Commutative and nilpotent automorphisms groups of hypermaps

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Abstract

For a hypermap of genus $g > 1$, we give the bounds on the order of commutative and nilpotent automorphisms group depending on the smallest divisor of its order. These bounds are sharp and reached for infinitely many g.

Résumé

L'ordre d'un groupe commutatif ou nilpotent d'automorphismes d'une hypercarte de genre $g > 1$ peut être borné en fonction du plus petit diviseur de cet ordre. Ces bornes sont atteintes pour une infinité de g.

1 Introduction

On a compact Riemann surface of genus $g > 1$ the maximal order for a commutative automorphism group is $4(g+1)$. This result was first proved in 1965 by C. Mac Lachlan (see [McL]). An improvement was obtained by R. Zomorrodian in 1988. He shows in [Zo] that if p is the smallest divisor of a nilpotent automorphism group G then $\mid G \mid \leq 2p(g-1)/(p-3)$ if $p \neq 2$ or $p \neq 3$ and that this bound is reached for at least one g when p is fixed. If $p = 3$, $\mid G \mid \leq 9(g-1)$ and if If $p = 2$, $\mid G \mid \leq 16(g-1)$. In this paper we show that these results can be seen as a generalization of results concerning a restricted type of surfaces: the so-called p-elliptic surfaces - these can be viewed as p-sheeted coverings of the sphere, where p is a prime- see below, or [Be2]. Our approach is combinatorial in nature; we represent a surface with a pair of permutations (α, σ) such that the group they generate is transitive; such a pair is called a *hypermap*. Then $Aut(S)$ becomes $Aut(\alpha, \sigma)$ the centralizer of the two permutations. As in the classical case, Machì in [Ma] proved that, for $g \geq 2$, $\mid Aut(\alpha, \sigma) \mid \leq 84(g-1)$ where g is the genus of the hypermap (see section 2). We proved in [Be5] that on a hypermap of genus $g > 1$, the bounds on the order of automorphisms group depend on the smallest divisor of its order. More precisely, using Machì's technique, we proved that if p is the smallest divisor of the order of $Aut(\alpha, \sigma)$, $\mid Aut(\alpha, \sigma) \mid \leq 2p(g-1)/(p-3)$ for $p \geq 5$, $\mid Aut(\alpha, \sigma) \mid \leq 15(g-1)$ for $p = 3$ and, if $p = 2$, the Hurwitz bound $84(g-1)$ cannot be improved.

We give here bounds for a commutative or nilpotent group of automorphisms still depending on the smallest divisor of its order. The order of a commutative group of automorphisms is bounded by $p(2 + 2g/(p-1))$ and this bound is sharp and reached for infinitely many g.

The order of a nilpotent group of automorphisms follows other laws. Namely, for $p > 3$, the bound is $2p(g-1)/(p-3)$; if $p = 3$, the bound is $9(g-1)$ and $16(g-1)$ if $p = 2$.

If we compare $p(2 + 2g/(p-1))$ and $2p(g-1)/(p-3)$, we find that $p(2 + 2g/(p-1)) \leq 2p(g-1)/(p-3)$ and that they happen to be equal for $p = (3 + \sqrt{8g+1})/2$; in which case, the automorphism group is of order p^2, thus commutative.

2 Hypermaps, automorphisms and induced automorphisms

For a general introduction to the theory of hypermaps see [CoMa]. In this section we recall a few definitions and results that will be needed in the sequel.

Definition 2.1 *A* hypermap *is a pair of permutations* (α, σ) *on* B *(the set of* brins*) such that the group they generate is transitive on* B. *When* α *is a fixed point free involution,* (α, σ) *is a* map. *The cycles of* α,σ *and* $\alpha^{-1}\sigma$ *are called* edges, vertices *and faces, respectively; but if there specification in termes of edges, vertices or faces is not needed, we will refer to them as* points.

