The cd-index of Eulerian cubical posets

RICHARD EHRENBORG* AND GÁBOR HETYEI†‡

Abstract

We show a cubical analogue of Stanley's theorem expressing the **cd**-index of an Eulerian simplicial poset in terms of its h-vector. Our result implies the **cd**-index conjecture for Gorenstein* cubical posets follows from Ron Adin's conjecture on the nonnegativity of his cubical h-vector for Cohen-Macaulay cubical posets. We show a cubical analogue of Stanley's conjecture about the connection between the **cd**-index of semisuspended simplicial shelling components and the reduced variation polynomials of certain subclasses of André permutations. The notion of signed André permutation used in this result is a common generalization of two earlier definitions of signed André-permutations.

Résumé

Nous démontrons un analogue cubique du théorème de Stanley, qui exprime l'index **cd** d'un ensemble partiellement ordonné simplicial eulérien en fonction de son vecteur h. Notre résultat implique que, si le vecteur h cubique de Ron Adin est positif pour les complexes cubiques de Cohen-Macaulay, alors la conjecture de positivité de l'index **cd** des ensembles partiellement ordonnés ayant la propriété Gorenstein* est vraie dans le cas cubique. Nous montrons un analogue cubique d'une conjecture de Stanley sur le rapport entre l'index **cd** de la semisuspension des composants d'effeuillage d'un complexe simplicial, et les polynômes de variation réduite de certaines sous-classes des permutations d'André. La notion de permutation d'André signée utilisée dans ce résultat est une généralisation commune de deux définitions antérieures des permutations d'André signées.

Introduction

In [14] Stanley expressed the **cd**-index of an Eulerian simplicial poset in terms of its h-vector. He conjectured that the **cd**-polynomials occurring in his formula are the **cd**-variation polynomials of certain classes of André permutations. (This conjecture was proved by G. Hetyei in [9].) In this paper we generalize Stanley's theorem and conjecture to cubical posets.

In Section 1 we recall the definition and fundamental properties of the **cd**-index of a graded poset, with a special focus on C-shellable CW-spheres. We draw attention to Stanley's [14, Lemma 2.1] which allows us to greatly simplify the calculation of the change in the **cd**-index of a CW-sphere when we subdivide a facet into two facets.

*Partially supported by CRM.

†Supported by the UQAM Foundation.

‡Both authors are postdoctoral fellows at LACIM, Université du Québec à Montréal, C.P. 8888, succ. Centre-Ville, Montréal (Québec) Canada, H3C 3P8.

In Section 2 we specialize the results of Section 1 to shellable cubical complexes. We indicate the reason why in this special case there is no difference between the usual notion of shelling (which we call C-shelling, following Stanley in [14]) and the notion of S-shelling or spherical shelling (also introduced in [14]). Using a consequence of [14, Lemma 2.1], we obtain a formula for the **cd**-index of a shellable cubical sphere.

In Section 3 we use [14, Lemma 2.1] to establish linear relations between the **cd**-indices of semisuspended cubical shelling components. These relations allow us to express the **cd**-index of a shellable cubical sphere in terms of the "long h-vector" suggested by Ron Adin in [1]. As in Stanley's [14, Theorem 3.1], every h_i is multiplied with a **cd**-polynomial with nonnegative coefficients. Ron Adin has asked whether the long h-vector of a Cohen-Macaulay cubical complex is nonnegative. An affirmative answer to his question would imply a new special case of Stanley's [14, Conjecture 2.1] about the nonnegativity of the **cd**-index of Gorenstein* posets.

There are two other h-vectors defined for cubical complexes which were studied before: the toric h-vector defined by Stanley for Eulerian posets [13] and the h-vector of the Stanley ring of cubical complexes introduced in [8]. Unfortunately none of them have been useful to prove nontrivial inequalities about the f-vector of cubical complexes. Our result indicates that Ron Adin's cubical h-vector might be a good candidate for this purpose.

In Section 4 we give recursion formulas for the **cd**-indices of both semisuspended simplicial and cubical shelling components. These formulas are useful in proving the results of Section 5.

Finally, in Section 5 we express the **cd**-index of the semisuspended cubical shelling components in terms of reduced variation polynomials of signed augmented André* permutations. Unsigned André* permutations may be obtained from André permutations by reversing the linear order of the letters and reading the permutation backwards. This small twist allows one to handle the signed generalizations more easily, as was first observed by Ehrenborg and Readdy in [6]. Our signed André* permutations generalize both the signed André permutations introduced by Purtill in [11] and studied in greater generality by Ehrenborg and Readdy in [6], and the signed André-permutations introduced by Hetyei in [9]. We prove not only a signed analogue of Stanley's [14, Conjecture 3.1], but as an auxiliary result we also obtain a new description of the **cd**-index of semisuspended simplicial shelling components.

1 On the cd-index and shellings

Let P be a graded poset of rank $n+1$, that is, P is ranked with rank function ρ, and has minimum element $\hat{0}$ and maximum element $\hat{1}$. The *flag f-vector* $(\alpha(S) \ :\ S \subseteq \{1,2,\ldots,n\})$ is defined by

$$\alpha(S) \stackrel{\text{def}}{=} \left|\left\{\{\hat{0} < x_1 < \ldots < x_k < \hat{1}\} \subseteq P \ :\ \{\rho(x_1),\ldots,\rho(x_k)\} = S\right\}\right|,$$

and the *flag h-vector* (also called the *beta-invariant*) is defined by the equation

$$\beta(S) \stackrel{\text{def}}{=} \sum_{T\subseteq S} (-1)^{|S\setminus T|} \cdot \alpha(T).$$

The **ab**-*index* $\Psi_P(\mathbf{a},\mathbf{b}) = \Psi(P)$ of the poset P is the following polynomial in the noncommuting variables $\mathbf{a}$ and $\mathbf{b}$:

$$\Psi_P(\mathbf{a},\mathbf{b}) = \sum_{S\subseteq\{1,2,\ldots,n\}} \beta(S)\cdot u_S, \tag{1}$$

where u_S is the monomial $u_1\cdots u_n$ satisfying $u_i = \mathbf{b}$ if $i\in S$ and $u_i = \mathbf{a}$ otherwise.

