms of multinomial sums and of numbers of standard tableaux. This phenomena is generalized to a large class of multinomial sums. The asymptotics of these numbers relate the Littlewood-Richardson rule to the Selberg integral.

Introduction. In [2] we saw that the free term in $(t+\frac{1}{t})^{2n}$ equals $\binom{2n}{n} = \sum_{(\lambda_1,\lambda_2)\vdash 2n} d_\lambda$. Note that $\frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan numbers. Here d_λ denotes the number of standard tableaux of shape λ. Similarly, the coefficient of $\frac{1}{t}$ in $(t+\frac{1}{t}+1)^{n+1}$ equals $(n+1)\mu_n$, where $\mu_n = \sum_{(\lambda_1,\lambda_2,\lambda_3)\vdash n} d_\lambda$ [2, pp. 135], and the μ_n's are the Motzkin numbers. Young's rule, which is a special case of the Littlewood-Richardson rule, was instrumental in deriving these relations.

As we show in theorem 1.2 below, these observations admit considerable generalizations: we show that the coefficient of a certain nonomials in the expansion

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX of a multinomial expression is equal to a similar sum of d_λ's. A key tool here is a certain consequence of the Littlewood-Richardson rule, and which is essentially due to R. Stanley [5].

In section 2 we calculate the asymptotics of these numbers. As a byproduct we deduce an "algebraic" evaluation of the integral

$$I(\gamma) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{k} t_i^\alpha \cdot \left[\prod_{1 \le i < j \le k} (t_i - t_j) \right]^{2\gamma} \cdot e^{-(t_1 + \cdots + t_k)} dt_1 \cdots dt_k$$

in the case $r = 1$. Note that the Selberg integral yields $I(\gamma)$ for all γ's.

1. The main theorem.

1.1 Notations. Identify a partition $\lambda \vdash n$ with its corresponding irreducible S_n character χ_λ and with its Young diagram D_λ.

Given a diagram D, not necessarily a Young diagram, let D^* denotes its double reflection with respect to both the x and the y axes. For example,

$$((4^3) - (3.2))^* =$$

Let k, h, v, w be integers, $0 \le k < h$, $0 \le v, w$. Let λ be a partition with diagram

$$\lambda \equiv$$

where

$$= (v^k)$$

Then $\lambda = (\lambda_1, \cdots, \lambda_{k+h}) \vdash kv + hw = n$, where $\lambda_{k+1} = \cdots = \lambda_h = w$ and $\lambda_i + \lambda_{k+h-i+1} = w + v$, $1 \le i \le k$, hence also $\lambda_i - \lambda_{i+1} = \lambda_{k+h-i} - \lambda_{k+h-i+1}$ $1 \le i \le k-1$. Conversely, let $\lambda = (\lambda_1, \cdots, \lambda_{k+h}) \vdash n$ satisfy $\lambda_{k+1} = \cdots = \lambda_h$ and

$\lambda_1 + \lambda_{k+h} = \lambda_2 + \lambda_{k+h-1} = \cdots = \lambda_k + \lambda_{h+1}$, then λ determines such $w = \lambda_h$ and $v = \lambda_1 + \lambda_{k+h} - \lambda_h$, and $n = kv + hw$. We denote

$\Lambda(k,h,n) = \{(\lambda_1, \cdots, \lambda_{k+h}) \vdash n \mid \lambda_1 + \lambda_{k+h} = \cdots = \lambda_k + \lambda_{h+1}\ ,\ \lambda_{k+1} = \cdots = \lambda_h\}$.

$A(k,h,n) = \{(v,w) \in \mathbb{N} \mid kv + hw = n\}$ and

$\Lambda(k,h,v,w) = \{\lambda \in \Lambda(k,h,kv+hw) \mid \lambda_h = w\ ,\ \lambda_1 + \lambda_{k+h} - \lambda_h = v\}$.

Since $k < h$, $\Lambda(k,h,n) = \bigcup_{(v,w)\in A(k,h,n)} \Lambda(k,h,v,w)$ (disjoin union).

Let $\chi(k,h,n) = \sum_{\lambda\in\Lambda(k,h,n)} \chi_\lambda$ (which is an S_n character), then $\deg \chi(k,h,n) = \sum_{\lambda\in\Lambda(k,h,n)} d_\lambda$, where $d_\lambda = \deg \chi_\lambda$.

