Affine Coxeter groups as infinite permutations (extended abstract)

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Abstract

We present a unified theory for permutation interpretations of the length function, the weak order and the Bruhat order of all the infinite families of finite and affine Coxeter groups.

Résumé

Nous présentons un théorie unifié pour interpréter le fonction de longueur, l'ordre faible et l'ordre de Bruhat de toutes les familles infinies de groupes de Coxeter finis et affines.

1 Introduction

The aim of this paper is to present a unified theory for Coxeter group aspects on permutation representations of the finite groups A_n, B_n, C_n, D_n, and the affine groups $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$.

The symmetric group S_n, that is, the group of permutations of $\{1, 2, \ldots, n\}$, is extremely well studied. If S_n is viewed as the group generated by adjacent transpositions, it is isomorphic to the Coxeter group A_{n-1}, and Coxeter group concepts such as length, weak order and Bruhat order have nice interpretations in permutation language.

Also the other families of finite Coxeter groups, B_n and D_n, have well-known representations by "signed" permutations; here, though, the meaning of length, weak order and Bruhat order is less well-known, although it has been around for a while, see e.g. Proctor [9] and Björner and Brenti [2].

In his 1994 thesis [5], H. Eriksson presented representations of all the affine groups by infinite periodic permutations (though some of these had been part of folklore before, known to people like Lusztig and Stanley). A permutation interpretation of the Bruhat order on $\tilde{A}_n$ will appear in the forthcoming book by Björner and Brenti [2].

We will here, in a unified way, describe the permutation interpretations of the length function, the weak order and the Bruhat order of *all* the families of finite and infinite Coxeter groups. Well, with one exception: the Bruhat order criteria for $\tilde{B}_n$ and $\tilde{D}_n$ are not described.

2 The affine groups as infinite permutations

The classification of finite and affine Coxeter groups (due to H.S.M.Coxeter himself in 1935) features the four infinite families defined by the Coxeter graphs in the table below.

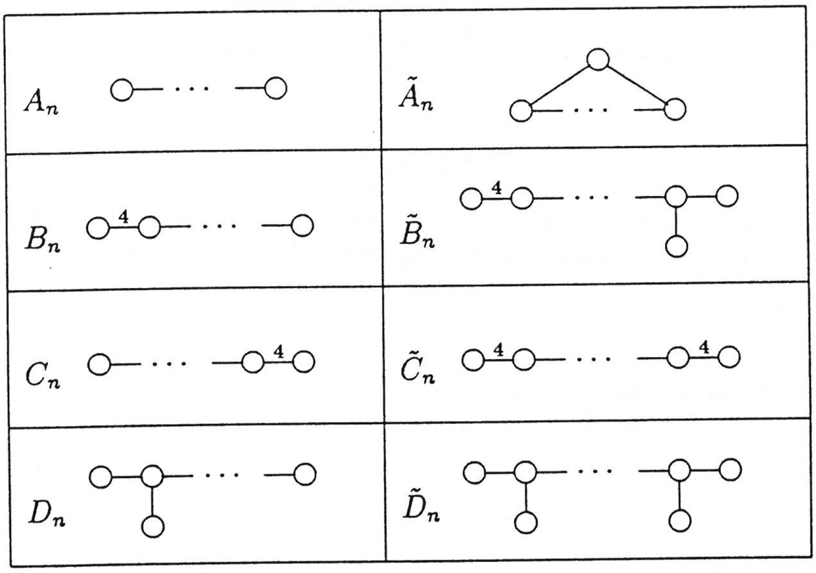


Table 1: ABCD-families of irreducible finite and affine Coxeter groups

Our theme is permutation representations of these groups, generalizing the ordinary model of A_{n-1} as permutations π of the set $\{1, \ldots, n\}$, with the i-th generator s_i corresponding to the adjacent transposition (π_i, π_{i+1}). Though we confine ourselves here to the $ABCD$-groups, it should be mentioned that similar things can be done with the sporadic $EFGH$-types as well as many other nameless groups, see [5] for details. For precise definitions and for Coxeter group theory in general, we refer to the book [7] by J.E.Humphreys.