Euler's formula gives the relationship between the numbers of cycles of these three permutations:

$$z(\alpha) + z(\sigma) + z(\alpha^{-1}\sigma) = n + 2 - 2g$$

where $n = card(B)$, g is a non-negative integer, called the *genus* of (α, σ), and where for any permutation θ, $z(\theta)$ denotes the number of its cycles (cycles of length 1 are included) (see [CoMa], p.422). If $g = 0$, then (α, σ) is *planar.*

Definition 2.2 *An* automorphism ϕ *of a hypermap* (α, σ) *is a permutation commuting with both* α *and* σ*:*

$$\alpha\phi = \phi\alpha \qquad \textit{and} \qquad \sigma\phi = \phi\sigma \quad .$$

Thus, the full automorphism group of (α, σ), denoted by $Aut(\alpha, \sigma)$, is the centralizer in $Sym(n)$ of the group generated by α and σ. A subgroup G of $Aut(\alpha, \sigma)$ is an *automorphism group* of (α, σ); the transitivity of (α, σ) implies that $Aut(\alpha, \sigma)$ is semi-regular. Recall here that a semi-regular group is defined by the fact that all its orbits are of the same length namely $| G |$.

We denote by $\chi_\theta(\phi)$ the number of cycles of a permutation θ fixed by an automorphism ϕ and by $\chi(\phi)$ the total number of cycles of α, σ, and $\alpha^{-1}\sigma$ fixed by ϕ; $o(\phi)$ will be the order of ϕ. If (α, σ) is planar ($g = 0$) then $\chi(\phi) = 2$ for all non trivial automorphisms ϕ. Moreover, $Aut(\alpha, \sigma)$ is one of C_n (cyclic), D_n (dihedral), A_4, S_4 and A_5 (see [CoMa] p.464). We shall need this result later.

We now define an equivalence relation R on the set B.

Definition 2.3 *Let* G *be an automorphism group of the hypermap* (α, σ). *Two brins* b_1 *and* b_2 *are* equivalent, $b_1 R b_2$, *if they belong to the same orbit of* G.

This leads to the following definition.

Definition 2.4 *The* quotient hypermap $(\overline{\alpha}, \overline{\sigma})$ *of* (α, σ) *with respect to an automorphism group* G*, is a pair of permutations* $(\overline{\alpha}, \overline{\sigma})$ *acting on the set* $\overline{B}$*, where* $\overline{B} = B/R$ *and* $\overline{\alpha}$*,* $\overline{\sigma}$ *are the permutations induced by* α *and* σ *on* $\overline{B}$*.*

The following Riemann-Hurwitz formula relates the genus γ of $(\overline{\alpha}, \overline{\sigma})$ to the genus g of (α, σ) (see [Ma]):

$$(RH1) \qquad 2g - 2 = card(G)(2\gamma - 2) + \sum_{\phi \epsilon G - \{id\}} \chi(\phi) \quad .$$

It follows that $\gamma \leq g$. In case G is a cyclic group, $G =< \phi >$, $(RH1)$ becomes

$$(RH2) \qquad 2g - 2 = card(G)(2\gamma - 2) + \sum_{i=1}^{o(\phi)-1} \chi(\phi^i) \quad .$$

As mentioned above one can prove that for $g \geq 2 \mid Aut(\alpha, \sigma) \mid \leq 84(g-1)$.

If ϕ is an automorphism of order m, then, for all integers i, $\chi(\phi) \leq \chi(\phi^i)$, and when m and i are coprime $\chi(\phi) = \chi(\phi^i)$.

Let (α, σ) be a hypermap, G an automorphism group of (α, σ) and let $(\overline{\alpha}, \overline{\sigma})$ be the quotient hypermap of (α, σ) with respect to G. The proof of the following results can be found in [Be3]. For any element ψ in the normalizer of G in $Aut(\alpha, \sigma)$, the permutation $\overline{\psi}$, defined as $\overline{\psi} = \psi/G$, is an automorphism of $(\overline{\alpha}, \overline{\sigma})$. The two following operations on (α, σ) are equivalent:

(i) quotienting (α, σ) first by G and then by $\overline{\psi}$

(ii) quotienting (α, σ) by $< G, \psi >$.

