Iterated Homology of Simplicial Complexes

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November 9, 1994
Revised March 9, 1995

Extended Abstract

Summary.

We use the exterior face ring of a simplicial complex to develop an iterated homology theory for simplicial complexes. Let Δ be a simplicial complex of dimension $d-1$. For each $r = 0, \ldots, d$, we define *rth iterated homology groups* of Δ, the $r = 0$ case corresponding to ordinary homology. (Our theory is different from Kalai's iterated homology theory introduced in [K2], but if all iterated homology groups vanish for $r \leq n$ in one theory, then they must vanish in the other theory as well.)

If a simplicial complex is shellable (in the generalized nonpure sense of Björner and Wachs [BW]), then its iterated Betti numbers (vector-space dimensions of the iterated homology groups over a field) give the restriction numbers, h_{ij}, of the shelling. Iterated Betti numbers are preserved by Kalai's algebraic shifting, and may be interpreted combinatorially in terms of the algebraically shifted complex in several ways. Iterated Betti numbers are also depth-sensitive.

Résumé.

Nous utilisons l'anneau des faces extérieures d'un complexe simplicial pour développer une théorie d'homologie itérée pour les complexes simpliciaux. Soir Δ un complexe simplicial de dimension $d-1$. Pour tout $r = 0, \ldots, d$, nous définissons les *r-ièmes groupes d'homologie*

*Partially supported by the NSF Regional Geometry Institute, July 1993, and by the University Research Institute at the University of Texas at El Paso.

†Partially supported by the NSF Regional Geometry Institute, July 1993.

itérée de Δ, le cas $r = 0$ correspondant à l'homologie ordinaire. (Notr théorie est différente de l'homologie itérée de Kalai, introduite en [K2], mais si tous les groupes d'homologie itérée s'annulent pour $r \leq n$ dans une théorie alors ils s'annulent aussi dans l'autre.)

Si un complexe simplicial est "shellable" (dans le sens non pur génŕalisé de Björner et Wachs [BW]), alors ses nombres de Betti itérés (dimensions des groupes d'homologie itérés sur un corps) donnent les nombres de restriction h_{ij} du shellage. Les nombres de Betti itérés sont préservés par le décalage algébrique de Kalai, et peuvent être interprétés combinatoirement en termes du complexe algébriquement décalé, ceci de plusieurs manières. Les nombres de Betti itérés sont aussi sensibles à la profondeur.

Notation.

Let Δ be a finite (abstract) simplicial complex. The **dimension** of $F \in \Delta$ is $\dim F = |F| - 1$, and the **dimension** of Δ is $\dim \Delta = \max\{\dim F \colon F \in \Delta\}$. The maximal faces of Δ are called **facets**, and Δ is **pure** if all the facets have the same dimension. Let Δ_k denote the set of k-faces (*i.e.*, k-dimensional faces) of Δ. The **f-vector** of Δ is the sequence $(f_1, \ldots, f_{d-1})$, where $f_k = \#\Delta_k$ and $d - 1 = \dim(\Delta)$. The same notion of $f_i(\Delta)$ and the f-vector will apply to every finite collection of sets.

We call $\beta_i(\Delta) = \dim_K \tilde{H}^i(\Delta; K)$ the **ith reduced Betti number** of Δ with respect to the field K. The **Betti sequence** of Δ is $\beta(\Delta) = (\beta_0, \ldots, \beta_{d-1})$. Recall that over a field $\dim_K \tilde{H}^i(\Delta; K) = \dim_K \tilde{H}_i(\Delta; K)$, so that the Betti sequence measures (reduced) homology as well as (reduced) cohomology of Δ.

Shifted complexes and near-cones.

Define the partial order $\leq_P$ on k-subsets of integers as usual: If $S = \{i_1 < \cdots < i_k\}$ and $T = \{j_1 < \cdots < j_k\}$ are two k-subsets of integers, then $S \leq_P T$ if $i_p \leq j_p$ for all p. A collection $\mathcal{C}$ of k-subsets is **shifted** if $S \leq_P T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$. A simplicial complex Δ is **shifted** if Δ_k is shifted for every k.

Björner and Kalai show in [BK] that shifted complexes are near-cones, defined as follows: Let $\Delta' \dot\cup B$ be a simplicial complex such that B is a set of maximal faces in $\Delta' \dot\cup B$ (so Δ' is a subcomplex and B is an antichain). Then $\Delta = (v_0 * \Delta') \dot\cup B$ is a **near-cone** (where v_0 is some new vertex not in $\Delta' \dot\cup B$ and $*$ denotes topological join). In this case, we define $B(\Delta) = B$. If $B(\Delta) = \emptyset$, then Δ is a **cone**; more generally, we have the following theorem.