2.1 The finite case: B_n, C_n, D_n.

Instead of representing the elements of these groups by signed permutations, we shall use symmetric permutations of the set of integers $[-n, \ldots, n]$. As sketched in Figure 1, the generator s_i, for $i = 1, \ldots, n-1$, transposes not only (π_i, π_{i+1}) but also (π_{-i}, π_{-i-1}). The action of the last generator s_n is different in B_n and D_n but symmetric in both cases, therefore only symmetric permutations will occur in the representations. In B_n, the generator s_n transposes (π_{-1}, π_1); in D_n, the generator s_n transposes (π_{-1}, π_2) as well as

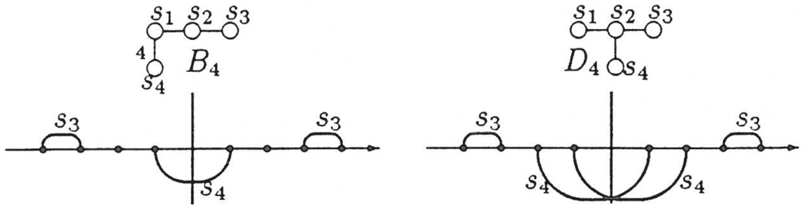


Figure 1: The actions of s_3 and s_4 in B_4 and D_4

(π_{-2}, π_1). In particular, zero is a fixed point and the symmetric action can be envisioned as a mirror at $x = 0$.

As a concrete example, consider s_3s_4, read from left to right. In B_4, this permutes the identity $(-4, -3, -2, -1, 0, 1, 2, 3, 4)$ as follows:

$$\rightarrow_{s_3} (-3, -4, -2, -1, 0, 1, 2, 3, 4) \rightarrow_{s_4} (-3, -4, -2, 1, 0, -1, 2, 3, 4).$$

In D_4, on the other hand, the permutation action of s_3s_4 on $(-4, -3, -2, -1, 0, 1, 2, 3, 4)$ is:

$$\rightarrow_{s_3} (-3, -4, -2, -1, 0, 1, 2, 3, 4) \rightarrow_{s_4} (-3, -4, 1, 2, 0, -2, -1, 3, 4).$$

2.2 The affine case: $\tilde{A}_n$.

For our representation of $\tilde{A}_{n-1}$, we shall use *n-periodic permutations*, that is permutations of $\mathbf{Z}$ generated by *periodic transpositions* $s_1, \ldots, s_n$. Here, s_i is the adjacent transposition $(i, i+1)$ together with all its n-translates $(kn+i, kn+i+1)$ for $k \in \mathbf{Z}$.

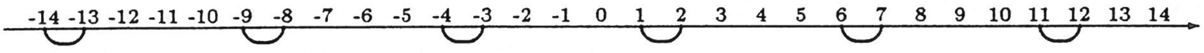


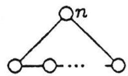
Figure 2: The action of $s_1 \in \tilde{A}_4$ as transpositions on $\mathbf{Z}$.

A natural mechanical model for this structure is a pile of n rulers, each with a protruding pin at every nth mark. The pinheads are round and so large that when a ruler is put on top of another, the pins must occupy different positions. In the complete ruler pile, the only movement possible is switching two neighbour pins by sliding their rulers one unit relative to the pile. The pinheads of ruler 1 are marked $\ldots, 1-2n, 1-n, 1, 1+n, 1+2n, \ldots$ etc, so a consecutive sequence of n pinhead numbers has got all congruence classes modulo n in it. Also, the sum of this consecutive sequence is invariant, for the only transposition that changes the set of numbers in the sequence is between the rulers of the leftmost and rightmost pins, but it increases the contribution from the first ruler by n and decreases the contribution from the second by the same amount.