Definition 2.5 *The permutation* $\overline{\psi}$ *is called the* induced automorphism *of* ψ *on* $(\overline{\alpha}, \overline{\sigma})$*. We also say* ψ induces $\overline{\psi}$ *on* $(\overline{\alpha}, \overline{\sigma})$*.*

We now give the theorem that counts the fixed cycles of an induced automorphism.(see [Be3])

Theorem 2.6 *Let* (α, σ) *be a hypermap admitting an automorphism* ψ *and an automorphism group* G *such that* G *is normal in* $< G, \psi >$*. Then:*

$$| G | \chi(\overline{\psi}) = \sum_{\phi \in G} \chi(\phi\psi)$$

where $\bar{\psi}$ is the permutation induced by ψ on the quotient hypermap of (α, σ) with respect to G.

When $G = C_p$, we have simpler formulas (see [Be2]). G is now generated by an automorphism ϕ of prime order p and ψ is an element of $Aut(\alpha, \sigma)$. Then there are two cases.

Proposition 2.7 *Let ψ commute with ϕ.*

i) If ψ is of order m where p and m are coprime, then

$$\chi(\overline{\psi})p = \chi(\psi) + (p-1)\chi(\phi\psi).$$

ii) If ψ is of order pn, p and n coprime, and ϕ belong to $< \psi >$, then

$$\chi(\overline{\psi})p = \chi(\psi^p) + (p-1)\chi(\psi).$$

iii) If ψ is of order $p^m n$, $m > 1$, p and n coprime, and ϕ belong to $< \psi >$, then

$$\chi(\psi) = \chi(\overline{\psi}).$$

iv) If ψ is of order pm, m being any integer, and ϕ does not belong to $< \psi >$, then

$$\chi(\overline{\psi})p = \sum_{i=0}^{p-1} \chi(\psi\phi^i)$$

and

$$\chi(\psi\phi^i) \equiv 0 (mod \quad p).$$

Proposition 2.8 *If ψ does not commute with ϕ, then*

$$\chi(\psi) = \chi(\overline{\psi}).$$

In the classical theory of Riemann surfaces, a hyperelliptic surface S is a surface admitting an involution which is central in $Aut(S)$ and fixes $2 + 2g$ points. This notion applies to hypermaps [CoMa]. In the next definition we consider automorphisms of prime order p to generalize the idea of hyperellipicity.

Definition 2.9 *A hypermap (α, σ) of genus $g > 1$ is said to be p-elliptic if it admits an automorphism ϕ of prime order p such that:*

(1) the quotient hypermap $(\overline{\alpha}, \overline{\sigma})$ with respect to ϕ is planar,

(2) $< \phi >$ is normal in $Aut(\alpha, \sigma)$.

Remark 2.10 *This definition is equivalent to that given in section 1*

Since an automorphism on the sphere fixes exactly 2 points, an automorphism ψ on a p-elliptic hypermap of genus g fixes $\chi(\psi) = 0, 1, 2, p, p+1, 2p$ or $2 + 2g/(p-1)$ points. It is a consequence of Proposition 2.7 together with the fact that a planar automorphism fixes exactly 2 points (see [Be4]).

Proposition 2.11 *Let (α, σ) be a hypermap and G an automorphism group; let $N(G)$ be the normalizer of G in $Aut(\alpha, \sigma)$ and $t > 0$ the number of points fixed by non trivial elements of G. Then there exists a homomorphism h from $N(G)$ to S_t and whose kernel is a cyclic group.*

We remark that when $G =< \phi >$, then the image of h is contained in $S_{\chi(\phi)}$.

For complete proofs of these results see [Be3].

Theorem 2.12 *Let (α, σ) be a p-elliptic hypermap. Then $Aut(\alpha, \sigma)$ is either C_{pn} (cyclic) where n is a divisor of $1 + 2g/(p-1)$; C_{pn} or D_{pn} (dihedral) where n is a divisor of $2g/(p-1)$; a semi-direct product of either C_n or a lifting of D_n by C_p, where n is a divisor of $2 + 2g/(p-1)$; or $Aut(\alpha, \sigma)$ is of order $12p$, $24p$ or $60p$ (liftings of A_4, S_4, A_5 respectively).*

Theorem 2.13 *Let (α, σ) be a hypermap of genus $g \geq 2$, p the smallest prime dividing the order of $Aut(\alpha, \sigma)$, where $Aut(\alpha, \sigma)$ is nilpotent. Let ϕ be an automorphism of order p such that $\chi(\phi) = 2 + 2g/(p-1)$. Then (α, σ) is p-elliptic for the automorphism ϕ.*

By the Riemann-Hurwitz formula , we know that if a hypermap (α, σ) of genus $g > 1$ admits an automorphism group G such that the quotient hypermap with respect to it is $\gamma > 1$, then $\mid G \mid \leq g - 1$;

We now give a bound when $\gamma = 1$.