Proposition 1 (Björner-Kalai [BK, Theorem 4.3]) *Let Δ be a near-cone. Then Δ is homotopy equivalent to the $f(B(\Delta))$-wedge of spheres. In particular,*

$$\beta_i(\Delta) = f_i(B(\Delta)).$$

□

A shifted simplicial complex is really an iterated near-cone.

Proposition 2 *If Δ is a non-empty shifted simplicial complex on vertices $\{1, 2, 3, \ldots\}$, then*

(a) *[BK]* Δ *is a near-cone with apex* 1, *so* $\Delta = (1 * \Delta') \dot\cup B$; *and*

(b) Δ' *is a shifted simplicial complex on vertices* $\{2, 3, \ldots\}$.

□

This means, for instance, that if $\Delta = (1 * \Delta') \dot\cup B$ is shifted, then $\Delta' = (2 * \Delta'') \dot\cup B_1$ for some B_1 and Δ'', and thus,

$$\Delta = (1 * ((2 * \Delta'') \dot\cup B_1)) \dot\cup B.$$

More generally, we have the following corollary.

Corollary 3 *Let* $\Delta = \Delta^{(0)}$ *be a shifted simplicial complex of dimension* $d-1$. *Then we may inductively define* $\Delta^{(r)} = (\Delta^{(r-1)})'$, i.e.,

$$\Delta^{(r-1)} = (r * \Delta^{(r)}) \dot\cup B_{r-1} \;\; (1 \leq r \leq d), \tag{1}$$

for some B_r. *Furthermore,*

$$\Delta = 1 * (2 * (3 * (\cdots (d-1) * ((d * \emptyset) \dot\cup B_{d-1}) \dot\cup B_{d-2} \cdots) \dot\cup B_2) \dot\cup B_1) \dot\cup B_0. \tag{2}$$

Proof: Proposition 2 shows, inductively, that $\Delta^{(r-1)}$ is a near-cone with apex r, allowing $\Delta^{(r)}$ to be defined by equation (1). Equation (2) then follows from iterating equation (1). □

By Proposition 1, then,

$$f_i(B_r) = \beta_i(\Delta^{(r)}). \tag{3}$$

Iterated homology is an algebraic way to recover this data, even when the simplicial complex is not shifted.

Iterated homology.

Before we can define iterated homology, we must define the exterior face-ring (see [BK] for more details). Let Γ be a simplicial complex with vertices $V = \{e_1, \ldots, e_n\}$ linearly ordered $e_1 < \cdots < e_n$. Let $\Lambda(KV)$ denote the exterior algebra of the vector space KV; it has a K-vector space basis consisting of all the monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $S = \{e_{i_1} < \cdots < e_{i_k}\} \subseteq V$ (and $e_\emptyset = 1$). Let I_Γ be the ideal of $\Lambda(KV)$ generated by $\{e_S \colon S \notin \Gamma\}$. The quotient algebra $\Lambda[\Gamma] := \Lambda(KV)/I_\Gamma$ is the **exterior face ring** of Γ (over K), an exterior algebra analogue to the Stanley-Reisner face ring [St, Re].

Cohomology is easy to compute with the exterior face ring. Let $\tilde{x}$ denote the image modulo I_Γ of $x \in KV$. If $f = \alpha_1 e_1 + \cdots + \alpha_n e_n$, then $\delta_f \colon \Lambda[\Gamma] \to \Lambda[\Gamma]$ defined by $\delta_f(x) = x \wedge \tilde{f}$ is a **weighted coboundary operator**, so-called because

$$\delta_f(\tilde{e}_S) = \tilde{e}_S \wedge \tilde{f} = \sum_{i=1}^{n} \alpha_i \tilde{e}_S \wedge \tilde{e}_i = \sum_{\substack{i \notin S \\ S \cup \{i\} \in \Delta}} \pm \alpha_i \tilde{e}_{S \cup \{i\}}.$$

Setting every $\alpha_i = 1$ gives the usual coboundary operator. Ordinary Betti numbers may be computed using weighted coboundary operators: $\beta_i(\Gamma) = \dim_K(\ker \delta_f)_i/(\operatorname{im} \delta_f)_i$, if $f = \alpha_1 e_1 + \cdots + \alpha_n e_n$ and every α_i is non-zero [BK, pp. 289–290].