The following characterization is more or less obvious by this pins and rulers model.

Proposition 1 An infinite integer vector $(\ldots, x_{-1}, x_0, x_1, \ldots)$ is an n-periodic permutation if and only if three conditions are satisfied

1. $x_{i+n} = x_i + n$ for all i
2. $x_1, \dots, x_n$ belong to different congruence classes modulo n
3. $x_1 + \cdots + x_n = n(n+1)/2$

The group of n-periodic permutations is isomorphic to the Coxeter group $\tilde{A}_{n-1}$: 

. This isomorphism is easily established via the *numbers game*, analysed by K.Eriksson [6]. In this game, numbers are to be placed on the nodes of the Coxeter graph, so on node i, we put the number $x_{i+1} - x_i$. The rules of the game say that node i can be *fired* by adding its number to the neighbouring numbers and then reversing the sign of the number on node i. But, as is easily verified, this is exactly what happens when the transposition s_i is performed. Also, the characterization in Prop. 1 implies a bijection between n-periodic permutations and numbers game positions. However, as shown in [6], the numbers game positions correspond bijectively to the elements of the Coxeter group, so we have the following.

Proposition 2 The group of n-periodic permutations is isomorphic to $\tilde{A}_{n-1}$.

Note. The n-vector $x_1, \dots, x_n$ determines the whole infinite permutation, so what we have here is an n-dimensional linear representation in disguise. By forgetting s_n, we retrieve the ordinary representation of the finite subgroup A_{n-1} as permutations of $1, \dots, n$, so $\mathbf{Z}$ may be viewed as countably many copies of the interval $[1, n]$, glued together by the action of s_n.

2.3 The affine case: $\tilde{B}_n, \tilde{C}_n, \tilde{D}_n$.

Before we move on to the affine groups $\tilde{B}_n, \tilde{C}_n, \tilde{D}_n$, recall that the corresponding finite Coxeter groups B_n, C_n, D_n were represented as symmetric permutations of the set of integers $[-n, \dots, n]$ and that the symmetric group action could be envisioned as a mirror at $x = 0$.

To obtain the affine groups, we start with the corresponding finite case and erect a second mirror at $x = n+1$. The transpositions $s_1, \dots, s_n$ now get infinitely many mirror images, all along $\mathbf{Z}$, with a period of $2n+2$. These intervals of length $2n+2$ are glued together by the action of the extra node, s_{n+1}, which is the single transposition $(n, n+2)$ for $\tilde{C}_n$ and the pair of transpositions $(n-1, n+2), (n, n+3)$ for $\tilde{B}_n$ and $\tilde{D}_n$. Thus, in $\tilde{C}_n$, both mirrors use the same glue as the mirror of C_n; similarly, in $\tilde{D}_n$, both mirrors use the same glue as the mirror of D_n. But in $\tilde{B}_n$, the two mirrors use different glue.

With these definitions of s_i as infinite collections of transpositions, it is evident that s_i commutes with mirror reflection and that therefore the following properties stay true for all infinite permutations obtained by application of the s_i.

$$\begin{aligned} x_{-i} &= -x_i \,, \\ x_{2n+2-i} &= 2n+2 - x_i \,. \end{aligned}$$

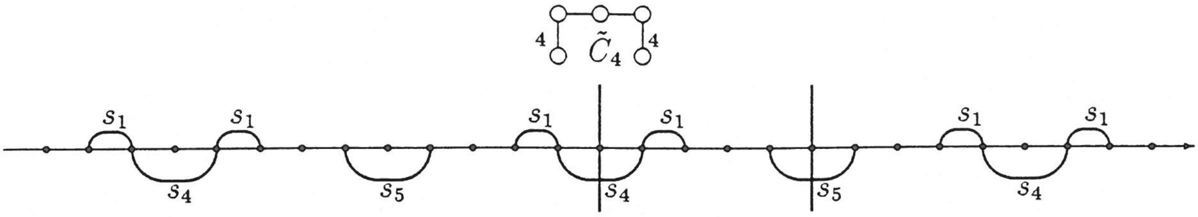


Figure 3: The actions of $s_1, s_4, s_5 \in \tilde{C}_4$ as transpositions on $\mathbf{Z}$.