Theorem 2.14 *Let (α, σ) be a hypermap of genus $g > 1$ and G an automorphism group such that the quotient hypermap with respect to it is of genus $\gamma = 1$ then:*

$\mid G \mid \leq \frac{2p}{p-1}(g-1)$ *where p is the smallest prime that divides the order of $Aut(\alpha, \sigma)$.*

The next theorem shows that if $Aut(\alpha, \sigma)$ is of odd order, then in the Hurwitz bound $84(g-1)$, 84 can be replaced by 15 if $\mid Aut(\alpha, \sigma) \mid$ is dividable by 3 and $\frac{2p}{p-3}$ if its smallest divisor $p \geq 5$.

Theorem 2.15 *Let (α, σ) be a hypermap of genus $g > 1$, G an automorphism group such that the quotient hypermap with respect to it is of genus $\gamma = 0$ and p the smallest prime that divides the order of $Aut(\alpha, \sigma)$. Then:*

If $p \geq 5$ $\mid G \mid \leq \frac{2p}{p-3}(g-1)$

If $p = 3$, $\mid G \mid \leq 15(g-1)$

If $p = 2$, *the Hurwitz bound* $\mid G \mid \leq 84(g-1)$ *cannot be improved.*

We now give a technical lemma which will help for the sequel:

Lemma 2.16 *Let (α, σ) be a hypermap of genus $g \geq 2$, G an automorphism group such that p is the smallest divisor of its order, ϕ an automorphism of order p, normal in G and fixing two points. Then either $G = C_n$ or $G = D_n$ with $n < 2pg/(p-1)$.*

For the proofs of these results see [Be5]

Finally, we give two results about automorphisms of the torus.

Proposition 2.17 *Let (α, σ) be a hypermap of genus 1 and ψ an automorphism; then only two cases can happen:*

i) either ψ fixes nothing and neither does any non trival power of ψ.

ii) or ψ fixes at least one point and then ψ is of order $2, 3, 4, 6$ with $4, 3, 2, 1$ fixed points respectively.

For a proof see [Be5].

Theorem 2.18 *Let (α, σ) be a hypermap of genus 1 then $Aut(\alpha, \sigma)$ is isomorphical to a semi-direct product of an abelien group H with a cyclic group C_m where $m = 1, 2, 3, 4, 6$ and all automorphisms of H fix no points.*

For a combinatorial proof see [Ca].

3 The commutative case

let us begin this section with an important remark.

Remark 3.1 *Let (α, σ) be a hypermap of genus $g \geq 2$. We know by Theorem 2.15 that an automorphim group G whose smallest divisor of the order is $p \geq 5$ satisfies $\mid G \mid \leq 2p(g-1)/(p-3)$. Let $\mid G \mid = pK$ with $K \neq 1$. Then, $p \leq K \leq 2(g-1)/(p-3)$. Thus, $p(p-3) \leq 2(g-1)$ that is $p^2-3p+2g-2 \leq 0$; it implies that $p \leq (3+\sqrt{8g+1})/2$. This remark can be summed up in the following way: either $\mid G \mid = p$ and $p \leq 2g+1$ or $\mid G \mid > p$ then p belongs to $\{5, \cdots, (3+\sqrt{8g+1})/2\}$.*

Theorem 3.2 *Let (α, σ) be a hypermap of genus $g \geq 2$, G a commutative automorphism group and p its smallest prime divisor, then:*

$$\mid G \mid \leq p(2 + 2g/(p-1))$$

*Proof.*We suppose that the quotient hypermap with respect to G is of genus $\gamma_0 = 0$ otherwise by Theorem 2.14 $\mid G \mid \leq 2p(g-1)/(p-1)$. Let us consider, ϕ an automorphism of prime order p central in the p-Sylow (we take the one fixing the most points in the center). Let γ be the genus of $(\overline{\alpha}, \overline{\sigma})$ the quotient hypermap with respect to $< \phi >$. Since G is commutative, $< \phi >$ is central in G and therefore all automorphisms of G are induced on the quotient w.r.t. $< \phi >$.

