Combinatorics of Fulton's ranked essential set (extended abstract)

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Abstract

For any permutation, Fulton introduced its ranked essential set. We give a new elementary proof of the fact that the ranked essential set of a permutation w determines w. We then characterize the class of ranked sets that arise as ranked essential sets of permutations by giving necessary and sufficient conditions. Several classes of permutations are characterized in terms of their essential set. Various enumerative results, on the number of elements of given rank, are obtained.

La notion d'ensemble essentiel gradué d'une permutation a été introduite par Fulton. Nous donnons une preuve nouvelle et élémentaire du fait que l'ensemble essentiel gradué d'une permutation w détermine w. Nous donnons ensuite des conditions nécéssaires et suffisantes pour qu'un ensemble gradué soit l'ensemble essentiel d'une permutation. Plusieurs classes de permutations sont caractérisées en termes de leur ensemble essentiel. Nous obtenons aussi plusieurs résultats énumératifs concernant le nombre déléments de rang donné.

1 The ranked essential set

The combinatorial object that we are studying is the essential set of a permutation, together with its rank function, as introduced by Fulton [4], 1992. They are defined as follows. First, let every permutation $w \in S_n$ be represented by its dotted permutation matrix, regarded as an $n \times n$-collection of squares in the plane, where square $(i, w(i))$ has a dot for all $i \in [1, n]$, and all other squares are white, so there is exactly one dot in each row and column.

We get the *diagram* of the permutation by shading the squares in each row from the dot and eastwards, and shading the squares in each column from the dot and southwards. Thus, we now have shaded squares and white, that is unshaded, squares. (To be precise, the diagram is what is made up of the white squares.)

We call a white square a *white corner* if it has no white neighbor either to the east or to the south. In other words, the white corners are the southeast corners of the components of the diagram. The *essential set* $\mathcal{E}(w)$ of a permutation w is defined to be the set of white corners of the diagram of w.

For every white corner (i,j) of w, its *rank* is defined by

$$r_w(i,j) \stackrel{\text{def}}{=} \#\{ \text{ dots northwest of } (i,j)\} = \#\{(i',j') \text{ with dot } : i' \leq i, j' \leq j\}$$

The name "rank" stems from the fact that $r_w(i,j)$ is equal to the matrix rank of the i by j upper left submatrix of the ordinary permutation matrix of w, where the dots are replaced by ones and the blank squares are zeros. Indeed, the ranked essential sets describe the irreducible loci of spaces of matrices where the upper left i by j submatrices have rank not greater than $r_w(i,j)$.

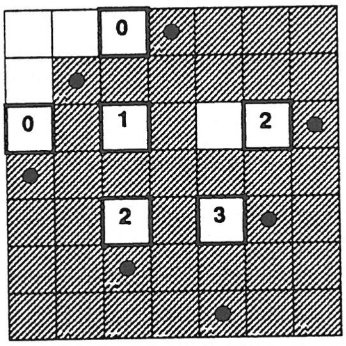

Figure 1: Diagram and ranked essential set of the permutation 4271635.

The fundamental property of the ranked essential set is the following.

Proposition 1.1 (Fulton) *A permutation $w \in S_n$ is determined by its ranked essential set.*

Fulton's proof is very short, but algebraic. We give here an elementary algorithm for explicitly determining the permutation from its ranked essential set, thereby giving a combinatorial explanation of the result.

Retrieval algorithm: Let B be a permutation matrix. Let B_0 be a copy of B where we have forgotten about the dots, but instead every square in the essential set of B_0 is labeled by its rank. We shall, dot by dot, recover B by constructing a sequence of labeled shapes $B_0, B_1, B_2, \ldots$, such that the labeled squares of every B_k will be the ranked essential set of the restriction of the dots of B to the subshape B_k. For i from 1 to r:

Step $2i-1$: The labeled shape B_{2i-1} is obtained from B_{2i-2} by removing every square c such that $c \leq c'$ where $c' \in B_{2i-2}$ is a square labeled zero. In these squares, there cannot be any dot in B, so no labels should be changed.