Denote $p(h,x) = \begin{cases} 1 & 0 \le h \le 1 \\ \frac{x_2\cdots x_h}{x_1^{h-1}} & 2 \le h \end{cases}$.

$$\mu_0(k,h) = p(h,x)p(k,y).$$

We shall write $\mu_0(k,h) = \frac{x_2\cdots x_h}{x_1^{h-1}} \cdot \frac{y_2\cdots y_k}{y_1^{k-1}}$.

Denote $s(h,x) = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{h-1}}{x_h} + \frac{x_h}{x_1} & h \ge 2 \end{cases}$

and $P(k,h,n) = (s(h,x) + s(k,y))^{n+\binom{k}{2}+\binom{h}{2}}$.

We shall write $P(k,h,n) = \left(\frac{x_1}{x_2} + \cdots + \frac{x_h}{x_1} + \frac{y_1}{y_2} + \cdots + \frac{y_k}{y_1}\right)^m$, $m = n + \binom{k}{2} + \binom{h}{2}$.

We denote by $K(k,h,n)$ the coefficient of $\mu_0(k,h)$ in $P(k,h,n)$.

With these notations we can now state

<u>**1.2 Theorem**</u>. a) *Let $0 \le k,h,n$ be integers with $k < h$. Denote $m = n + \binom{k}{2} + \binom{h}{2}$, $a(k,h) = \prod_{j=0}^{k-1}(j!)\prod_{j=0}^{h-1}(j!)$, and let $K(k,h,n)$ be the coefficient of $\mu_0(k,h) = \frac{x_2\cdots x_h}{x_1^{h-1}} \cdot \frac{y_2\cdots y_k}{y_1^{k-1}}$ in $P(k,h,n) = \left(\frac{x_1}{x_2} + \cdots + \frac{x_h}{x_1} + \frac{y_1}{y_2} + \cdots + \frac{y_k}{y_1}\right)^m$. Then*

$$K(k,h,n) = \sum_{(v,w)\in A(k,h,n)} \frac{m!}{\prod_{i=0}^{k-1}(v+i)! \cdot \prod_{j=0}^{h-1}(w+j)!} = \frac{m!}{a(k,h)\cdot n!} \sum_{\lambda\in(k,h,n)} d_\lambda$$

b) Let $0 \le k,d$ $(k = h)$ and denote $n = 2kd$, $a(k) = \prod_{j=0}^{k-1}(j!)^2$, $m = n + 2\binom{k}{2} =$

$2kd + k(k-1)$, and

$$\overline{K}(k,n) = \frac{(2kd + k(k-1))!}{\left[\prod_{j=0}^{k-1}(kd+j)!\right]^2}.$$

Then $\overline{K}(k,n)$ is the free term in $\left\{t\left(\frac{x_1}{x_2} + \cdots + \frac{x_k}{x_1}\right) + \frac{1}{t}\left(\frac{y_1}{y_2} + \cdots + \frac{y_k}{y_1}\right)\right\}^m$. and

$a(k) \cdot \frac{n!}{m!} \cdot \overline{K}(k,n) = n! \cdot \left[\prod_{j=0}^{k-1} \frac{(j!)}{(kd+j)!}\right]^2 = \sum_{\lambda \in \Lambda(k,k,n)} d_\lambda$

(here $\lambda = (\lambda_1, \cdots, \lambda_{2k}) \in \Lambda(k,k,n)$ if and only if $\lambda \vdash n$ and $\lambda_1 + \lambda_{2k} = \cdots = \lambda_k + \lambda_{k+1}$).

c) Let $n = 2kd$, $a(k)$ and $\Lambda(k,k,n)$ as in b). and denote

$$K'(k,n) = \sum_{v=0}^{d} \frac{(n + k(k-1))!}{\prod_{j=0}^{k-1}[(v+j)!(d-v+j)!]}.$$

Then $K'(k,n)$ is the coefficient of $\mu_0(k,k) = \frac{x_2 \cdots x_k}{x_1^{k-1}} \cdot \frac{y_2 \cdots y_k}{y_1^{k-1}}$ in

$$\left(\frac{x_1}{x_2} + \cdots + \frac{x_k}{x_1} + \frac{y_1}{y_2} + \cdots + \frac{y_k}{y_1}\right)^{n+k(k-1)}.$$

Moreover, denote $\chi(k,k,n) = \sum_{\lambda \in \Lambda(k,k,n)} (\lambda_k - \lambda_{k+1} + 1)\chi_\lambda$, then

$K'(k,n) = \frac{(n+k(k-1))!}{a(k)\cdot n!} \cdot \deg \chi(k,k,n) = \frac{(n+k(k-1))!}{a(k)n!} \sum_{\lambda \in \Lambda(k,k,n)} (\lambda_k - \lambda_{k+1} + 1)d_\lambda.$

We give below the proof of a). The proof of b) is similar and simpler, and is left for the reader. The proof of c) is also similar. and below we indicate the few changes that occur in it.