Note that as a consequence of these two mirror relations, the $(2n+2)$-translative property $x_{2n+2+i} = 2n+2 + x_i$ holds!

It is now possible to characterize the infinite permutations that can arise.

Proposition 3 The infinite permutation vectors $(\ldots, x_{-1}, x_0, x_1, \ldots)$ obtainable by application of the s_i in the $\tilde{C}_n$-case are exactly those that satisfy the mirror conditions

1. $x_{-i} = -x_i$ for all i.
2. $x_{2n+2-i} = 2n+2 - x_i$ for all i.

For $\tilde{B}_n$ and $\tilde{D}_n$ there is one more condition, namely

3. Among $x_1, \ldots, x_n$, an even number have odd $\left\lfloor \frac{x_i}{2n+2} \right\rfloor$. ($\tilde{B}_n$ only)
3. Among $x_1, \ldots, x_n$, an even number have odd $\left\lfloor \frac{x_i}{n+1} \right\rfloor$. ($\tilde{D}_n$ only)

PROOF. (Sketch) We first check that the conditions are invariant, then assume that there are vectors outside the representation and satisfying the conditions, select such a vector with minimal $(x_1, \ldots, x_n)$-span and derive a contradiction. □

Remark 1 It is clear that $\tilde{B}_n$ is $\tilde{C}_n$-like at one end and $\tilde{D}_n$-like at the other. Depending on which end goes to 0 and which goes to $n+1$, we get different representations. The reader should have no difficulty in finding big-endian versions of the little-endian ones given here. For instance, in the third condition above, the fraction is modified to $\left\lfloor \frac{x_i+n+1}{2n+2} \right\rfloor$.

Proposition 4 The groups $\tilde{B}_n, \tilde{C}_n, \tilde{D}_n$ are isomorphic to the groups of infinite permutations defined in Proposition 3.

PROOF. Again, the easiest connection goes via a numbers game. The details are omitted in this extended abstract but can be found in [5]. □

3 Length, class inversions, and weak order

The *length* of a group element w is the length of the shortest word for w in the generators s_i. As we shall see, given the permutation corresponding to w, it is easy to compute its length $l(w)$: it is the number of "class inversions", as will be defined below. Closely connected to the length function $l(w)$ is the *weak order*, in which $w \geq u$ if there is a factorization $w = uv$ with $l(w) = l(u) + l(v)$. For a general Coxeter group, deciding whether $w \geq u$ involves computing $u^{-1}w$ and its length, but for our permutations, we can give a direct criterion.

3.1 Class inversions in the finite case: A_n, B_n, C_n, D_n.

For an ordinary permutation π, the length is of course the number of inversions $\pi_i > \pi_j, i < j$. Something similar is true for the symmetric permutations in D_n, but now inversions occur in pairs. For instance, if $\pi_1 > \pi_2$, then necessarily $\pi_{-2} > \pi_{-1}$. Another such pair would be $\pi_{-1} > \pi_2$ and $\pi_{-2} > \pi_1$. If we agree to count an inversion and its mirror inversion as one, then it is clear that every s_i in a reduced word for w will produce exactly one inversion, so $l(w)$ will be the number of inversions, exactly as in A_n. Note that inversions between an element and its mirror image, such as $\pi_{-1} > \pi_1$ are not counted at all, since they do not appear in pairs.

For the groups B_n and C_n, the only difference is that inversions of the form $\pi_{-i} > \pi_i$ must now be counted, otherwise the action of s_n would go unnoticed in the length calculation. In order to clarify these slightly different inversion concepts and give them a form that carries over to the infinite permutations, we introduce the notion of *class inversion*. A class consists of an element and its mirror images, so A_{n-1} has n single-element classes while B_n, C_n, D_n have n two-element classes. The class consisting of zero only may be considered as an artificial class.