A poset P is *Eulerian* if it is graded and the Möbius function of any interval $[x,y]$ is $\mu(x,y) = (-1)^{\rho(y)-\rho(x)}$. Fine observed (see [3]) that the **ab**-index of an Eulerian poset can be written uniquely as a non-commutative polynomial in the variables $\mathbf{c} \stackrel{\text{def}}{=} \mathbf{a}+\mathbf{b}$ and $\mathbf{d} \stackrel{\text{def}}{=} \mathbf{ab}+\mathbf{ba}$. For an inductive proof of this fact see Stanley [14]. In this case we call $\Psi_P(\mathbf{a},\mathbf{b}) = \Phi_P(\mathbf{c},\mathbf{d}) = \Phi(P)$.

Stanley [14] introduced the polynomial $\Upsilon_P(\mathbf{a},\mathbf{b}) \stackrel{\text{def}}{=} \sum_{S\subseteq\{1,\ldots,n\}} \alpha(S)\cdot u_S$, and he observed that

$$\Psi_P(\mathbf{a},\mathbf{b}) = \Upsilon_P(\mathbf{a}-\mathbf{b},\mathbf{b}) \qquad \text{and} \qquad \Upsilon_P(\mathbf{a},\mathbf{b}) = \Psi_P(\mathbf{a}+\mathbf{b},\mathbf{b}). \tag{2}$$

We associate to every chain $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ in P a *weight* $w(c) = z_1\cdots z_n$, where $z_i = \mathbf{b}$ if $i \in \{\rho(x_1),\ldots,\rho(x_k)\}$ and $z_i = \mathbf{a}-\mathbf{b}$ otherwise. By the first equation in (2), the **ab**-index $\Psi_P(\mathbf{a},\mathbf{b})$ is the sum of the weights of all chains in P,

$$\Psi(P) = \Psi_P(\mathbf{a},\mathbf{b}) = \sum_c w(c), \tag{3}$$

where c ranges over all chains $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ in the poset P.

A poset P is called *near-Eulerian* if it may be obtained from an Eulerian poset $\tilde{\Sigma}P$ by removing one coatom. The poset $\tilde{\Sigma}P$ may be uniquely reconstructed from P by adding a coatom x which covers all $y \in P$ for which $[y,\hat{1}]$ is the three element chain. Following Stanley [14] we call $\tilde{\Sigma}P$ the *semisuspension* of P. In this paper we focus on *cubical posets*, with an eye on *simplicial* posets.

Definition 1.1 *A* simplicial poset *is a graded poset such that for all $\hat{0} \le x \le y < \hat{1}$ the interval $[x,y]$ is a boolean algebra. A* cubical poset *is a graded poset such that for all $x < \hat{1}$ the interval $[\hat{0},x]$ is the face lattice of a cube, and for all $\hat{0} < x \le y < \hat{1}$ the interval $[x,y]$ is a boolean algebra.*

When P is also a meet-semilattice then $P - \{\hat{1}\}$ is the face poset of a *simplicial or cubical complex*, with a maximal element $\hat{1}$ added. They may both be realized as regular CW-complexes. Following Björner [5] we call a poset P with $\hat{0}$ a *CW-poset* when for all $x > \hat{0}$ in P the geometric realization $|(\hat{0},x)|$ of the open interval $(\hat{0},x)$ is homeomorphic to a sphere. By [5], P is a CW-poset if and only if it is the face poset $P(\Omega)$ of a regular CW-complex Ω. As Stanley does in [14], we use $P_1(\Omega)$ to denote the face poset $P(\Omega)$ adjoined with $\hat{1}$. If Ω is homeomorphic to a sphere then $P_1(\Omega)$ is Eulerian. If Ω is homeomorphic to a ball then $P_1(\Omega)$ is near-Eulerian. For a regular CW-ball Ω the semisuspension $\tilde{\Sigma}P_1(\Omega)$ of $P_1(\Omega)$ is of the form $P_1(\tilde{\Sigma}\Omega)$ where $\tilde{\Sigma}\Omega$ is the regular CW-sphere obtained from Ω by adding an extra facet, the boundary of which is identified with the boundary $\partial\Omega$ of Ω.

Stanley observed the following, see [14, Lemma 2.1]. Let Ω be an n-dimensional CW-sphere, and σ an (open) facet of Ω. Let Ω' be obtained from Ω by subdividing $\overline{\sigma}$ into a regular CW-complex with two faces σ_1 and σ_2 , such that $\partial\sigma$ remains the same and $\overline{\sigma_1}\cap\overline{\sigma_2}$ is a regular

$(n-1)$-dimensional CW-ball Γ. Then we have

$$\Phi(P_1(\Omega')) - \Phi(P_1(\Omega)) = \Phi(P_1(\tilde{\Sigma}\Gamma)) \cdot \mathbf{c} - \Phi(P_1(\partial\Gamma)) \cdot (\mathbf{c}^2 - \mathbf{d}). \tag{4}$$

In particular, if we take another n-dimensional CW-sphere and subdivide it isomorphically, the **cd**-index changes by the same amount.