Let $\gamma = 0$, then the hypermap is p-elliptic and the maximal order of G is $p(2 + 2g/(p-1))$ (Theorem 2.12).

Let $\gamma = 1$. Let $\overline{\psi}$ be an automorphism of prime order. Because it equals the quotient hypermap of (α, σ) w.r.t. G, the quotient hypermap of $(\overline{\alpha}, \overline{\sigma})$ w.r.t. $< \overline{\psi} >$ is of genus 0. Thus, $\overline{\psi}$ fixes points on the torus (immediate application of (RH)). Now, by Proposition 2.17, $o(\overline{\psi}) = 2$ or 3;

If $p = 3$, $o(\phi) = 3$, $\chi(\phi) = g-1$ $o(\overline{\psi}) = 3$ and $\chi(\overline{\psi}) = 3$. Since the quotient hypermap with respect to $\overline{\psi}$ is a commutative subgroup of S_3 of odd order, it is either the identity or C_3; thus, $G/ < \phi > = C_3, C_3 \times C_3$. Thus $\mid G \mid \leq 27 \leq 3(g+2)$ for $g \geq 5$. If $g \leq 4$ then $\chi(\phi) \leq 3$ thus $\mid G/ < \phi > \mid \leq 3$ i.e. $\mid G \mid \leq 9 \leq 3(g+2)$.

If $p = 2$, $o(\phi) = 2$, $\chi(\phi) = 2g-2$, $o(\overline{\psi}) = 2$ and $\chi(\overline{\psi}) = 4$ or $o(\overline{\psi}) = 3$ and $\chi(\overline{\psi}) = 3$. Thus, the quotient hypermap with respect to $\overline{\psi}$ is a commutative subgroup of S_3 or S_4. Hence, $\mid G/ < \phi > \mid \leq 8$ or $G/ < \phi > = C_3 \times C_3$. Thus,

$\mid G \mid \leq 18 \leq 4(g+1)$ for $g > 2$. If $g = 2$ and $\mid G \mid = 18$, then an automorphism of order 3 fixes 4 points (it can not fix only one point) and thus the other automorphism group of order 3 would have a fixed point in common with it which is impossible. If $g = 2$ and $\mid G \mid = 16$, then an automorphism of order 2 fixes either 2 or 6 points that is $\mid G \mid \leq 4$ or the hypermap is hyperellliptic and then $\mid G \mid \leq 12$.

Let $\gamma \geq 2$.

If $\chi(\phi) = 0$, then all automorphisms of order a multiple of p fix no point. Now by (RH), we know that $\sum_{\psi \in G-id} = 2g - 2 + 2 \mid G \mid$ but $\sum_{\psi \in G-id} = \sum_{\psi \in H-id} = 2g - 2 + 2 \mid H \mid$ where H is the product of the q-Sylows subgroups of G where q is a prime strictly greater than p; hence the contradiction.

If $\chi(\phi) = 2$, then by Lemma 2.16, $\mid G \mid < 2pg/(p-1)$.

If $2 < \chi(\phi) < p$, then evry element of G fixes all the points fixed by ϕ. Thus, G is cyclic and by $(RH2)$, we have $\mid G \mid \leq 2g + 1$.

If $p \leq \chi(\phi) < 2p$, let q be a divisor of $\mid h(G) \mid$ the image of G in $S_{\chi(\phi)}$; q is greater then p and divides $(2p-1)!$. Since no prime greater then p is repeated twice in $(2p-1)!$, q is square-free. If $q_1 \neq q_2$ are two different primes dividing q, then, since $h(G)$ is commutative, there exists a cyclic subgroup of order $q_1 q_2$ in $h(G)$; but $q_1 q_2 \geq p^2 > 2p$ a contradiction. Thus, $h(G)$ is a commutative group of order q, where q is a prime such that $p \leq q < 2p$. By (RH), $\mid kerh \mid \leq 1 + 2g/(q-2)$ i.e. $\mid G \mid \leq q(1 + 2g/(q-2))$. Now, $q(1 + 2g/(q-2))$ is a decreasing function on the intervalle $\{5, \cdots, (3 + \sqrt{8g+1})/2\}$. Thus, $\mid G \mid \leq p(1 + 2g/(p-2))$ and $\mid G \mid \leq p(2 + 2g/(p-1))$.