Definition: Let $\{f_1, \ldots, f_n\}$ be a "generic" basis of KV, *i.e.*, $f_i = \sum_{j=1}^n \alpha_{ij} x_j$, where the α_{ij}'s are n^2 transcendentals, algebraically independent over K. We define $f_S := f_{i_1} \wedge \cdots \wedge f_{i_k}$ for $S = \{i_1 < \cdots < i_k\}$ and set $f_\emptyset = 1$. If Γ is a simplicial complex and $0 \le r \le k \le d$, define

$$\begin{aligned}
\Lambda_k[r](\Gamma) &= \tilde{f}_{[r]} \wedge \Lambda_{k-r}[\Gamma] = \tilde{f}_1 \wedge \cdots \wedge \tilde{f}_r \wedge \Lambda_{k-r}[\Gamma], \\
Z^{k-1}[r](\Gamma) &= \{x \in \Lambda_k[r](\Gamma) \colon \tilde{f}_{r+1} \wedge x = 0\}, \\
B^{k-1}[r](\Gamma) &= \begin{cases} \tilde{f}_{r+1} \wedge \Lambda_{k-1}[r](\Gamma) & \text{if } r < k \\ 0 & \text{if } r = k \end{cases}, \\
H^k[r](\Gamma) &= Z^k[r](\Gamma)/B^k[r](\Gamma).
\end{aligned}$$

Notice that $B^{k-1}[r](\Gamma) = \Lambda_k[r+1](\Gamma)$. The $H^k[r](\Gamma)$ are the **rth iterated cohomology groups** of Γ. Define the **rth iterated Betti numbers** by

$$\beta^k[r](\Gamma) = \dim H^k[r](\Gamma).$$

□

The $r = 0$ case is just ordinary (reduced) cohomology. The iterated Betti numbers can also be interpreted as

$$\beta_k[r](\Gamma) = \beta_k(\Lambda[r](\Gamma)).$$

Remark: Kalai [K2] defined a similar, but not identical, iterated cohomology. We distinguish between the two definitions by putting bars over his. First let $F_r = \operatorname{span}\{\tilde{f}_1, \ldots, \tilde{f}_r\}$. Then define

$$\begin{aligned}
\bar{Z}^k[r](\Gamma) &= \{x \in \Lambda_k[\Gamma] \colon \tilde{f}_1 \wedge \cdots \wedge \tilde{f}_r \wedge x = 0\}, \\
\bar{B}^k[r](\Gamma) &= \operatorname{span}\{F_r \wedge \Lambda_{k-1}[\Gamma]\},
\end{aligned}$$

and define $\bar{H}^k[r](\Gamma)$ and $\bar{\beta}^k[r](\Gamma)$ in terms of $\bar{B}^k[r](\Gamma)$ and $\bar{Z}^k[r](\Gamma)$ as above. We show below (using a combinatorial characterization of iterated cohomology) that the two iterated cohomology definitions are different. □

Algebraic shifting.

Algebraic shifting (introduced by Kalai [K1]; see also [BK, K2]) transforms a simplicial complex Γ into a shifted simplicial complex $\Delta(\Gamma)$ with the same f-vector, and also preserves many algebraic properties of the original complex. Algebraic shifting provides much of the motivation for iterated homology.

For a set of integers F, define

$$\mathrm{init}(F) = \min\{r \colon r \notin F\} - 1,$$

which measures the largest "initial segment" in F, and is 0 if there is no initial segment (*i.e.*, $1 \notin F$).

Theorem 4 *Let Γ be a simplicial complex. Then*

$$\beta^{k-1}[r](\Gamma) = \#\{\text{facets } F \in \Delta(\Gamma) \colon |F| = k,\ \mathrm{init}(F) = r\},$$

where $\Delta(\Gamma)$ is the result of applying algebraic shifting to Γ.

Proof: Very similar to the proof of the $r = 0$ case by Björner and Kalai, Claim 2 in [BK, Theorem 3.1]. □

Corollary 5 *Let Γ be a simplicial complex. Then*

$$\beta^{k-1}[r](\Gamma) = \beta^{k-1}[r](\Delta(\Gamma)).$$

where $\Delta(\Gamma)$ is the result of applying algebraic shifting to Γ.

