Maple Umbral Calculus Package

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Abstract

Rota's Umbral Calculus uses sequences of Sheffer polynomials to count certain combinatorial objects. We have developed a Maple package that implements Rota's Umbral Calculus and some of its generalizations. A Mathematica version of this package is being developed in parallel.

Résumé

Le calcul ombral de Rota utilise des suites de polynômes de Sheffer pour énumérer certains objets combinatoires. Nous avons développé une bibliothèque de fonctions Maple qui implémentent cette théorie ainsi que ses généralisations. Une version Mathematica de ce package est développée en parallèle.

AMS classification: 05A40.
Hardware requirements: A computer equipped with Maple V.2 or V.3.

1 Introduction

Umbral calculus is the study of the analogies between various polynomial sequences and the powers sequence x^n. For example, x^n has many parallels with the lower factorial sequence $(x)_n = x(x-1)\cdots(x-n+1)$:

- The forward difference operator $\Delta : p(x) \mapsto p(x+1) - p(x)$ plays a role with respect to $(x)_n$ analogous to that played by the derivative d with respect to x^n.

- Taylor's theorem is analogous to Newton's theorem.

- The binomial theorem for $(x+y)^n$ is replaced by Vandermonde's identity for $(x+y)_n$.

Although Umbral Calculus dates back to the 18th century, it was only put on a rigorous foundation by Gian-Carlo Rota and his collaborators [6, 12] in the 1970's. We now characterize each polynomial sequence under study by one or more polynomial operators (usually shift-invariant[1]) associated with it. The duality between operators and polynomials is the key tool to deriving umbral calculus results.

Umbral Calculus has many applications in enumerative combinatorics. The powers x^n counts all functions from an n-element set to an x-element set, while the lower factorials $(x)_n$ count injections. Similarly, given any species of combinatorial structures (or quasi-species), let $p_n(x)$ be the number of functions from an n-element set to an x-element set enriched by this species. A function is enriched by associating a (weighted) structure with each of its fibers. The resulting sequence of polynomials $(p_n)_{n \in N}$ is said to be of binomial type since it obeys the "binomial" identity

$$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y).$$

For example, given the species of rooted forests, the enriched functions are called persistent functions and are enumerated by the Abel polynomials $A_n(x) = x(x + n)^{n-1}$. Other applications include lattice path counting [7, 8, 15].

Our Maple package provides a number of different tools by which to enter operators. These operators can then be manipulated in many different ways. In particular, the polynomial sequences associated with them can be explicitly calculated.

This package has already aided us in our research [3]; we hope that it will help you too. We expect to release a Mathematica version of this package in the near future.

2 Polynomial Operators

Polynomial operators (shift-invariant or not) can be specified in several convenient manners:

Explicitly by their action on polynomials. For example, the shift operator is defined
`< subs(x = x+a, p) | p >` using the "angle-bracket" notation for functional operators. (See `?operators[functional]` and `?unapply` for details.) Similarly, the Bernoulli operator $p(x) \mapsto \int_x^{x+1} p(t)dt$ is defined `<int(subs(x=t, p), t=x..(x+1)) | p | t>`.

As an analytic function of the derivative. By the expansion theorem (see [6, Theorem 2, p. 185] or [12, Theorem 2, p. 691]), any shift-invariant operator can be expanded as a formal power series in d where d is a special reserved symbol representing the derivative. For example, the shift operator is defined `exp(a*d)`.

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[1]Shift-invariant operators commute with the shift operator $E^a : p(x) \to p(x + a)$.

Abstractly as an unspecified function of d. For example, `f(d)` or `f(d,x)` in the case of a non-shift-invariant operator [5].

Using the `powseries` package. If the coefficients of the formal power series given by the expansion theorem are all known, then use `powcreate`. For example, `powseries [powcreate] (f(n) = a^n/n!);`.

As a series. If only finitely many terms are known, then use `series`. For example, `series(1 + a*d + a^2*d^2/2 + c*d^3, d, 3);`.

See `?linear` for more information.

Operators can be converted easily from one form to another with `convert`. A delta operator `Q` is a shift-invariant operator such that `Q (x)` is a non-zero constant. An abstract operator is assumed to be invertible unless indicated otherwise. (For example, `Q := d*f(d);` or `Q := f(d); f(0) := 0;`.)