If $\chi(\phi) \geq 2p$ we have by (RH) that $2g - 2 \geq p(2\gamma - 2) + 2p(p-1)$. Thus, after computation, $2 + 2g/(p-1) \geq p(2 + 2\gamma/(p-1))$. Now, $\mid G \mid /p = \mid G/ < \phi > \mid \leq p(2 + 2\gamma/(p-1))$ by induction on γ. Hence, $\mid G \mid \leq p(2 + 2g/(p-1))$.$\diamond$

Example 3.3 *Let $g = 2$, (α, σ) is defined on 12 brins with 6 edges, 2 vertices and 2 faces:*

$\sigma = (1,3,5,7,9,11)(2,4,6,8,12)$
$\alpha = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$
$\alpha\sigma = (1,4,5,8,9,12)(2,3,6,7,10,11)$
$Aut(\alpha, \sigma) = C_2 \times C_6$.

In Figure 1 the cuts have been made along the edges. Edges bearing the same number are actually identified. The hyperelliptic involution is a rotation around an axis perpendicular to the figure going through the edge

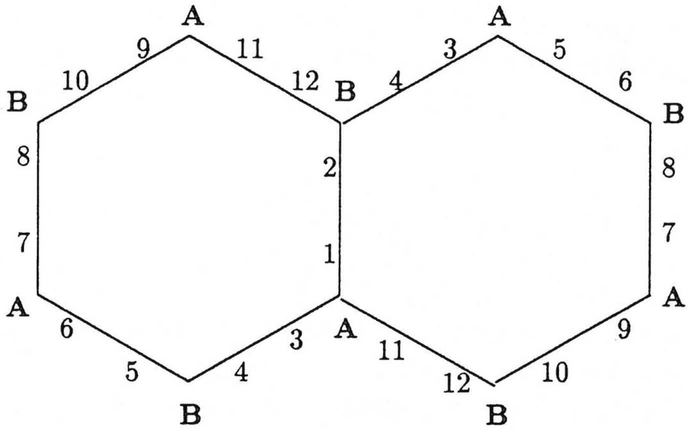

Figure 1: The double hexagon

$(1,2)$ and exchanging the two faces. The automorphism of order 6 can be viewed as a rotation around an axis going through the midpoints of the two faces. It fixes both faces while its square also fixes the vertices A and B.

Now, we show that this bound is sharp, since for infinitly many $g > 1$ it can be found a commutative group of order $p(2+2g/(p-1))$ as the following proposition shows:

Proposition 3.4 *For all prime p and for all $g \geq 2$ such that $p-1$ divides $2g$ and p divides $2+2g/(p-1)$,that is $g = (kp-2)(p-1)/2$ k being any integer, there exists a p-elliptic map such that its automorphism group is exactly $C_p \times C_{2+2g/(p-1)}$.*

Proof Let $g > 1$ be an integer and (α, σ) the following hypermap on $p(2+2g/(p-1))$ brins with $2+2g/(p-1)$ edges, p vertices and p faces:

$\sigma = (1, p+1, 2p+1, \cdots, p(1+2g/(p-1))+1) \cdots (p, 2p, \cdots, p(2+2g/(p-1)))$

$\alpha = (1, 2, 3 \cdots p) \cdots (p(1+2g/(p-1))+1), \cdots, p(2+2g/(p-1)))$

Since $\alpha\sigma = \sigma\alpha$, $\alpha^{-1}\sigma$ is of order $2+2g/(p-1)$ (it has p cycles) and $Aut(\alpha, \sigma) =< \alpha, \sigma >$. Thus, $Aut(\alpha, \sigma) = C_p \times C_{2+2g/(p-1)}$. $\diamondsuit$.

Remark 3.5 *Note that when $p = 3$, the condition on g is reduced to $g = 1 \mod 3$ and when $p = 2$ no condition at all is required so that for all*

$g \geq 2$ it can be found a hyperelliptic hypermap of automorphism group equal to $= C_2 \times C_{2g+2}$.