Lemma 1.2 *Let Ω_1 and Ω_2 be n-dimensional CW-spheres. Assume that we subdivide a facet σ^i of Ω_i $(i = 1, 2)$ into two facets σ_1^i and σ_2^i such that $\partial(\sigma_i)$ is unchanged and $\overline{\sigma_1^i} \cap \overline{\sigma_2^i}$ is a regular $(n-1)$-dimensional CW-ball Γ_i. Then $P_1(\Gamma_1) = P_1(\Gamma_2)$ and $P_1(\partial\Gamma_1) = P_1(\partial\Gamma_2)$ imply*

$$\Phi(P_1(\Omega_1')) - \Phi(P_1(\Omega_1)) = \Phi(P_1(\Omega_2')) - \Phi(P_1(\Omega_2)).$$

Fine [3, Conjecture 3] conjectured that the **cd**-index of the face lattice of a convex polytope is nonnegative. Stanley proved this in greater generality [14, Theorem 2.2] for *S-shellable*, or spherically shellable, regular CW-spheres.

Definition 1.3 *Let Ω be an n-dimensional Eulerian regular CW-complex. We call Ω or $P_1(\Omega)$* S-shellable *(or spherically shellable) if either $\Omega = \{\emptyset\}$ (and so $P_1(\Omega)$ is a two-element chain with* **cd***-index 1), or else we can linearly order the facets (open n-cells) of Ω $F_1, F_2, \ldots, F_r$, such that for all $1 \leq i \leq r$ the following two conditions hold (both $^-$ and* cl *denote closure operation).*

(S-a) *$\partial\overline{F_1}$ is S-shellable of dimension $n-1$.*

(S-b) *For $2 \leq i \leq r-1$, let $\Gamma_i \stackrel{\text{def}}{=} \mathrm{cl}\,[\partial\overline{F_i} - ((\overline{F_1} \cup \cdots \cup \overline{F_{i-1}}) \cap \overline{F_i})]$. Then $P_1(\Gamma_i)$ is near-Eulerian of dimension $n-1$, and the semisuspension $\tilde{\Sigma}\Gamma_i$ is S-shellable, with the first facet of the shelling being the facet $\tau = \tau_i$ adjoined to Γ_i to obtain $\tilde{\Sigma}\Gamma_i$.*

As a consequence of Lemma 1.2, the **cd**-index of an S-shellable regular CW-sphere may be computed from just knowing $\partial\overline{F_1}$ and the complexes Γ_i.

The definition of S-shellability is different from the usual notion of shellability (given e.g. in [5, Definition 4.1]), which is called C-shellability in [14]. It is trivially true, however, that the two notions of shellability coincide for the geometric realizations of simplicial spheres. We show in Section 2 that the same holds for cubical complexes.

2 Shellable cubical complexes

Let C^n denote the complex of faces of an n-cube with vertex set $V(C^n)$. We may geometrically realize any n-cube in $\mathbf{R}^n$ as the convex hull of the vertex set $\{0,1\}^n$. We call such a realization $\phi : V(C^n) \longrightarrow \mathbf{R}^n$ of a cube a *standard geometric realization.* By abuse of notation we also denote by ϕ the map associating the convex hull of $\{\phi(v) \,:\, v \in \sigma\}$ to a face $\sigma \in C^n$. Using ϕ we may define the boundary ∂C^n as the inverse image under ϕ of the boundary of $[0,1]^n$.

Following Metropolis and Rota in [10], given a standard geometric realization ϕ, we encode the nonempty faces σ of our n-cube with vectors $(u_1, u_2, \ldots, u_n) \in \{0, 1, *\}^n$ such that for every $i \in \{1, 2, \ldots, n\}$ we set $u_i = 0$ or 1 respectively if the i-th coordinate of every element of $\phi(\sigma)$ is 0 or 1 respectively and $u_i = *$ otherwise. Using this coding, the facets of ∂C^n correspond to those vectors $(u_1, \ldots, u_n)$ for which exactly one u_i is not the $*$-sign.

Definition 2.1 *Let A_i^0 respectively A_i^1 denote the facet $(u_1, u_2, \ldots, u_n)$ with $u_i = 0$ respectively, $u_i = 1$ and $u_k = *$ for $k \neq i$. Let $\{F_1, \ldots, F_k\}$ be a collection of facets of $\partial(C^n)$. Let r be the number of indices i such that $|\{A_i^0, A_i^1\} \cap \{F_1, \ldots, F_k\}| = 1$, and let s be the number of indices j such that such that $\{A_j^0, A_j^1\} \subseteq \{F_1, \ldots, F_k\}$. We call (r, s) the* type *of $\{F_1, \ldots, F_k\}$.*

Clearly the type does not depend on the choice of the standard geometric realization. The following observation is originally due to Ron Adin and Clara Chan.

Lemma 2.2 *Let $\{F_1, \ldots, F_k\}$ be a collection of facets of ∂C^n and ϕ a standard geometric realization of C^n. Then $\phi(F_1) \cup \phi(F_2) \cup \cdots \cup \phi(F_k)$ is an $(n-1)$-sphere if and only if it has type $(0, n)$ and it is an $(n-1)$-ball if and only if its type (r, s) satisfies $r > 0$.*

It is easy to see by induction on the dimension that there exists a C–shelling of the boundary of $[0, 1]^n$ starting with the facets $\{\phi(F_1), \phi(F_2), \ldots, \phi(F_k)\}$ if and only if $\phi(F_1) \cup \phi(F_2) \cup \cdots \cup \phi(F_k)$ is an $(n-1)$-sphere or an $(n-1)$-ball. Thus we may rephrase the definition of C–shellability for finite cubical complexes in a purely combinatorial way as follows.

Lemma 2.3 *Let C be an n-dimensional pure cubical complex (i.e. a cubical complex with equidimensional maximal faces). An enumeration $F_1, \ldots, F_m$ of the facets of C induces a C-shelling of its geometric realization if and only if for every $k \in \{2, \ldots, m\}$ the following two conditions hold:*

(i) The set of faces contained in $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ is a pure complex of dimension $(n-1)$.