The proof of a) is done in steps 1. 2 and 3 below, assuming $2 \le k$. The remaining (small) cases can easily be verified similarly.

Step 1: Denote $\delta(k,h,n) = \sum_{(v,w) \in A(k,h,n)} \frac{n!}{\prod_{i=0}^{k-1}(v+i)! \prod_{j=0}^{h-1}(w+j)!}$, then verify that $K'(k,h,n) = \frac{m!}{n!}\delta(k,h,n)$.

Indeed, let $\mu = \left(\frac{x_1}{x_2}\right)^{a_1} \cdots \left(\frac{x_h}{x_1}\right)^{a_h} \cdot \left(\frac{y_1}{y_2}\right)^{b_1} \cdots \left(\frac{y_k}{y_1}\right)^{b_k}$ be a monomial in the expansion of $p = P(k,h,n)$; $0 \le a_i, b_j$ and $\Sigma a_i + \Sigma b_j = m$. The coefficient of μ in the expansion of P is the multinomial coefficient $\frac{m!}{a_1! \cdots a_h!\, b_1! \cdots b_k!}$.

Now, $\mu = \frac{x_2^{a_2-a_1} \cdots x_h^{a_h-a_{h-1}}}{x_1^{a_h-a_1}} \cdot \frac{y_2^{b_2-b_1} \cdots y_k^{b_k-b_{k-1}}}{y_1^{b_k-b_1}}$, hence $\mu = \mu_0$ if and only if $a_i = a_1+i-1,\ 2 \le i \le h$ and $b_j = b_1+j-1,\ 2 \le j \le k$. In that case, $m = n+\binom{h}{2}+\binom{k}{2} = \Sigma a_i + \Sigma b_j = a_1 h + b_1 k + \binom{h}{2} + \binom{k}{2}$, so $(b_1, a_1) \in A(k, h; n)$ and, denoting $b_1 = v$ $a_1 = w$, the proof follows.

<u>Step 2</u>: Verify that

$$a(k,h) \cdot \delta(k,h,n) = \sum_{(v,w)\in A(k,h,n)} \deg((v^k)\widehat{\otimes}(w^h)).$$

Here $(v^k)\widehat{\otimes}(w^h)$ is the outer product of the (irreducible) S_{kv} and the S_{hw} characters (v^k) and (w^h). The above follows from the "hook" formula and since $\deg((v^k)\widehat{\otimes}(w^h)) = \binom{n}{kv}\deg(v^k) \cdot \deg(w^h)$.

<u>Step 3</u>: The proof of Theorem 1.2.(a) will clearly follow (by taking degrees) once we show that

$$\sum_{(v,w)\in A(k,h,n)} (v^k)\widehat{\otimes}(w^h) = \sum_{\lambda\in\Lambda(k,h,n)} \chi_\lambda.$$

This follows from the following theorem, which is a slight generalization of a result of R. Stanley [5, Lemma 3.3].

<u>**1.3 Theorem.**</u>

$$(v^k)\widehat{\otimes}(w^h) = \sum_{\lambda\in\Lambda(k,h,v,w)} \chi_\lambda$$

We remark that one can deduce 1.3 from the Littlewood-Richardson (L-R) rule [1] *or from the Remmel-Whitny rule* [3], *which is a variant of the L-R rule.*

Note that since $\Lambda(k,h,n) = \biguplus_{(v,w)\in A(k,h,n)} \Lambda(k,h,v,w)$, a disjoint union, hence $\sum_{(v,w)\in A(k,h,n)} (v^k)\widehat{\otimes}(w^h) = \sum_{\lambda\in\Lambda(k,h,n)} \chi_\lambda$, and the proof of 1.2.(a) is complete.