Corollary 3.6 *Let (α, σ) be a hypermap of genus $g \geq 2$ and G a commutative automorphism group then $\mid G \mid \leq 4(g+1)$*

Proof Immediate since the function $f(p) = p(2+2g/(p-1))$ is decreasing and reaches its maximum for $p = 2.\diamondsuit$

4 The nilpotent case

Let us also begin this section with an important remark.

Remark 4.1 *The bound $2p(g-1)/(p-3)$ of Theorem 2.15 follows immediately from the Riemann-Hurwitz formula and can also be found in [Zo]. The bound $p(2+2g/(p-1))$ of Theorem 3.2 is new. The two bounds are equal if and only if $p = (3+\sqrt{8g+1})/2$ in which case $2p(g-1)/(p-3) = p(2+2g/(p-1)) = p^2$. So that $\mid G \mid = p^2$ and G is abelien. Thus the two bounds meet, so that $2p(g-1)/(p-3)$ is a bound reached for even commutative groups and therefore for nilpotent groups.*

Here is an example where the bound $2p(g-1)/(p-3)$ is reached.

Example 4.2 *Consider (α, σ) is defined on 25 brins with 5 edges, 5 vertices and 5 faces:*

$\sigma = (1,6,11,16,21)(2,7,12,17,22)(3,8,13,18,23)$
$(4,9,14,19,24)(5,10,15,20,25)$,

$\alpha = (1,2,3,4,5)(6,7,8,9,10)\cdots(21,22,23,24,25)$,

$\alpha\sigma = (1,10,14,18,22)(2,6,15,19,23)$
$(3,7,11,20,24)(4,8,12,16,25)(5,9,13,17,21)$.

Then $g = 6$ and $Aut(\alpha, \sigma) = C_5 \times C_5$.

Here, the two bounds are reached.

$5^2 = 2 \cdot 5(6-1)/(5-3) = 5(2+2 \cdot 6/(5-1)$

When p is either 2 or 3 the bounds don't derive from a general formula as the two following theorems show.

Theorem 4.3 *Let (α, σ) be a hypermap of genus $g \geq 2$, G a nilpotent automorphism group and 3 its smallest prime divisor then*

$$\mid G \mid \leq 9(g-1)$$

Proof Let us consider, ϕ an automorphism of prime order 3 such that it is in the center of the 3-sylow subgroup of G. Let γ be the genus of the quotient hypermap $(\overline{\alpha}, \overline{\sigma})$ with respect to $< \phi >$. Since G is nilpotent, all automorphisms are induced on the quotient. We consider that the quotient hypermap with respect to G is of genus 0 otherwise $\mid G \mid \leq 3(g-1)$.

If $\gamma = 0$, then the hypermap is hyperelliptic. Thus, G is of maximal order $3(g+2)$ (Theorem 2.12).

Let $\gamma = 1$, then $\chi(\phi) = 2g - 2$. There exists an automorphism $\overline{\psi}$ of prime order fixing points on the torus (because the quotient by G is of genus 0). By Proposition 2.17, 2.18, we know that $G/ < \phi >$ is the semi direct product of H a group regular on $2g-2$ points (since all automorphisms fix nothing) and a C_m where $m = 1, 2, 3, 4, 6$. Thus, $\mid G/ < \phi > \mid \leq 3(g-1)$ because the smallest divisor is 3. Hence, $\mid G \mid \leq 9(g-1)$.

Let $\gamma \geq 2$, then we may procceed by induction on γ. (RH) gives that $2g - 2 \geq 3(2\gamma - 2)$. Thus, $g - 1 \geq 3(\gamma - 1)$. Now, $\mid G \mid /p \leq 9(\gamma - 1)$. Hence, $\mid G \mid \leq 9(g-1)$.

Let us now give the last bound on nilpotent hypermaps:

Theorem 4.4 *Let (α, σ) be a hypermap of genus $g \geq 2$ and G a nilpotent automorphism group then $\mid G \mid \leq 16(g-1)$*

Proof. Let us consider, ϕ an automorphism of prime order 2 such that it is in the center of the 2-sylow subgroup of G. Let γ be the genus of the quotient hypermap $(\overline{\alpha}, \overline{\sigma})$ with respect to $< \phi >$. Since G is nilpotent, all automorphisms are induced on the quotient. We consider that the quotient hypermap with respect to G is of genus 0 otherwise $\mid G \mid \leq 4(g-1)$.