Definition. An inversion between two elements together with all its mirror images constitute a *class inversion* between the classes of these elements.

Note two things: First, inversions within a class never have to be considered; instead one can look at inversions between this class and the artificial zero class. Second, between two classes, there may be *two* class inversions. For example, in $(\ldots, 2, 1, 0, -1, -2, \ldots) \in D_n$ the pair 2, 1 and –1, –2 constitute one class inversion and the pair 2, –1 and 1, –2 a second class inversion between the same classes.

Proposition 5 The length of an element in A_n, B_n, C_n or D_n is the number of class inversions in the corresponding permutations. For B_n and C_n, one should consider zero as a class in counting class inversions.

Other versions of length formulas, not introducing class inversions, have been given by Deodhar and Brenti.

Example. What is the length of $(3,2,1,0,-1,-2,-3) \in B_3$? There are double class inversions between all three classes and each class also has an inversion with zero, so the length is 9.

3.2 Class inversions in the affine case: $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n$.

In the affine groups, if $x_i > x_j, i < j$ is an inversion pair, so are infinitely many other pairs, namely those generated by n-translations in the $\tilde{A}_{n-1}$-case and those generated by mirror reflections in the other cases. In analogy with B_n, C_n, D_n above, if we count such an infinite set of translated or mirrored pairs as one *class inversion*, the length function will again be the inversion count. Note that a pair of classes may contribute arbitrarily much to the class inversion count, as illustrated below. In the second case, $(5,1)$ and $(5,4)$ are two different class inversions.

For $\tilde{A}_n$, it is clear that a translated inversion is still an inversion, e.g. $2>1 \Rightarrow 5>4 \Rightarrow 8>7$ in the first example, but in the mirror models, this is less evident. Can we be sure that $x_{-2} > x_1 \Rightarrow x_{-1} > x_2$, for example? Yes, the mirror conditions of Prop. 3 imply this!

For $\tilde{B}_n$ and $\tilde{C}_n$, s_n creates an inversion *within* a class. Instead of counting these, we can clearly count inversions *between* that class and the artificial class $\ldots, -c, 0, c, 2c, \ldots$, where $c = 2n+2$. The same trick can be used for class-internal inversions brought about by s_{n+1} in $\tilde{C}_n$.

Proposition 6 The length of an element in $\tilde{A}_{n-1}, \tilde{B}_n, \tilde{C}_n$ or $\tilde{D}_n$ is the number of class inversions in the corresponding infinite permutation. By translations or mirror reflections, $x_1, \ldots, x_n$ define one class each, and these are the classes used for $\tilde{A}_{n-1}$ and $\tilde{D}_n$. For $\tilde{B}_n$ and $\tilde{C}_n$, the class generated by 0 should also be considered in counting class inversions and for $\tilde{C}_n$ also the class generated by $n+1$.

PROOF. The proof is omitted in this extended abstract, but can be found in [5]. □

3.3 Weak order

Generalizing the case of the symmetric group, the weak order is encoded by the sets of class inversions:

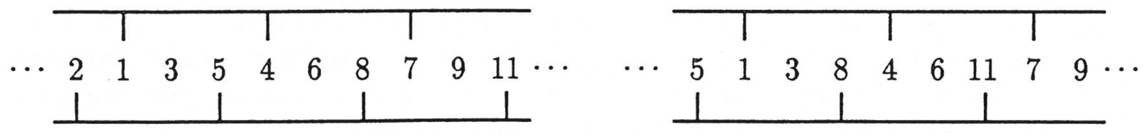


Figure 4: Single and double class inversion in $\tilde{A}_2$.