Step $2i$: After the previous step, B_{2i-1} has no square labeled zero. We can now be sure that every minimal square of B_{2i-1} must have a dot in B. We now obtain B_{2i} by removing from B_{2i-1}, for every minimal square c, all slices containing c (since c has a dot, none of the other squares in a slice can have a dot in a proper dotting), and decreasing the label by one for every labeled square $c' \geq c$ (since the removed c had one of the dots counted by these labels). Since we are removing squares that would have been shaded, the set of white corners is unchanged.

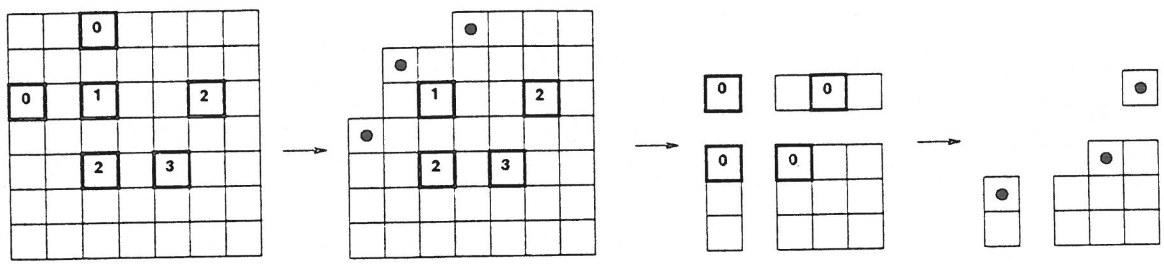

Figure 2: A run of the algorithm on the essential set of the permutation in Figure 1. B_0, B_1, B_2 and B_3 depicted, with the revealed dots indicated. B_4 will then be a single square that must have a dot.

2 Characterization of ranked essential sets

Fulton [4] posed the problem of finding a characterization of what ranked essential sets that can arise from arbitrary permutations. In this section we present such a characterization. Define $R(i,j)(i',j')$ to mean the rectangle with corners $(i,j),(i,j'),(i',j),(i',j')$.

Theorem 2.1 (Main Theorem) *Let $E \subseteq [1,n] \times [1,n]$ be a set of squares with rank function $r(i,j)$. E is the essential set of an $n \times n$ permutation matrix if and only if:*

C1. For each $(i,j) \in E$ we have

(a) $r(i,j) \geq 0$ and

(b) $r(i,j) + n \geq i + j$.

C2. For every distinct pair $(i,j),(i',j') \in E$ such that $i \geq i', j \leq j'$ and $E \cap [i',i] \times [j,j'] = \{(i,j),(i',j')\}$ we have

$$i - i' > r(i,j) - r(i',j') > j - j'$$

C3. For every pair $(i,j),(i',j') \in E$ such that $i < i', j < j'$ and $E \cap [i+1,i'] \times [j+1,j'] = \{(i',j')\}$, let $(i'',j'') \in E$ be the square of E with the largest i'' satisfying $i'' \leq i$, $j'' \geq j'$ and $E \cap [i'',i] \times [j',j''] = \{(i'',j'')\}$ (if no such square exists let $r(i'',j'') = 0$); symmetrically, let (i''',j''') be the square of E with the largest j''' satisfying $j''' \leq j$, $i''' \geq i'$ and $E \cap [i',i'''] \times [j''',j] = \{(i''',j''')\}$ (if no such square exists let $r(i''',j''') = 0$). We have

$$r(i',j') + r(i,j) \geq i - i'' + j - j''' + r(i'',j'') + r(i''',j'''). \ \square$$

Sometimes it is interesting to consider the situation where we do not have full rank and the matrix might be a rectangle. The proof of Theorem 3.2 (omitted here) gives immediately the following useful corollary.