<u>To prove (c)</u>, note that when $k = h$, the sets $\Lambda(k,k,v,w)$ are not necessarily disjoint. In fact, it is easy to verify the following: Let $\lambda = (\lambda_1, \cdots, \lambda_{2k}) \in \Lambda(k,k,n)$ $(n = 2kd)$, and let $v + w = 2d$ (i.e. $kv + kw = n$). Then $\lambda \in \Lambda(k,k,v,w)$ if and only if $\lambda_{k+1} \le v \le \lambda_k$ (and similarly for w).

Imitate now the proof of (a)! It clearly follows that

$$\chi(k,k,n)=\sum_{(v,w)\in A(k,k,n)}\ \sum_{\lambda\in\Lambda(k,k,v,w)}\chi_\lambda=\sum_{\lambda\in\Lambda(k,k,n)}(\lambda_k-\lambda_{k+1}+1)\chi_\lambda.$$

The rest of the proof of (c) is similar to that of (a). Q.E.D.

2. Asymptotics.

We calculate the asymptotics of $K(k,h,n)$, $n\to\infty$, using first the expression for $K(k,h,n)$ in 1.2 as a sum of multinomial coefficients. We then sketch a second approach which is based on the tableaux-interpretation of $K(k,h,n)$, also in 1.2. That second calculation involves an integral which can be evaluated by the Selberg integral. A comparison between these two computations yields an "algebraic" evaluation of that integral.

2.1 The Selberg Integral [4].

Denote $D_n(x)=D(x_1,\cdots,x_n)=\prod_{1\le i<j\le n}(x_i-x_j)$, $D_1(x)=1$. Then $I(\alpha,\beta,\gamma;n)=$ $\int_0^1\cdots\int_0^1\prod_{j=1}^{n}[x_j^{\alpha-1}(1-x_j)^{\beta-1}]\cdot|D_n(x)|^{2\gamma}dx=$

$$=\prod_{j=0}^{m-1}\frac{\gamma(1+\gamma+\gamma j)\cdot\Gamma(\alpha+\gamma j)\cdot\Gamma(\beta+\gamma j)}{\Gamma(1+\gamma)\cdot\Gamma(\alpha+\beta+\gamma(n+j-1))}$$

By setting $\beta=m$, dilating each variable by the factor $1/m$ and then letting $m\to\infty$, one obtains

$$\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^{n}x_j^{\alpha-1}\cdot|D_n(x)|^{2\gamma}\cdot e^{-\Sigma x_i^2}dx=\prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+\gamma j)\Gamma(\alpha+\gamma j)}{\Gamma(1+\gamma)}$$

We turn next to the asymptotics of $K(k,h,n)$, first in 1.2.a and in 1.2.b. Recall that $a_n\underset{n\to\infty}{\simeq}b_n$ if $\lim_{n\to\infty}a_n/b_n=1$.

2.2. Lemma. *As in 1.2, let*

$$K(k,h,n)=\sum_{(v,w)\in A(k,h,n)}\frac{m!}{\prod_{i=0}^{k-1}(v+i)!\prod_{j=0}^{h-1}(w+j)!},$$

where $m = n+\binom{k}{2}+\binom{h}{2}$. Then $K(k,h,n) \underset{n\to\infty}{\simeq} \sqrt{kh}\cdot\left(\frac{1}{2\pi}\right)^{(k+h-2)/2}\cdot\frac{(k+h)^{m+(k+h-2)/2}}{m^{(k+h-2)/2}}$.

Proof. Denote $\alpha = \binom{k}{2}+\binom{h}{2}$, so $m = n+\alpha$. The sum $K(k,h,n)$ is dominated by the summands with large v and w. For such a summand, let

$$v = \frac{m-\alpha}{k+h} + \frac{\lambda_1}{k}\sqrt{m} + \frac{\gamma_1}{k} \quad \text{and}$$
$$w = \frac{m-\alpha}{k+h} + \frac{\lambda_2}{h}\sqrt{m} + \frac{\gamma_2}{h},$$

with $\lambda_1+\lambda_2 = \gamma_1+\gamma_2 = 0$. By Stirlings' formula, such a summand is asymptotic to