(ii) The collection of the facets of ∂F_k contained in $F_1 \cup \cdots \cup F_{k-1}$ has type (r, s) with $r > 0$ or $s = n - 1$.

Definition 2.4 *We call the cubical complex of faces contained in $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ the kth* shelling component, *and the type (r, s) associated to it the* type *of the shelling component. The empty cubical complex is also a shelling component of type $(0, 0)$.*

We may also show by induction on the dimension that the boundary of $[0, 1]^n$ has an S–shelling and that, given a collection of facets $F_1, \ldots, F_k$ of ∂C^n of type (r, s) with $r > 0$, the semisuspension of $\phi(F_1) \cup \cdots \cup \phi(F_k)$ has an S–shelling starting with the facet which was added to obtain $\tilde{\Sigma}(\phi(F_1) \cup \cdots \cup \phi(F_k))$.

Corollary 2.5 *Let Ω be the geometric realization of a cubical complex C as a regular CW-complex. Assume that Ω is an n-sphere. Then an enumeration $F_1, \ldots, F_m$ of the facets of C induces a C-shelling if and only if it induces an S–shelling.*

In fact, in both cases we have the same types of allowed shelling components: for $k \in \{2, \ldots, m-1\}$ the type of the kth shelling component must be (r, s) with $r > 0$ and the last shelling component must have type $(n, 0)$. Thus in the case of cubical spheres we may simply speak about shellings, without any reference to C-shellings and S-shellings.

Definition 2.6 *Given a shellable cubical n-sphere C and a shelling $F_1, \ldots, F_m$ of it, we denote the number of shelling components of type (r, s) by $g_{r,s}$. (In particular we have $g_{0,0} = g_{0,n} = 1$.) We call the vector $(\ldots, g_{r,s}, \ldots)$ the g-vector of the shelling.*

Similar to the way Stanley treated the simplicial case in [14], we may express the **cd**-index of a shellable cubical sphere in terms of the numbers $g_{i,j}$, and the **cd**-indices of (semisuspended) shelling components of one dimension higher. For this purpose we introduce the following notation.

Definition 2.7 *Let B_n be the boolean algebra and C_n the cubical lattice of rank n. That is, B_n is the face lattice of the $(n-1)$-dimensional simplex Δ^n while C_n is that of the cube C^n. We denote $\Phi(B_n)$ and $\Phi(C_n)$ by U_n and V_n respectively. In particular, for $n = 1$ we have $U_1 = V_1 = 1$.*

Given a collection $F_1, \ldots, F_k$ of $k \leq n-1$ facets of $\partial\Delta^{n-1}$, we denote the semisuspension of the poset $[\hat{0}, F_1] \cup \cdots \cup [\hat{0}, F_k] \cup \{\hat{1}\} \subset B_n$ by $B_{n,k}$ and its **cd**-index by $U_{n,k}$. Given a collection $F_1, \ldots, F_{r+2s}$ of facets of ∂C^{n-1} of type (r, s), where r is positive, we denote the semisuspension $[\hat{0}, F_1] \cup \cdots \cup [\hat{0}, F_{r+2s}] \cup \{\hat{1}\} \subset C_n$ by $C_{n,r,s}$ and its **cd**-index by $V_{n,r,s}$.

Proposition 2.8 *Let C be an $(n-1)$-dimensional shellable cubical sphere which has a shelling with g-vector $(\ldots, g_{r,s}, \ldots)$. Then the **cd**-index of $P_1(C)$ is given by*

$$\Phi(P_1(C)) = V_{n+1,1,0} + \sum_{\substack{r,s \\ r>0}} g_{r,s} \cdot (V_{n+1,r+1,s} - V_{n+1,r,s}). \tag{5}$$

3 Ron Adin's h-vector and the cd-index

Definition 3.1 *Let P be a graded simplicial or cubical poset of rank $n+1$. For $i = -1, 0, \ldots, n-1$ we denote the number of elements of rank $i+1$ in P by f_i. The vector $(f_{-1}, f_0, \ldots, f_n)$ is called the f-vector of P. When P is simplicial we define its h-vector by*

$$\sum_{i=0}^{n} h_i \cdot x^{n-i} \stackrel{\text{def}}{=} \sum_{j=0}^{n} f_{j-1} \cdot (x-1)^{n-j}.$$

From now on let C be an $(n-1)$-dimensional cubical sphere. It is well known that for the face poset of an $(n-1)$-dimensional C–shellable simplicial complex, h_i is the number of facets F_j in any shelling $F_1, \ldots, F_m$, for which $F_j \cap (F_1 \cup \cdots \cup F_{j-1})$ is a collection of i facets of ∂F_j. In this sense the g-vector of a shelling of a shellable C is an analogue of the h-vector. We have, however, only the following straightforward *nonbijective* relationship between the f-vector and the g-vector.

$$\sum_{k=0}^{n-1} f_k \cdot x^k = \sum_{r,s} g_{r,s} \cdot x^s \cdot (x+1)^r \cdot (x+2)^{n-1-r-s}. \tag{6}$$

By Lemma 1.2 we have $V_{n,r+1,s} - V_{n,r,s} = V_{n,r,s+1} - V_{n,r+1,s}$ for $n > 1$, $r > 0$, $s \geq 0$ and $r + s \leq n - 1$. Repeated use of this equation allows us to obtain the following formula.

$$\Phi(P_1(\mathcal{C})) = V_{n+1,1,0} + \sum_{l=1}^{n-1} \left(\sum_{\substack{r,s \\ r \geq \max(1,l-s)}} \frac{g_{r,s}}{2^r} \cdot \binom{r-1}{l-1-s} \right) \cdot (V_{n+1,1,l} - V_{n+1,1,l-1}).$$