Proof: Using the stability of algebraic shifting, *i.e.*, that $\Delta(\Delta(\Gamma)) = \Delta(\Gamma)$ [BK, p. 291], and Theorem 4 twice,

$$\begin{aligned}\beta^{k-1}[r](\Gamma) &= \#\{\text{facets } F \in \Delta(\Gamma) \colon |F| = k,\ \mathrm{init}(F) = r\} \\ &= \#\{\text{facets } F \in \Delta(\Delta(\Gamma)) \colon |F| = k, \mathrm{init}(F) = r\} \\ &= \beta^{k-1}[r](\Delta(\Gamma)).\end{aligned}$$

□

Corollary 6 *Let Γ be a simplicial complex, let $\Delta = \Delta(\Gamma)$ be the result of applying algebraic shifting to Γ, and define $B_1, \ldots, B_d$ as in Corollary 3. Then*

$$\beta^{k+r-1}[r](\Gamma) = f_{k-1}(B_r) = \beta^{k-1}(\Delta^{(r)}).$$

Proof: It is easy to see by equation (2) that

$$f_{k-1}(B_r) = \#\{\text{facets } F \in \Delta(\Gamma) \colon |F| = k + r, \mathrm{init}(F) = r\}.$$

Then apply Theorem 4 and equation (3). □

Remark: We can now show that Kalai's iterated cohomology is different from the one presented here. In [K2], Kalai gives the formula

$$\bar{\beta}^{k-1}[r](\Gamma) = \#\{F \in \Delta(\Gamma) \colon |F| = k,\ F \cap [r] = \emptyset,\ F \cup [r] \notin \Delta(\Gamma)\}.$$

To see that the definitions are essentially different, consider the (shifted) simplicial complex on n vertices whose maximal faces are

$$\{1,2\}, \{1,3\}, \{1,4\}, \dots, \{1,n\}.$$

In Kalai's theory, each face $\{m\}$ contributes $m-2$ ($m \geq 3$) to the iterated Betti numbers (with $r = 2, \dots, m-1$), for a grand total of $\binom{n-1}{2}$. However, our iterated Betti numbers in this case only have a grand total of $n-1$, *one* each for each of the $n-1$ maximal faces.

On the other hand, it is not hard to check that if a simplicial complex is "s-fold acyclic" (*i.e.*, all the rth iterated homology groups vanish for $r = 0, \dots, s$) under either definition, then it is "s-fold acyclic" under the other definition (both conditions correspond to the algebraically shifted complex Δ being an "s-fold cone", *i.e.*, $\Delta = [s] * \Delta'$ for some Δ'). □

Shelling.

A simplicial complex Γ is **shellable** [BW] if there is a map R: {facets of Γ} $\to \Gamma$ called the **restriction map** and an ordering (called the **shelling ordering**) of the facets $F_1, \dots, F_t$ of Γ such that:

$$\Gamma = \bigcup_{1 \leq i \leq t} [R(F_i), F_i]; \text{ and} \tag{4}$$

$$R(F_i) \subseteq F_j \Rightarrow i \leq j. \tag{5}$$

Note that condition (5) implies that the union in equation (4) is disjoint. The **restriction numbers** are defined by

$$h_{ij}(\Gamma) = \{\text{facets } F\colon |F| = i,\ |R(F)| = j\}$$

and are independent of the shelling order.

Alternatively, a simplicial complex Δ is shellable if it can be constructed by adding one facet at a time, so that as each facet F is added, the restriction face $R(F)$ is the unique new minimal face added. Traditionally, Δ also had to be pure, but in [BW], Bjorner and Wachs generalize the definition of shelling by dropping the assumption of purity, and prove basic results about general (*i.e.*, non-pure) shellability. For instance, if Δ is shellable, then

$$\beta_{k-1}(\Delta) = h_{kk}(\Delta), \tag{6}$$

generalizing earlier results about homology of pure shellable complexes. Iterated homology provides an algebraic interpretation of the non-diagonal restriction numbers (*i.e.*, $h_{ij}(\Delta)$, where $i \neq j$).

Lemma 7 (Björner-Wachs [BW, Corollary 11.4]) *If Δ is a shifted simplicial complex, it is shellable with restriction numbers given by*

$$h_{ij}(\Delta) = \#\{\text{facets } F \in \Delta\colon |F| = i,\ \text{init}(F) = i - j\}.$$

□

Theorem 8 *Let Γ be simplicial complex. Then*

$$\beta^{k-1}[r](\Gamma) = h_{k,k-r}(\Delta(\Gamma))$$

where $\Delta(\Gamma)$ is the result of applying algebraic shifting to Γ.