$$\left(\frac{1}{\sqrt{2\pi}}\right)^{h+k-1} m^{m+\frac{1}{2}} \left(\frac{m}{h+k}\right)^{-\sum_{i=0}^{h-1}\left(\frac{m-\alpha}{h+k}+\frac{\lambda_1}{h}\sqrt{m}+\frac{\gamma_1}{h}+i+\frac{1}{2}\right)\sum_{j=0}^{k-1}\left(\frac{m-\alpha}{h+k}+\frac{\lambda_2}{k}\sqrt{m}+\frac{\gamma_2}{k}+j+\frac{1}{2}\right)}$$
$$\times\left[\left(1+\frac{h+k}{h}\,\frac{\lambda_1}{\sqrt{m}}\right)^{h\left(\frac{m}{h+k}+\frac{\lambda_1}{h}\sqrt{m}\right)}\left(1+\frac{h+k}{k}\,\frac{\lambda_2}{\sqrt{m}}\right)^{k\left(\frac{m}{h+k}+\frac{\lambda_2}{k}\sqrt{m}\right)}\right]^{-1}$$
$$\simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{h+k-1}\frac{(h+k)^{m+(h+k)/2}}{m^{(h+k-1)/2}}\exp\left[-\frac{1}{2}\,\frac{(h+k)^2}{hk}\,\lambda_1^2\right]$$

using the relation $\lambda_1+\lambda_2 = 0$. Then

$$K(k,h,n) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{h+k-1}\frac{(h+k)^{m+(h+k)/2}}{m^{(h+k-2)/2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(h+k)^2}{hk}\lambda_1}\,d\lambda_1$$
$$= \left(\frac{1}{2\pi}\right)^{(h+k-2)/2}\sqrt{hk}\,\frac{(h+k)^{m(h+k-2)/2}}{m^{(h+k-2)/2}}.$$

Next, we calculate the asymptotics of $K(k,h,n)$ from the presentation

$$K(k,h,n) = \sum_{\lambda\in\Lambda(k,h,n)} d_\lambda\ , \qquad \text{in 1.2.a. and 1.2.b.}$$

Note that $\Lambda(k,h,n)$

$$= \{(\lambda_1,\cdots,\lambda_{k+h}) \mid \lambda_i+\lambda_{k+h-i+1} = w+v \qquad 1\le i\le k;$$
$$\lambda_j = \lambda_{j+1} = w \qquad k+1\le j\le h-1\,, \quad \lambda_i \ge \lambda_i+1 \qquad 1\le i\le h+k-1,$$
$$\text{with } n = kv+hw\}.$$

If $k = h$, then $v = w$, $n = 2kv$, $v = \frac{n}{2k}$, hence

$$\Lambda(k,k,n) = \Lambda(k,k,2kv) = \{(\lambda_1,\ldots,\lambda_{2k}) \mid \lambda_i+\lambda_{2k-i+1} = 2v \qquad 1\le i\le k,$$
$$\lambda_i \ge \lambda_{i+1}, \quad 1\le i\le 2k-1, \text{ with } n = 2kv\}.$$

To estimate d_λ, set $w = \frac{n}{k+h} + \frac{c_*\sqrt{n}}{h} + \frac{\gamma}{h}$, $v = \frac{n}{k+h} - \frac{c_*\sqrt{n}}{k} - \frac{\gamma}{k}$. The factors involving γ play no role in the calculations, so they are dropped in further discussion. Set $\lambda_i = \frac{n}{k+h} + c_i\sqrt{n} - \frac{c_*(h-k)}{2hk}\sqrt{n}$ for $i = 1, \ldots, k$. Then $\lambda_{k+h-i+1} = \frac{n}{k+h} - c_i\sqrt{n} - \frac{c_*(h-k)}{2hk}\sqrt{n}$. Further

$$c_1 \geq c_2 \geq \cdots \geq c_k \geq |c_*| \left(\frac{h+k}{2hk}\right)$$

Set $\lambda_i = \frac{n}{k+h} + x_i\sqrt{n}$ with $\sum_{i=1}^{k+h} x_i = 0$ and apply (F.1.2) from [2], keeping in mind the above restrictions relating the x_i's to the variables $\{c_i, c_*\}$. Then in the resulting integral set $y_i = x_i + \frac{c_*(h-k)}{2hk}$ for all i. Then $y_{k+1} = \frac{h+k}{2hk} c_*$

$$d_\lambda \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{k+h-1} \left(\prod_{j=1}^{h-k-1} j!\right) (k+h)^{n+(k+h)^2/2} D(x_1, \ldots, x_{k+h}) e^{-(k+h)x^2/2} \left(\frac{1}{n}\right)^{(k^2+h^2-1)/2} \left(\frac{1}{\sqrt{}}\right.$$

$$D(x_1, \ldots, x_{k+h}) = D(y_1, \ldots, y_{k+h}) = \prod_{i=1}^{k} (2y_i)(y_i^2 - y_{k+1}^2)^{h-k} [D_k(y_1^2, \ldots, y_k^2)]^2$$