If $\gamma = 0$, then the hypermap is hyperelliptic. Thus, G is of maximal order $8(g+1)$(Theorem 2.12).

Let $\gamma = 1$, then $\chi(\phi) = 2g - 2$. There exists an automorphism $\overline{\psi}$ of prime order fixing points on the torus (because the quotient by G is of genus 0). By Proposition 2.17 and Theorem 2.18, we know that $G/ < \phi >$ is the semi direct product of H a group regular on $2g-2$ points (since all automorphisms fix nothing) and a C_m where $m = 1, 2, 3, 4, 6$. In the case

$m = 6$, we have G is a semi direct product of $C = C_6$ by H, where H is normal in G. Let us show that H is reduced to the identity. The elements of H of order prime to 3 commute with the C_3 contained in C; therefore they are induced on the quotient of (α, σ) w.r.t. C_3 which is of genus 0. Now, each of these induced automorphisms fix two points on the sphere; This is not possible by Proposition 2.17. Hence, H is a 3-group. The element τ of order 2 in C commutes with all elements of H and since the quotient w.r.t. τ is planar, the elements induced on this quotient by the elements of H must fix two points. As before this is not possible. Thus, $G = C_6$. If $m \leq 4$, then $\mid G/ < \phi > \mid \leq 4(2g-2)$ and $\mid < \phi > \mid \leq 2$ implies $\mid G \mid \leq 16(g-1)$.

Let $\gamma \geq 2$, then we may procceed by induction on γ. (RH) gives that $2g - 2 \geq p(2\gamma - 2)$. Thus, $g - 1 \geq p(\gamma - 1)$. Now, $\mid G \mid /p \leq 16(\gamma - 1)$. Hence, $\mid G \mid \leq 16(g-1)$. $\diamondsuit$

Corollary 4.5 *Let (α, σ) be a hypermap of genus $g \geq 2$ and G a nilpotent automorphism group then $\mid G \mid \leq 16(g-1)$*

Proof Immediate since the function $f(p) = 2p(g-1)/(p-3)$ is decreasing and reaches its maximum for $p = 5$ where $f(5) = 5(g-1)$. Thus, the result is smaller than $16(g-1)$ $\diamondsuit$

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References

[Be1] L.Bessis, *A note on the fixed points of a hypermap automorphism*, Europeen Journal of combinatorics,(13),1992, p.65-69.

[Be2] L.Bessis, *Induced automorphisms of hypermaps and p-elliptic hypermaps*, accepted for publication in the Int. J. of Algebra and Computation.

[Be3] L.BESSIS, *Fixed points of induced automorphisms of hypermaps*, accepted for publication in Discrete Mathematics.

[Be4] L.BESSIS, *p-elliptic hypermaps and the Klein hypermap*, submitted.

[Be5] L.BESSIS, *Harvey-Wiman hypermaps*, submitted.

[BiCo] G. BIANCHI and R. CORI, *Colorings of hypermaps and a conjecture of Brenner and Lyndon*, Pac. J. of Math., (110), 1, 1984, 41-48.

[Ca] L. CACCIARI, *Automorphisms of the torus*, RAIRO 1992.

[CoMa] R.CORI AND A.MACHÌ, *Maps, hypermaps and their automorphisms: a survey*,I,II,III, Expositiones Matematicae, (10),1992, 403-427,429-447,449-467.

[JoSi] G.A.JONES AND D.SINGERMAN, *Theory of maps on an orientable surface*, Proc. London Math. Soc. 3 (37) 1978, 273-307.

[McL] C. MAC LACHLAN, *Abelien groups of automorphisms of compact Riemann surfaces*, Proc. London Math. Soc.(3),15 (1965) 699-712

[Ma] A.MACHÌ, *The Riemann-Hurwitz formula for the centrilizer of a pair of permutations*, Arch. d. Math. (42), 1984, 280-288.

[Zo] R. ZOMORRODIAN, *Classification of p-groups of automorphisms of Riemann surfaces and their lower central series*, Glasgow math. J. 29 (1987), 237-244