Keeping in mind Stanley's [14, Theorem 3.1], our last equality suggests to define $h_0 \stackrel{\text{def}}{=} 1$ and

$$h_l \stackrel{\text{def}}{=} \sum_{\substack{r,s \\ r \geq \max(1,l-s)}} \frac{g_{r,s}}{2^r} \cdot \binom{r-1}{l-1-s} \quad \text{for } 1 \leq l \leq n-1 \tag{7}$$

to be the first n entries of the cubical h-vector. This, by $g_{0,0} = g_{0,n-1} = 1$, is equivalent to

$$(1+x) \cdot \sum_{l=0}^{n-1} h_l \cdot x^l = 1 - x^n + x \cdot \sum_{r,s} \frac{g_{r,s}}{2^r} \cdot (1+x)^r \cdot x^s.$$

Proposition 3.2 *For a shellable $\mathcal{C}$ we have $\sum_{r,s} \frac{g_{r,s}}{2^r} \cdot (1+x)^r \cdot x^s = \sum_{k=0}^{n-1} f_k \cdot x^k \cdot \left(\frac{1-x}{2}\right)^{n-1-k}$, and so the expression (7) is independent of the choice of the shelling.*

Definition 3.3 *Let P be an Eulerian cubical poset of rank $n+1$, with f-vector $(f_{-1}, f_0, \dots, f_n)$. We define the h-*vector *of P by the following polynomial equation.*

$$\sum_{l=0}^{n} h_l \cdot x^l \stackrel{\text{def}}{=} \frac{1 + x^{n+1} + \sum_{k=0}^{n-1} f_k \cdot x^{k+1} \left(\frac{1-x}{2}\right)^{n-1-k}}{1+x}$$

The right hand side is a polynomial because of the Eulerian equation $\sum_{j=-1}^{n} f_j \cdot (-1)^j = 0$. The same equation allows us to show that for cubical $(n-1)$-spheres, this h-vector is identical with the 2^{-n+1}-multiple of the *"long h-vector"* suggested by Ron Adin [1] for cubical complexes. By Proposition 3.2, for a shellable $\mathcal{C}$ the h_l's given by this definition satisfy $h_0 = 1$ and equation (7). Using this h-vector, we have the following cubical analogue of Stanley's [14, Theorem 3.1].

Theorem 3.4 *For an Eulerian cubical poset P of rank $n+1$, with h-vector $(h_0, h_1, \dots, h_n)$,*

$$\Phi(P) = h_0 \cdot V_{n+1,1,0} + \sum_{l=1}^{n-1} h_l \cdot (V_{n+1,1,l} - V_{n+1,l-1}) \quad \textit{holds.}$$

By the proof of Stanley's [14, Theorem 2.2], the differences $V_{n+1,1,l} - V_{n+1,1,l-1}$ have nonnegative coefficients. Hence the nonnegativity of the h-vector of an Eulerian cubical poset P implies the nonnegativity of $\Phi(P)$. Ron Adin asked whether the "long h-vector" of a Cohen-Macaulay cubical complex is nonnegative ([1, Question 1]). Because of the analogy to the simplicial case we state Conjecture 3.5. This conjecture, if true, implies the truth of [14, Conjecture 2.1] for cubical posets.

Conjecture 3.5 *The h-vector of every Gorenstein* cubical poset is nonnegative.*

4 The cd-index of semisuspended shelling components

We wish to remind the reader of the following formulas (cf. [6, Section 3, equation (1) and [Section 5, equation (3)] or [9, equations (5) and (16)].)

Proposition 4.1 *We have for $n \geq 1$,*

$$U_{n+2} = \sum_{i=1}^{n} \binom{n}{i} \cdot U_i \cdot \mathbf{d} \cdot U_{n+1-i} + \mathbf{c} \cdot U_{n+1} \quad \textit{and} \quad V_{n+2} = \sum_{i=0}^{n-1} \binom{n}{i} \cdot 2^{n-i} \cdot V_{i+1} \cdot \mathbf{d} \cdot U_{n-i} + V_{n+1} \cdot \mathbf{c}.$$

Proposition 4.1 gives the recursion formula for $U_{n,n-1} = U_n$ and $V_{n,1,n-1} = V_n$. In the following theorems these special cases will not be covered.

Theorem 4.2 *For $2 \leq k \leq n$ we have*

$$\begin{aligned} U_{n+2,k} &= \sum_{\substack{i \leq k-2 \\ j \leq n-k}} \binom{k-1}{i} \binom{n-k+1}{j} \cdot U_{i+j+1} \cdot \mathbf{d} \cdot U_{n-i-j,k-i-1} \\ &+ \sum_{i \leq k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{d} \cdot U_{k-i+1} + U_{n+1} \cdot \mathbf{c}. \end{aligned}$$

Proof: *(Sketch)* We calculate the total weight of the chains in $B_{n+2,k} \setminus \{\hat{0}, \hat{1}\}$. We assume that $B_{n+2,k}$ was obtained by adding an extra coatom E to the poset $\bigcup_{i=1}^{k} [\hat{0}, \{1, 2, \ldots, n+2\} \setminus \{i\}] \cup \{\hat{1}\}$.