Using the generalized expansion theorem, shift-invariant operators can be expanded with `operatorExpansion` as a formal power series in an arbitrary delta operator.

```
> oe(d,delta,3);
```

$$\exp(d) - 1 - 1/2\ (\exp(d) - 1)^2 + O((\exp(d) - 1)^3)$$

Such operator expansions are practical for numerical calculations. For example, expanding the Bernoulli operator into powers of the forward difference operator `delta` yields the classical Newton-Cotes formula of numerical integration [6, p. 186].

Our package also allows the expansion of linear operators which are not shift-invariant. Such expansions [5] express the operator as a formal power series in d whose coefficients are polynomials in x. For example, if `Q` is the operator $Q: p(x) \mapsto \int_0^x p(t)dt$, then the expansion `convert(Q,function,5,x)` or `convert(Q,powseries,x)` of `Q` in terms of multiplication by x and the derivative d gives an elementary proof of Bourbaki's method of asymptotic integration [1, Sections 3.5 and 3.6].

Operators can be applied to polynomials with `dp` by specifying the free variable of the polynomial:

```
> dp(delta,x^3,x);
```

$$3\ x^2 + 3\ x + 1$$

In case the degree of the polynomial is not explicitly given, the program will calculate the `Order` most significant terms of the answer. (`Order` is a system variable whose default value is 6.)

```
> dp(delta,x^n,x);
```

$$n\ x^{(n-1)} + 1/2\ n\ (n-1)\ x^{(n-2)} + 1/6\ n\ (n-1)\ (n-2)\ x^{(n-3)}$$

$$+ \ 1/24 \ n \ (n-1) \ (n-2) \ (n-3) \ x^{(n-4)}$$

$$+ \ 1/120 \ n \ (n-1) \ (n-2) \ (n-3) \ (n-4) \ x^{(n-5)} \ + \ O(x^{(n-6)})$$

3 Polynomial Sequences

Given the necessary operators, the program can calculate polynomials of binomial type (bfo), Sheffer sequences (sfo), Steffensen sequences (steff), and cross-sequences (cseq) (see [12, Sections 5 an 8]). For example, the sequence of binomial type bfo(p(d), x, n) associated with a delta operator p(d) is defined by the conditions

$$
\begin{aligned}
dp(p(d), bfo(p(d), x, n)) &= n * bfo(p(d), x, n) \quad \text{for } n>0 \\
bfo(p(d), 0, n) &= 0 \quad\quad\quad\quad\quad\ \text{for } n>0 \\
bfo(p(d), x, 0) &= 1,
\end{aligned}
$$

or equivalently by its exponential generating function $\exp(q(x)t)$ where q is the compositional inverse of p. Note that q(t) is the generating function of the associated species.

For example, the lower factorial $(x)_n$ is the basic sequence of binomial type for the forward difference operator Δ.

```
> factor(bfo(delta,x,4));
                    x (x - 1) (x - 2) (x - 3)
```

If the degree is not explicitly given, then only the most significant terms will be computed.

Several functions in the package do further operations on polynomial sequences. Arbitrary polynomials can be expressed in terms of such sequences (polynomialExpansion, shefferExpansion, basicExpansion). For example,

```
> p := randpoly(x);
                  5       4       3       2
          p := 79 x  + 56 x  + 49 x  + 63 x  + 57 x - 59
> be(delta,p,x);
        - 59 bfo(exp(d) - 1, x, 0) + 304 bfo(exp(d) - 1, x, 1)
           + 1787 bfo(exp(d) - 1, x, 2) + 2360 bfo(exp(d) - 1, x, 3)
           + 846 bfo(exp(d) - 1, x, 4) + 79 bfo(exp(d) - 1, x, 5)
> ";
                5       4       3       2
            79 x  + 56 x  + 49 x  + 63 x  + 57 x - 59
```

Connection constants can be determined between arbitrary polynomial sequences. For example, the Stirling numbers are given by cc(topseq(powerx, 5, x), topseq(lower, 5, x), x) where powerx(n,x) is x^n and lower(n,x) is $(x)_n$. Other features include umbral composition (uc), and umbral inversion (ui).