Proof: Apply Theorem 4 and Lemma 7. □

A different kind of decomposition, collapsing [K2, §4], is needed to fully generalize equation (6). A face R of a simplicial complex Γ is **free** if it is included in a unique facet F (the empty set is a free face of Γ if Γ is a simplex). A **collapse step** is the deletion from Γ of a free face and all faces containing it (*i.e.*, the deletion of the interval $[R, F]$, a Boolean algebra). A **collapsing sequence** is a sequence of collapse steps that reduce Γ to the empty simplicial complex. If the maximal face of each collapse step is a facet of Γ, then the collapsing sequence corresponds to a shelling of Γ, with the intervals of the collapse steps corresponding to the shelling decomposition in equation (4).

Theorem 9 *If Γ is a shellable simplicial complex, then*

$$h_{ij}(\Gamma) = h_{ij}(\Delta(\Gamma))$$

where $\Delta(\Gamma)$ is the result of applying algebraic shifting to Γ.

Proof: (sketch) Algebraic shifting preserves collapsibility [K2, Theorem 4.2]; applying this one collapse at a time to Γ, using the shelling of Γ, we see that we can collapse $\Delta(\Gamma)$ into Boolean algebras that have the same dimensions as the shelling decomposition of Γ. Now, by [BW, Theorem 2.6], we may assume that the shelling order of the facets of Γ is in order of decreasing dimension (*i.e.*, if F_i and F_j are facets of Γ and $i < j$, then $\dim F_i \geq \dim F_j$), so the maximal face in each step of the collapse of $\Delta(\Gamma)$ is a facet in $\Delta(\Gamma)$. Thus, the collapsing sequence of $\Delta(\Gamma)$ corresponding to the shelling of Γ gives a shelling order of $\Delta(\Gamma)$ with the same restriction numbers as Γ. □

Corollary 10 *If Γ is a shellable simplicial complex, then*

$$\beta^{k-1}[r](\Gamma) = h_{k,k-r}(\Gamma).$$

Proof: By Theorem 8 and Theorem 9,

$$\beta^{k-1}[r](\Gamma) = h_{k,k-r}(\Delta(\Gamma)) = h_{k,k-r}(\Gamma).$$

□

Corollary 10 is the desired generalization of equation (6), providing an algebraic interpretation of restriction numbers and a combinatorial interpretation of iterated Betti numbers of shellable complexes.

Depth.

A sequence $(x_1, \ldots, x_k)$ of elements of a ring R is a **regular sequence** on R if each x_i is not a zero divisor on the quotient $R/(x_1, \ldots, x_{i-1})$. The **depth** of a ring is the length of the longest regular sequence on R. The depth of a simplicial complex Δ is defined to be the depth of $K[\Delta]$, the face ring of Δ over K. Smith [Sm] and Munkres [Mu] have described the depth of Δ in terms of combinatorial and topological properties of Δ. In [Bj], Björner gives a description of the depth of a shellable complex Δ in terms of the shelling restriction numbers $h_{ij}(\Delta)$. We use this to describe depth in terms of iterated homology.

Theorem 11 *Let* Γ *be a simplicial complex; then* $\text{depth}(\Gamma) = k$ *if and only if:*

(a) $\beta^i[r](\Gamma) = 0$ *for* $i < k$*; and*

(b) $\beta^k[r](\Gamma) \neq 0$ *for some* r*.*

Proof: Using [Sm, Theorem 4.8] (see also Hibi [Hi]), we know that $\text{depth}(\Gamma) = k$ if and only if k is the largest integer such that the k-skeleton of Γ is Cohen-Macaulay. From [K2, Theorem 5.3], this is equivalent to k being the largest integer such that the k-skeleton of the shifted complex $\Delta(\Gamma)$ is pure. This means that all facets of $\Delta(\Gamma)$ have dimension at least k, and there exists a facet of dimension exactly k. Thus, in any shelling of $\Delta(\Gamma)$, we have $h_{ij} = 0$ whenever $i \leq k$, but $h_{k+1,j} \neq 0$ for some j. By Theorem 8, this is equivalent to $\beta^i[r](\Gamma) = 0$ if $i < k$, for any r, but $\beta^k[r](\Gamma) \neq 0$, for some r. □

Acknowledgements.

We were helped greatly by conversations with Anders Björner, Michelle Wachs, and Gil Kalai.

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