The total weight of all chains c of which every element is either E or a set not containing 1 is

$$U_{n+1} \cdot \mathbf{c} - \sum_{i=0}^{k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{k-i-2} \cdot \mathbf{b}. \tag{8}$$

From now on we may assume that every chain considered contains a set λ with $1 \in \lambda$. Let us compute first the total weight of all chains containing a set λ with $1 \in \lambda$ and $n+2 \notin \lambda$. Their total weight is

$$\begin{aligned} &\sum_{\substack{i \leq k-2 \\ j \leq n-k}} \binom{k-1}{i} \binom{n-k+1}{j} \cdot U_{i+j+1} \cdot \mathbf{b} \cdot \mathbf{a} \cdot U_{n-i-j,k-i-1} \\ &+ \sum_{i=0}^{k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{b} \cdot \mathbf{a} \cdot U_{k-i-1} + \sum_{i=0}^{k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{k-i-2} \cdot \mathbf{b}. \end{aligned} \tag{9}$$

For all remaining chains c the smallest set $\lambda \in c$ with $1 \in \lambda$ contains $n+2$. Their total weight is

$$\begin{aligned} &\sum_{\substack{i \leq k-2 \\ j \leq n-k}} \binom{k-1}{i} \binom{n-k+1}{j} \cdot U_{i+j+1} \cdot \mathbf{a} \cdot \mathbf{b} \cdot U_{n-i-j,k-i-1} \\ &+ \sum_{i=0}^{k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{a} \cdot \mathbf{b} \cdot U_{k-i-1}. \end{aligned} \tag{10}$$

Adding the weights (8), (9), and (10) we obtain the statement of the theorem. □

Theorem 4.3 *For* $1 \leq i$ *we have*

$$V_{n+2,i,j} = \sum_{\substack{i_0,i_1,i_*,j_*,k_* \\ i_0+j-j_*>0 \\ i_1+k-k_*>0}} \binom{i-1}{i_0\ i_1\ i_*}\binom{j}{j_*}\binom{k}{k_*} \cdot 2^{j-j_*+k-k_*} \cdot V_{i_*+j_*+k_*+1} \cdot \mathbf{d} \cdot U_{n-i_*-j_*-k_*,i_0+j-j_*}$$

$$+ \sum_{\substack{i_0,j_* \\ i_0+j-j_*>0}} \binom{i-1}{i_0}\binom{j}{j_*} \cdot 2^{j-j_*} \cdot V_{n+2-i_0+j_*-j} \cdot \mathbf{d} \cdot U_{i_0+j-j_*} + V_{n+1} \cdot \mathbf{c}.$$

where $k = n + 1 - i - j$.

The proof is similar to the one of Theorem 4.2 and is therefore omitted.

5 Augmented André* signed permutations

Let X be a finite (possibly empty) linearly ordered set with m elements and linear order Λ. A *permutation* on X is a list $(\tau_1, \ldots, \tau_m)$ such that every letter of X occurs exactly once. We say that $i \in \{2, \ldots, m\}$ is a *descent* of τ if we have $\tau_{i-1} > \tau_i$ (otherwise i is an *ascent*). The *descent set* $D_\Lambda(\tau)$ of τ is the set $D_\Lambda(\tau) = \{i : \tau_{i-1} > \tau_i\}$. We say that τ has a *double descent* if there is an index i, where $2 \leq i \leq m-1$, such that τ has a descent at the ith and $(i+1)$st positions. In other words, both i and $i+1$ belong to $D_\Lambda(\tau)$. Given a (possibly empty) subinterval $[i, j] \subseteq \{1, 2, \ldots, m\}$, we define the *restriction of* τ *to* $[i, j]$ to be the permutation $\tau\big|_{[i,j]} = (\tau_i, \tau_{i+1}, \ldots, \tau_j)$.

Definition 5.1 *Let* X *be a finite linearly ordered set with linear order* Λ. *A permutation* $\tau = (\tau_1, \ldots, \tau_m)$ *on* X *is an* André* permutation *if it satisfies the following:*

1. *The permutation* τ *has no double descents.*
2. *For all* $2 \leq i < j \leq m$, *if* $\tau_{i-1} = \max_\Lambda\{\tau_{i-1}, \tau_i, \tau_{j-1}, \tau_j\}$ *and* $\tau_j = \min_\Lambda\{\tau_{i-1}, \tau_i, \tau_{j-1}, \tau_j\}$, *then there exists a* k, *with* $i < k < j$, *such that* $\tau_{i-1} <_\Lambda \tau_k$.

*We call an André * permutation* augmented *if its first letter is* $\min_\Lambda X$. *We denote the set of augmented André * permutations by* $\mathcal{A}(X)$.

Observe that we obtain the usual definition of André permutations (as it is given in [7] or in [11]) if we read the permutations backwards and reverse the linear order. In analogy with [11, Corollary 5.6] we have the following recursive description of augmented André* permutations.

Proposition 5.2 *Let* X *be a finite set with linear order* Λ *and* $|X| = n$. *A permutation* $\tau = (\tau_1, \ldots, \tau_n)$ *on* X *is an augmented André * permutation if and only if for* $m \stackrel{\text{def}}{=} \tau^{-1}(\max_\Lambda X)$ *the permutations* $\tau\big|_{[1,m-1]}$ *and* $\tau\big|_{[m+1,n]}$ *are augmented André permutations and* $\tau_1 = \min_\Lambda X$.

Definition 5.3 *Let* N *be a subset of* $\mathbb{P}$ *of cardinality* n. *Define* $-N = \{-i : i \in N\}$. *A* (non-augmented) signed permutation σ *on the set* N *is a list of the form* $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ *such that for all* i, $\sigma_i \in N \cup -N$ *and* $(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_n|)$ *is a permutation on* N. *An* augmented signed permutation σ *on* N *is a list* $(0, \sigma_1, \sigma_2, \ldots, \sigma_n)$ *such that* $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ *is a signed permutation on* N. *We write* $\sigma_0 = 0$.

As in the signless case, we use the notation $\sigma\Big|_{[i,j]}$ to denote the restricted permutation $\sigma\Big|_{[i,j]} = (\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$.