4 Generalizations

Several authors (eg. [10, 14]) have generalized the umbral calculus by considering not only sequences of binomial type with generating function $\exp(g(x)t)$ but also those whose generating function is $\Phi(g(x)t)$ where $\Phi(t) = \sum_{n=0}^{\infty} t^n/[n]!$ and $[n]!$ denotes the generalized factorial $[n]! = \mathtt{a}(1)\mathtt{a}(2)\cdots\mathtt{a}(n)$. Most of the functions in the umbral calculus package allow an optional argument $\mathtt{a}$ which is either left undefined, or defines the coefficients used by the "generalized derivative." Thus,

```
> dp(d, x^3, x, proc(n) 1 end);
                    2
                   x
```

The following possible choices for $\mathtt{a}$ are predefined.

Umbral Calculus		dp(d,p,x,a)	a
Classical	[6, 12]	$\dfrac{dp(x)}{dx}$	classical(n) = n
q-umbral calculus	[10, 11]	$\dfrac{p(qx)-p(x)}{(q-1)x}$	gaussian(n) = (q^n-1)/(q-1)
Divided Difference	[4, 13]	$\dfrac{p(x)-p(0)}{x}$	divided(n) = 1
Hyperbolic	[3]	$\left(\dfrac{d}{d\sqrt{x}}\right)^2 p(x)$	hyperbolic(n) = 2*n*(2*n-1)

See ?genderiv for details.

Generalizations of the umbral calculus to several variables [9, 15] are supported. Most functions included in the package have an alternate syntax for use in multivariate umbral calculi. In particular, d[i] represents the partial derivative with respect to the ith variable. Instead of a single delta operator, a collection of operators are required to define a sequence of binomial type. This generalization is completely compatible with the above generalization. See ?multilinear and ?moe for details.

For further instructions consult the on-line help and examples provided in the package. For help, type ?*key-word*. An index of key-words is available via ?umbral.

See [2] for an extensive survey and bibliography of the umbral calculus.

References

[1] N. Bourbaki. *Fonctions d'une Variable Réelle*, volume IV of *Eléments de Mathématique*, chapter 5 of Fasc. XII: Etude locale des fonctions. Hermann, 1951. (MR 13,631a).

[2] A. Di Bucchianico and D.E. Loeb. A selected survey of the umbral calculus. 1995. WWW URL: http://www.labri.u-bordeaux.fr/~loeb/survey/a.html

[3] A. Di Bucchianico and D.E. Loeb. Sequences of binomial type with persistent roots. Submitted to J. Math. Anal. Appl., 1995.

[4] P.S. Hirschhorn and L.A. Raphael. Coalgebraic foundations of the method of divided differences. *Adv. Math.*, 91:75–135, 1992. (MR 92m:05012).

[5] S.G. Kurbanov and V.M. Maksimov. Mutual expansions of differential operators and divided difference operators. *Dokl. Akad. Nauk UzSSR*, 4:8–9, 1986. (MR 87k:05021).

[6] R. Mullin and G.-C. Rota. On the foundations of combinatorial theory III. Theory of binomial enumeration. In *Graph theory and its applications*, ed. Harris, pages 167–213. Academic Press, 1970. (MR 43#65).

[7] H. Niederhausen. The enumeration of restricted random walks by Sheffer polynomials with applications to statistics. *J. Stat. Plan. Inf.*, 14:95–114, 1986. (MR 87j:05015).

[8] M. Razpet. An application of the umbral calculus. *J. Math. Anal. Appl.*, 149:1–16, 1990. (MR 91i:05018).

[9] S.M. Roman. The algebra of formal series. III. Several variables. *J. Approx. Theory*, 26:340–381, 1979. (MR 81i:05023b).

[10] S.M. Roman. *The umbral calculus*. Academic Press, 1984. (MR 87c:05015).

[11] S.M. Roman. More on the umbral calculus, with emphasis on the q-umbral calculus. *J. Math. Anal. Appl.*, 107:222–254, 1985. (MR 86h:05024).

[12] G.-C. Rota, D. Kahaner, and A. Odlyzko. On the foundations of combinatorial theory VII. Finite operator calculus. *J. Math. Anal. Appl.*, 42:684–760, 1973. (MR 49#10556).

[13] L. Verde-Star. Divided differences and combinatorial identities. *Stud. Appl. Math.*, 85:215–242, 1991. (MR 92i:65027).

[14] O.V. Viskov. Operator characterization of generalized Appell polynomials. *Sov. Math. Dokl.*, 16:1521–1524, 1975. (MR 52#14416).

[15] T. Watanabe. On a generalization of polynomials in the ballot problem. *J. Stat. Plan. Inf.*, 14:143–152, 1986. (MR 87j:05024).