Proposition 7 For any one of the groups A_n, B_n, D_n, $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$, we let $I(\pi)$ denote the set of class inversions in the infinite permutation corresponding to π. Then $\pi \geq \sigma$ in the weak order if and only if $I(\pi) \supseteq I(\sigma)$.

PROOF. First assume that $\pi \geq \sigma$ in the weak order, so $\pi = \sigma s_{i_1} s_{i_2} \cdots s_{i_k}$ with $l(\pi) = l(\sigma) + k$. Then each multiplication by a generator introduces a new class inversion, but the class inversions already in $I(\sigma)$ are still there when we reach π, so $I(\pi) \supseteq I(\sigma)$.

For the converse, assume that $I(\pi) \supseteq I(\sigma)$ and show that there is a factorization $\pi = \pi' s$ with $l(\pi) = l(\pi') + 1$ and $I(\pi') \supseteq I(\sigma)$; induction would then give $\pi \geq \sigma$. Let (π_i, π_j) be a representative of a class inversion in $I(\pi) \setminus I(\sigma)$, such that $\pi_i > \pi_j$, $i < j$, and the difference $j - i$ is minimal among such inversions. By considering the possible configurations, it is easy to see that (π_i, π_j) then must be "adjacent", that is, the transposition (π_i, π_j) is a generator s. (Sometimes, as we have seen, this means that $j = i + 2$ or even $j = i + 3$.) Hence $\pi = \pi' s$ will do as factorisation. □

Finally, we look at the interpretation of *descent* in our permutation models. In an ordinary permutation, a descent is any occurrence of $x_i > x_{i+1}$, but for an element w of an arbitrary Coxeter group, the *descent set* is defined as

$$D(w) = \{s_i \mid l(ws_i) < l(w)\}.$$

In the terminology of permutations, we can say that the descent set consists of all s_i that *resolve an inversion*. For most s_i, this simply means that $x_i > x_{i+1}$, but for s_n and s_{n+1}, the interpretation is different for different groups. For example, in $\tilde{C}_n$ we have $s_n \in D(w)$ if $x_{-1} > x_1$, but in $\tilde{D}_n$ we have $s_n \in D(w)$ if $x_{-1} > x_2$.

4 Bruhat order

In this concluding section of the paper, we are going to present a generalization of the tableau criterion for the Bruhat order in the symmetric group to all the finite and affine groups that we have been considering.

In general Coxeter group theory, conjugates of generators are called *reflections*. A reflection can always be written as a palindrome $t = s'_k \cdots s'_2 s'_1 s'_2 \cdots s'_k$, with $s'_i \in S$. In the symmetric group, a reflection is a transposition, not necessarily adjacent. The weak order, generated by $w < ws$, where $s \in S$ and $l(ws) = l(w) + 1$, can be expanded to the *Bruhat order*, generated by $w < wt$, where t is any reflection such that $l(wt) = l(w) + 1$. For permutations, the following criterion can decide whether a permutation π precedes another permutation σ in Bruhat order. It is due to Ehresmann [4] and can be seen as a special case of Deodhar's criterion [3] for general Coxeter groups.

Tableau criterion: Let π_{ij} be the element obtained by sorting the first j symbols of π in increasing order and then picking the ith symbol. Then $\pi \leq \sigma$ in Bruhat order if and only if $\pi_{ij} \leq \sigma_{ij}$ whenever $1 \leq i \leq j \leq n$.

Example. Let $\sigma = (2,1,3,4)$ and $\pi = (3,1,4,2)$. We have $l(\sigma) = 1$ and $l(\pi) = 3$ and a transposition chain $(2,1,3,4) \mapsto (3,1,2,4) \mapsto (3,1,4,2)$ demonstrating that $\sigma < \pi$ in Bruhat order. (But there is no such chain using adjacent transpositions, so no weak order relation exists.) The tableau criterion involves sorting all initial segments and comparing them: $(2) \le (3)$, $(1,2) \le (1,3)$, $(1,2,3) \le (1,3,4)$, $(1,2,3,4) \le (1,2,3,4)$. The conclusion is that $\sigma < \pi$. The *dual tableau criterion* is equivalent, it sorts final segments instead: $(4) \geq (2)$, $(3,4) \geq (2,4)$, $(1,3,4) \geq (1,2,4)$, $(1,2,3,4) \geq (1,2,3,4)$.