Corollary 2.2 *Given $k \leq m \leq n$, let $E \subseteq [m] \times [n]$ be a set of squares with rank function $r(i,j)$. Augment $(0, n+m-k)$ and $(n+m-k, 0)$, to E both with rank zero. E is the essential set of a properly dotted $m \times n$ rectangle with k dots if and only if conditions $1(a), 2(a), 2(b)$ and 3 of Theorem 3.2 are fulfilled and for every square $(i,j) \in E$ we have*

$$n + m - k - i - j + r(i,j) \geq 0.$$

In the proof we need to consider also three alternative rank functions, namely $r^{\text{ne}}(i,j) \stackrel{\text{def}}{=} i - r(i,j)$, $r^{\text{sw}}(i,j) \stackrel{\text{def}}{=} j - r(i,j)$ and $r^{\text{se}}(i,j) \stackrel{\text{def}}{=} n - i - j + r(i,j)$. Clearly, if $w \in S_n$, then for any $i,j \in [1,n]$ we have the following interpretation of the rank funtions:

$$r_w^{\text{ne}}(i,j) = \#\{\text{ dots northeast of } (i,j)\} = \#\{(i',j') \text{ with dot } : i' \leq i, j' > j\}$$

$$r_w^{\text{sw}}(i,j) = \#\{\text{ dots southwest of } (i,j)\} = \#\{(i',j') \text{ with dot } : i' > i, j' \leq j\}$$

$$r_w^{\text{se}}(i,j) = \#\{\text{ dots southeast of } (i,j)\} = \#\{(i',j') \text{ with dot } : i' > i, j' > j\}$$

It obviously suffices to know $r_w(i,j)$ but it is convenient to work with all four rank functions.

3 Combinatorial aspects and enumerative results

We will here discuss the combinatorial meaning of white squares and white corners, and then state and prove some enumerative results on the distribution of white corners with certain rank.

As mentioned in Macdonald's book [5], the white squares of the diagram of a permutation w correspond exactly to the inversions of w: (i,j) is a white square exactly when both $w(i) > j$ and $i < w^{-1}(j)$. As observed by Fulton, every row with a white corner corresponds to a descent: if (i,j) is a white corner, then $w(i+1) \leq j$ while $w(i) > j$, so $w(i+1) < w(i)$; conversely, if $w(i+1) < w(i)$, then the square $(i, w(i+1))$ must be white, so there must be a white corner in row i.

3.1 No restrictions

We shall begin by studying the distribution of ranks of white corners for permutations in S_n without restrictions. Define $P_n(x)$ to be the polynomial that keeps track of the distribution of ranks:

$$P_n(x) \stackrel{\text{def}}{=} \sum_{w \in S_n} \sum_{c \in \mathcal{E}(w)} x^{r_w(c)}$$

Define $P_n^{\text{ne}}(x)$, $P_n^{\text{sw}}(x)$ and $P_n^{\text{se}}(x)$ in the analogous way, that is, with the rank function taken to be r_w^{ne}, r_w^{sw} and r_w^{se} respectively. One can prove that $P_n(x) = P_n^{\text{se}}(x)$ and $P_n^{\text{sw}}(x) = P_n^{\text{ne}}(x)$ by considering the two involutions $w \mapsto w^{-1}$ (transposition of the permutation matrix) and $w \mapsto \text{rt}w$ (rotation of the matrix 180°).

Proposition 3.1 *The total number of white corners in S_n is*

$$P_n(1) = (n-1)!\frac{\binom{n-1}{3} + 6\binom{n}{2}}{6}.$$

Note that by dividing with $n!$, the number of permutations in S_n, we obtain $(\binom{n-1}{3} + 6\binom{n}{2})/(6n)$ as the average number of white squares.