Let Λ be a linear order on the set $N \cup \{0\} \cup -N$. The *descent set* of a signed permutation σ (augmented or non-augmented) with respect to Λ is the set $D_\Lambda(\sigma) = \{i \ : \ \sigma_{i-1} >_\Lambda \sigma_i\}$. Here $D_\Lambda(\sigma)$ is a subset of $\{1, 2, \ldots, n\}$ for augmented permutations, and it is a subset of $\{2, \ldots, n\}$ for non-augmented permutations. As before, we say that a signed permutation σ as a *double descent* if there is an i such that $2 \le i \le n-1$ in the nonaugmented case and $1 \le i \le n-1$ in the augmented case, and both i and $i+1$ are contained in the descent set $D_\Lambda(\sigma)$ of σ.

Assume from now on that $i >_\Lambda 0$ and $i >_\Lambda -i$ holds for all $i \in N$.

Definition 5.4 *Let N be a subset of the positive integers $\mathbf{P}$ of cardinality n. We say an augmented signed permutation $\sigma = (0 = \sigma_0, \sigma_1, \ldots, \sigma_n)$ on the set N is an* augmented André* signed permutation *if the following three conditions are satisfied:*

1. *The permutation σ has no double descents.*
2. *For all $1 \le i < j \le n$, if $\sigma_{i-1} = \max_\Lambda\{\sigma_{i-1}, \sigma_i, \sigma_{j-1}, \sigma_j\}$ and $\sigma_j = \min_\Lambda\{\sigma_{i-1}, \sigma_i, \sigma_{j-1}, \sigma_j\}$, then there exists a k, with $i < k < j$, such that $\sigma_{i-1} <_\Lambda \sigma_k$.*
3. *For $x = \max N$, there exists $1 \le m \le n$ such that $\sigma_m = x$ and that $\sigma\Big|_{[0,m-1]}$ is an augmented André * signed permutation on the set J, where $J = \{|\sigma_k| \ : \ 1 \le k \le m-1\}$.*

*The permutation (0) is allowed to be an augmented André * signed permutation on the set $N = \emptyset$.*

Observe that conditions 1 and 2 of Definition 5.4 are equivalent to the following:

1'. $(0 = \sigma_0, \sigma_1, \ldots, \sigma_n)$ is an André permutation on the set $\{0 = \sigma_0, \sigma_1, \ldots, \sigma_n\}$ linearly ordered by the restriction of Λ.*

A non-augmented signed permutation satisfying conditions 1 and 2 in Definition 5.4 is called a *non-augmented André * signed permutation.* We denote the set of all augmented André* signed permutations on the set N by $\mathcal{A}^\pm(N)$ and the set of all non-augmented André* signed permutations on the set N by $\mathcal{N}^\pm(N)$. Furthermore, we denote the set of those non-augmented André* signed permutations which begin with their smallest element (with respect to the linear order Λ) by $\mathcal{N}_0^\pm(N)$. That is, $\mathcal{N}_0^\pm(N) \stackrel{\text{def}}{=} \{(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{N}^\pm(N) \ : \ \sigma_1 = \min_\Lambda\{\sigma_1, \sigma_2, \ldots, \sigma_n\}\}$.

Examples

1. Let $N \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ and consider the linear order $-n <_\Lambda -n+1 <_\Lambda \cdots <_\Lambda -1 <_\Lambda 0 <_\Lambda 1 <_\Lambda \cdots <_\Lambda n-1 <_\Lambda n$ on $-N \cup \{0\} \cup N$. Then $\mathcal{A}^\pm(N)$, $\mathcal{N}^\pm(N)$, and $\mathcal{N}_0^\pm(N)$ are the same as the similarly denoted sets of augmented (respectively non-augmented) r-signed André-permutations studied in [6] for $\mathbf{r} = (2, 2, \ldots, 2)$. The set $\mathcal{A}^\pm(N)$ may be obtained from Purtill's set of augmented signed André permutations defined in [11] by reversing the permutations and replacing each entry k with $-k$.
2. Let $N \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ and consider the linear order $0 <_\Lambda -1 <_\Lambda 1 <_\Lambda -2 <_\Lambda 2 <_\Lambda \cdots <_\Lambda -n <_\Lambda n$ on $-N \cup \{0\} \cup N$. Then $\mathcal{A}^\pm(N)$ and $\mathcal{N}^\pm(N)$ may be obtained from the corresponding sets of augmented (respectively non-augmented) signed André permutations defined in [9] on the set $\{1, 2, \ldots, n+1\}$ by reading each permutation backwards, and replacing each letter k with $n+1-k$, while keeping its sign.

We define the *variation* $U(\pi)$ of a signed or unsigned permutation π as $U(\pi) = u_S$, where S is the descent set of π and u_S is the **ab**-word defined in Section 1. In the case when π contains no double descents (e.g., when π is a signed or unsigned, augmented or non-augmented André* permutation), the *reduced variation* of π, which we denote by $V(\pi)$, is formed by replacing each **ab** in $U(\pi)$ with **d** and then replacing each remaining letter by **c**. Given a set $\mathcal{P}$ of signed or unsigned permutations, we denote the sums $\sum_{\pi\in\mathcal{P}} U(\pi)$ and $\sum_{\pi\in\mathcal{P}} V(\pi)$ respectively by $U(\mathcal{P})$ and $V(\mathcal{P})$.

Note that the (reduced) variation of an unsigned André* permutation τ is the reverse of the (reduced) variation of the reversed permutation τ^{rev} which is an André permutation with respect to the reversed order. Hence we may reformulate Purtill's [11, Theorem 6.1] as follows.