In all cases, a reflection element $t = wsw^{-1}$ is a not-necessarily-adjacent transposition, together with its symmetric transpositions. This is clear, as the action is "permute, transpose, unpermute". So the Bruhat order can be described combinatorially easily enough. But is there a generalization also of the tableau criterion? Yes, there is; the following result is due to Proctor [9]:

Proposition 8 [Proctor] For a finite Coxeter group of type C_n, represented as permutations of $-n, \ldots, n$, the Bruhat relation $\sigma < \pi$ holds when the following criterion is satisfied. Any initial segment $(\sigma_{-n}, \ldots, \sigma_i)$, $i = -n, \ldots, -1$, sorted in increasing order must be componentwise less than or equal to the corresponding sorted initial segment of π.

For D_n, the sorted initial segments of σ and π must additionally satisfy that no pair of corresponding subsegments (of length, say, k) both constitute a signed permutation of $1, \ldots, k$ such that the number of negative elements is odd in one segment and even in the other.

We would like to extend the result to the infinite permutations, but there seem to be complications. Is it possible to sort an infinite interval? Yes, it is! Assuming that the $\mathbf{Z}$-axis has ben cut in two between x_0 and x_1, the right half-axis is sorted by putting its smallest element in x_1, its next smallest in x_2 etc. And the left half-axis sorts its largest element into x_0, its next largest into x_{-1} etc. Thus, it is possible to formulate Bruhat order criteria analogous to the tableau criteria of the finite groups. For $\tilde{A}_n$ and $\tilde{C}_n$, it looks as the simple criterion for A_n and C_n.

Proposition 9 For an affine Coxeter group of type $\tilde{A}_n$ or $\tilde{C}_n$, represented as infinite permutations of $\mathbf{Z}$, the Bruhat relation $\sigma < \pi$ holds when the following criterion is satisfied. Any initial half-infinite segment $(\ldots, \sigma_i)$, sorted in increasing order must be componentwise less than or equal to the corresponding sorted initial segment of π.

PROOF. In this extended abstract, we just sketch the proof: The necessity is simple, for a transposition that creates inversions replaces some numbers by greater numbers in some of the initial segments. The sufficiency is proved roughly as follows: find a suitable transposition τ that resolves an inversion, and check that the criterion is still satisfied with σ and $\pi\tau$. $\square$

Remark 2 *Another combinatorial Bruhat order criterion for $\tilde{A}_n$ has been developed by Björner and Brenti [2].*

Remark 3 *For $\tilde{B}_n$ and $\tilde{D}_n$, the criteria are more complicated, as can be understood from Proctor's criterion for D_n.*

References

[1] A. Björner, *Orderings of Coxeter groups*, Combinatorics and Algebra, Contemporary Math. **34**, Amer. Math. Soc., 1984, 175–195.

[2] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, book manuscript, 1994.

[3] V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group*, Invent. Math. **39**, 1977, 187–198.

[4] C. Ehresmann, *Sur la topologie de certains espaces homogènes*, Ann. Math. **35** (1934) 396–443.

[5] H. Eriksson, *Computational and combinatorical aspects of Coxeter groups*, PhD thesis, KTH, Stockholm, Sweden (1994).

[6] K. Eriksson, *Strongly convergent games and Coxeter groups*, PhD thesis, KTH, Stockholm, Sweden (1993).

[7] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Univ. Press, Cambridge, MA, 1990.

[8] G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. **277**, 1983, 623–653.

[9] R. Proctor, *Classical Bruhat orders and lexicographic shellability*, J. Algebra **77**, 1982, 104–126.