Another curious result in this context:

Proposition 3.2 *Let $\mathcal{E}'(w)$ be the set of white corners of w that are the last white corners of their rows. Then*

$$\sum_{w\in S_n} (\ \#\ of\ c \in \mathcal{E}'(w) : r_w^{\mathbf{ne}}(c) = t) = (n-t)(n-1)!$$

Observe the interpretation in terms of descents that follows from

$$r_w^{\mathbf{ne}}(i,j) = \#\ \{k < i : w(k) > w(i)\}.$$

For a given descent i in a permutation w, that is, $w(i) > w(i+1)$ we have that $r_w^{\mathbf{ne}}(i,j)$ counts the number of inversions having $w(i)$ as the smaller element. So, looking at all possible descents in all permutations of S_n, the number of them having exactly t larger predecessors is $(n-t)(n-1)!$.

3.2 Vexillary permutations

Let $\mathcal{V}_n$ denote the set of vexillary permutations in S_n. By summing only over permutations in $\mathcal{V}_n$ we get another polynomial:

$$V_n(x) \stackrel{\text{def}}{=} \sum_{w\in\mathcal{V}_n} \sum_{c\in\mathcal{E}(w)} x^{r_w(c)}$$

As we did for $P_n(x)$ in the S_n case, define $V_n^{\mathbf{ne}}(x)$, $V_n^{\mathbf{sw}}(x)$ and $V_n^{\mathbf{se}}(x)$ in the analogous way.

In analogy with the previous case, we have $V_n^{\mathbf{sw}}(x) = V_n^{\mathbf{ne}}(x)$, and $V_n^{\mathbf{se}}(x) = V_n(x)$.

We would like very much an expression for $V_n(1)$, the total number of white corners of permutations in $\mathcal{V}_n$, but it has eluded us; maybe there is none. At least, we have obtained some partial results. Let v_n denote $|\mathcal{V}_n|$, the number of permutations in S_n that are vexillary. The number sequence $\{v_n\}_{n\geq 1}$ starts $1, 2, 6, 23, 103, 513, \ldots$ There is no exact formula known for v_n, but an asymptotic, see Macdonald [5].

Proposition 3.3 *In $V_n^{\mathbf{ne}}(x)$, the coefficients of x^{n-1} and x^{n-2} is v_{n-1} and $2v_{n-1}$ respectively.*

From the data we have computed, the following statement seems to hold.

Conjecture 3.4 *For fixed integer $k \geq 2$ and variable n, the coefficient of x^{n-k} in $V_n(x)$ can be expressed as a polynomial in n of degree $k-2$.*

3.3 321-avoiding permutations

As the last item in this section we shall discuss 321-*avoiding* permutations. We say that w contains a 321-pattern if there are indices $i_1 < i_2 < i_3$ such that $w(i_1) > w(i_2) > w(i_3)$), and we say that w is 321-avoiding if it does not contain a 321-pattern. Define $A_n(x)$ (and $A_n^{\mathrm{ne}}(x)$ etc. in analogy) by summing the ranked white squares over all 321-avoiding permutations. It should be quite obvious that the property of being 321-avoiding is invariant under transposition and rotation, so once again we need only study the two polynomials $A_n(x)$ and $A_n^{\mathrm{ne}}(x)$.

n	$A_n^{\mathrm{ne}}(x)$
2	$1x$
3	$3x + 1x^2$
4	$10x + 5x^2 + 1x^3$
5	$35x + 21x^2 + 7x^3 + 1x^4$
6	$126x + 84x^2 + 36x^3 + 9x^4 + 1x^5$
7	$462x + 330x^2 + 165x^3 + 55x^4 + 11x^5 + 1x^6$
8	$1716x + 1287x^2 + 715x^3 + 286x^4 + 78x^5 + 13x^6 + 1x^7$

In the table above one quickly recognizes binomial coefficients from every other row of Pascal's triangle. We have the following theorem:

Theorem 3.5 *The coefficients of $A_n^{\mathrm{ne}}(x)$ come from the last half of row $2n-3$ of Pascal's triangle, that is,*

$$A_n^{\mathrm{ne}}(x) = \sum_{k=1}^{n-1} \binom{2n-3}{k+n-2} x^k.$$

By summing these binomial coefficients, we immediately get the following appealing result.