$$e^{-(k+h)x^2/2} = e^{-(h+k)(y_1^2+\cdots+y_k^2) - k(h-k)y_{k+1}^2}$$

$$\sum d_\lambda \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{k+h-1} \left(\prod_{j=1}^{h-k-1} j!\right) (k+h)^{n+(k+h)^2/2} \left(\frac{1}{n}\right)^{(k^2+h^2-1)/2} \left(\frac{2hk}{h+k}\right)$$

$$\int_{y_1 \geq y_2 \geq \cdots \geq y_k \geq |y_{k+1}|} \prod_{i=1}^{k} (2y_i)(y_i^2 - y_{k+1}^2)^{h-k} [D_k(y_1^2, \ldots, y_k^2)]^2$$

$$e^{-(h+k)(y_1^2+\cdots+y_k^2) - k(h-k)y_{k+1}^2} dy_1 \ldots dy_{k+1}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{k+h-1} \left(\prod_{j=1}^{h-k-1} j!\right) (k+h)^{n+(k+h)^2/2} \left(\frac{1}{n}\right)^{(k^2+h^2-2)/2} \frac{2hk}{h+k}$$

$$\frac{1}{k!} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{k} t_i^{h-k} [D_k(t_1, \ldots, t_k)]^2 e^{-(h+k)(t_1+\cdots+t_k)} dt \int_{-\infty}^{\infty} e^{-2hk\, y_{k+1}^2} dy_{k+1}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{k+h-2} \left(\prod_{j=1}^{h-k-1} j!\right) \sqrt{hk}(k+h)^{(k^2+h^2-2)/2} \frac{1}{n^{(k^2+h^2-2)/2}} (k+h)^n \frac{1}{k!}$$

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{k} t_i^{h-k} [D_k(t_1 \ldots, t_k)]^2 e^{-(t_1+\cdots+t_k)} dt_1 \cdots dt_k$$

The value for this last integral is given, by the Selberg integral, in 2.1. However, a comparison of the two asymptotics of $K(k,h,n)$ also yields the value for that integral.

Finally, we indicate the asymptotics corresponding to 1.2.c: we only need to calculate the "tableaux" asymptotics. The computations here are the same as before, except that the multiplicities $\lambda_k - \lambda_{k+1} + 1$ in this case, lead to the integral

$$I(k) = \int_{y_1 \geq \cdots \geq y_k \geq 0} 2y_k \prod_{j=1}^{k} (2y_j)[D_k(y_1^2, \cdots, y_k^2)]^2 e^{-2(y_1^2 + \cdots + y_k^2)} dy_1 \cdots dy_k$$

We show next that $I(k)$ is also a consequence of the Selberg integral.

First, set $t_j = y_j^2$, $j = 1, \cdots, k$; then

$$I(k) = 2 \int_{t_1 \geq \cdots \geq t_k \geq 0} \sqrt{t_k}\ [D_k(t_1, \ldots, t_k)]^2 e^{-2(t_1 + \cdots + t_k)} dt_1 \cdots dt_k .$$

Now make the translation $t_j = w_j + t_k$, $j = 1, \cdots, k-1$; then

$$I(k) = 2 \int_0^{\infty} \sqrt{t_k}\ e^{-2kt_k} dt_k \int_{w_1 \geq \cdots \geq w_{k-1} \geq 0} \prod_{j=1}^{k-1} w_j^2 \cdot$$
$$[D_{k-1}(w_1, \cdots, w_{k-1})]^2 e^{-2(w_1 + \cdots + w_{k-1})} dw_1 \ldots dw_{k-1}$$
$$= \frac{\sqrt{\pi}}{2^{3/2}\sqrt{k}} \frac{1}{k!} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{k-1} w_j^2 [D_{k-1}(w_1, \ldots, w_{k-1})]^2 e^{-2(w_1 + \cdots + w_{k-1})} dw_1 \ldots dw_{k-1}$$
$$= \frac{\sqrt{\pi}}{2k} \frac{1}{2^{k^2}} \frac{1}{k!} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{j=1}^{k-1} w_j^2 [D_{k-1}(w_1, \ldots, w_{k-1})]^2 e^{-2(w_1 + \cdots + w_{k-1})} dw_1 \ldots dw_{k-1}$$

and this is a special case of the previous integral.

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