Proposition 5.5 $U_n = V(\mathcal{A}(\{1,2,\ldots,n\}))$ *holds for all* $n \in \mathbf{P}$.

Corollary 5.6 *We have* $V\left(\mathcal{N}_0^{\pm}(N)\right) = 2^{|N|} \cdot U_n$.

The following description of the polynomials $U_{n,k}$ is analogous to Stanley's [14, Conjecture 3.1].

Theorem 5.7 *Let* $\mathcal{A}_{n,k}$ *denote the set* $\{\tau \in \mathcal{A}(\{0,1,\ldots,n-1\}) : \tau_n \in \{n-1,n-2,\ldots,n-k\}\}$. *Then we have* $U_{n,k} = V(\mathcal{A}_{n,k})$.

Proof: *(Sketch)* Let us denote $\mathcal{A}(\{1,2,\ldots,n\})$ by $\mathcal{A}_n$. Using Proposition 5.2 it is easy to show that $V(\mathcal{A}_{n+2,k})$ satisfies the recursion formula given in Theorem 4.2 for $2 \le k \le n$. After this we are done by induction, where our induction basis is formed by the following results:

- Proposition 5.5 which implies our statement for $U_{n,n-1} = U_n$ and $V(\mathcal{A}_{n,n-1}) = V(\mathcal{A}_n)$,
- The relations $U_{n+1,1} = U_n \cdot \mathbf{c}$ and $V(\mathcal{A}_{n+1,1}) = V(\mathcal{A}_{n,1}) \cdot \mathbf{c}$ which may be seen directly. □

Remark Observe that in terms of "usual" André permutations, Theorem 5.7 expresses the polynomials $U_{n,k}$ as the reduced variation of augmented André permutations *starting* with given letters, while Stanley's [14, Conjecture 3.1] ([9, Theorem 2]) partitions the augmented André permutations depending on their *second to last letter*.

Proposition 5.2 has the following signed analogue.

Proposition 5.8 *There exists a bijection between the two sets*

$$\mathcal{A}^{\pm}([n+1]) \quad \textit{and} \quad \mathcal{A}^{\pm}([n]) \ \dot{\cup} \ \dot{\bigcup_{\substack{I+J=[n]\\ I\neq\emptyset}}} \mathcal{A}^{\pm}(J) \times \mathcal{N}_0^{\pm}(I),$$

where all the unions are disjoint and $\times$ *is the Cartesian product.*

Theorem 5.9 *We have* $V_n = V(\mathcal{A}^{\pm}([n]))$.

Proof: *(Sketch)* It is enough to show that $V(\mathcal{A}^{\pm}([n]))$ satisfies the same recurrence as the one given for V_n in Proposition 4.1. This formula follows by the bijection given in Proposition 5.8. □

Finally, we describe the polynomials $V_{n,i,j}$ in terms of the reduced variation of signed augmented André * permutations.

Theorem 5.10 *Let $N \subset \mathbf{P}$ be an n-element set and Λ a linear order on $N \cup -N \cup \{0\}$ such that $0 <_\Lambda i$ and $-i <_\Lambda i$ for all $i \in N$. Assume that A and B are disjoint subsets of N such that $A \cup B \cup -B$ is an upper segment in $N \cup -N$, and all the elements of A are larger than the elements of $B \cup -B$ with respect to Λ. Let us denote $|A|$ by i and $|B|$ by j, where we assume $i > 0$ or $j = n$. Then $V_{n+1,i,j}$ is the total reduced variation of all those signed augmented André *-permutations with respect to Λ which end with a letter from $A \cup B \cup -B$.*

References

[1] R. M. ADIN, A new cubical h-vector, *Conference Proceedings at the 6th International Conference on Formal Power Series and Algebraic Combinatorics,* DIMACS/Rutgers 1994.

[2] M. BAYER AND L. BILLERA, Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143-157.

[3] M. BAYER AND A. KLAPPER, A new index for polytopes, *Discrete Comput. Geom.* **6** (1991), 33-47.

[4] A. BJÖRNER, Shellable and Cohen-Macaulay partially ordered sets, *Trans. Amer. Math. Soc.* **260** (1980), 159-183.

[5] A. BJÖRNER, Posets, regular CW-complexes and Bruhat order, *European J. of Combin.* **5** (1984), 7-16.

[6] R. EHRENBORG, AND M. READDY, The **r**-cubical lattice and a generalization of the **cd**-index, preprint 1994.

[7] D. FOATA AND M.P. SCHÜTZENBERGER, Nombres d'Euler et permutations alternantes. In: J.N. Srivastava et al., "A Survey of Combinatorial Theory," Amsterdam, North–Holland, 1973 (pp. 173–187).

[8] G. HETYEI, On the Stanley ring of a cubical complex, to appear in *Discrete Comput. Geom.*.

[9] G. HETYEI, On the *cd*-variation polynomials of André and simsun permutations, preprint 1994.

[10] N. METROPOLIS, AND G.-C. ROTA, Combinatorial structure of the faces of the n-cube, *SIAM J. Appl. Math.* **35** (1978), 689-694.

[11] M. PURTILL, André permutations, lexicographic shellability and the *cd*-index of a convex polytope, *Trans. Amer. Math. Soc.* **338** (1993), 77-104.

[12] R. STANLEY, "Enumerative Combinatorics, Vol. I," Wadsworth and Brooks/Cole, Pacific Grove, 1986.

[13] R. STANLEY, Generalized h-vectors, intersection cohomology of toric varieties, and related results, pp. 187–213 in: "Commutative Algebra and Combinatorics," M. Nagata, H. Matsumura, eds., Advanced Studies in Pure Math. **11**, North-Holland, New York, 1987.

[14] R. STANLEY, Flag f-vectors and the *cd*-index, *Math. Z.* **216** (1994), 483-499.