Corollary 3.6 *The total number of white corners in 321-avoiding permutations in S_n is*

$$A_n(1) = A_n^{\mathrm{ne}}(1) = 2^{2n-4}.$$

4 Essential sets of certain classes of permutations

An important example of characterization by essential set is Fulton's description of the vexillary permutations as having no white corners (i,j) and (i',j') such that $i < i'$ and $j < j'$. His proof is algebraic in nature, but we would like to point out here that the result can be obtained elementarily from the alternative characterization of vexillary permutations as 2143-avoiding, see Macdonald [5].

In a similar way we shall prove some other connections between certain shapes of essential sets and permutations avoiding certain patterns. Let 2(41)3 denote the pattern 2413 where the two elements corresponding to 4 and 1 are neighbors in the permutation, that is, there are no elements inbetween in the word-form.

Proposition 4.1 *A permutation w is* 2(41)3*-avoiding if and only if it has at most one white corner in each row.*

Now, let us return to 321-avoiding permutations. Billey, Jockush and Stanley [1] obtained a "curious" enumerative consequence of their analysis of the Schubert polynomials of 321-avoiding permutations: the number of $w \in S_n$ that are both vexillary and 321-avoiding is $1+2(2^n-(n+1))-\binom{n+1}{3}$ (when written in a suitable form). By characterizing these permutations in terms of the essential set, we can give an immediate interpretation of each of these terms.

Proposition 4.2 *A permutation w is* 321*-avoiding and* 2143*-avoiding (vexillary) if and only if it has all its white corners either in one single row or in one single column.*

The identity permutation is both 321-avoiding and vexillary, so it takes care of the first term, 1, in the expression $1+2(2^n-(n+1))-\binom{n+1}{3}$. All other permutations have at least one white corner. Having all white corners in one single row is equivalent to having exactly one descent. The number of permutations with one descent is easily seen to be $2^n-(n+1)$: choose any subset of $[1,n]$ except for intervals $[1,k], k=0,1,\ldots,n$, and order it in increasing order, then continue with the complement in increasing order. By transposition, there are equally many permutations with all white corners in one single column, so this takes care of the second term, $2(2^n-(n+1))$. We must now subtract the number of permutations that have been added twice; they are those with only one white square all together. As is most easily seen from the picture (Figure 3), these are the permutations of the word-form

$$1\ldots i(j+1)\ldots k(i+1)\ldots j(k+1)\ldots n.$$

We can choose $i<j<k$ arbitrarily in the interval $[0,n]$, so this takes care of the last term, $\binom{n+1}{3}$, of the expression.

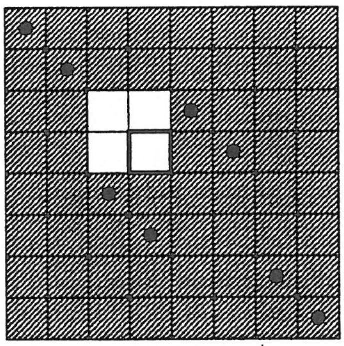

Figure 3: Permutation with only one white corner: the dots lie in four diagonal slopes, northwest, east, south, and southeast of the white area.

Let us conclude this section with a discussion of *antivexillary* permutations. By antianalogy with the vexillary case, we define a permutation to be antivexillary if it has no white corners (i, j) and (i', j') such that $i < i'$ and $j > j'$. Thus, an antivexillary permutation has its white corners spread in the northwest–southeast direction, while the vexillary permutations have their white corners spread in the southwest–northeast direction. The antivexillary permutations admit a surprising characterization in terms of forbidden patterns:

Proposition 4.3 *A permutation w is antivexillary if and only if it is* 321*-avoiding and* 351624*-avoiding.*

5 Remarks

All the results in this paper, and a few more, with details and proofs, are available in our two preprints [2] and [3]. We thank Dan Laksov for drawing our attention to this problem.

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