Formal Power Series, Operator Calculus, and Duality on Lie Algebras

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Abstract. This paper presents an operator calculus approach to computing with non-commutative variables. First, we recall the product formulation of formal exponential series. Then we show how to formulate canonical boson calculus on formal series. This calculus is used to represent the action of a Lie algebra on its universal enveloping algebra. As applications, Hamilton's equations for a general Hamiltonian, given as a formal series, are found using a double dual representation, and a formulation of the exponential of the adjoint representation is given. With these techniques one can represent the Volterra product acting on the enveloping algebra. We illustrate with a 3-step nilpotent Lie algebra.

I. Introduction

The foundations of a theory of non-linear causal functionals were laid by M.Fliess [8] using non-commutative indeterminates and formal power series. The observability of a class of systems for which the state space is a Lie group and the output space is a coset space was studied recently by D. Cheng, W.P. Dayawansa, and C.F. Martin [3]. H. Hermes [12] investigated structurally stable properties associated with systems described by real analytic vector fields approximated by appropriately chosen nilpotent systems. Finite Volterra series which admit Hamiltonian realizations were studied by P.E. Crouch and M. Irving [4] with the help of nilpotent endomorphisms on symplectic vector spaces. For aspects relating to controllability see Sussmann [14] and Jakubczyk and Sontag [13]. Calculations with enveloping algebras have been considered by Duchamp and Krob [5].

In this paper we present an operator calculus approach to these problems, particularly in the Lie context based on representations on the universal enveloping algebra. The paper is organized as follows. Section II recalls the product formulation of Volterra series. In §III, we consider the case of abelian (commuting) increments. Next we formulate the boson calculus on formal series. §V shows how to represent the action of a Lie algebra on its universal enveloping algebra in terms of canonical boson operators. As applications, Hamilton's equations for a general Hamiltonian, given as a formal series of monomials in the enveloping algebra, are found in terms of the double dual, then it is shown how to compute the exponential of the adjoint representation. Of particular note is the fact that the calculations can be done readily using matrix realizations and thus can be readily implemented using a package such as MAPLE. We remark that one can use the double dual for representing the Volterra product acting on the enveloping algebra. §VI illustrates these methods with some calculations for a 3-step nilpotent algebra.

II. Products and iterated sums

A (discrete) Volterra series in the variables $\{X_1, X_2, \ldots\}$ is of the form $\sum v^n I_n(X_1, X_2, \ldots)$, where the I_n are iterated sums of the variables X_i. This has been studied since the time of Volterra as the basic construction. The principal connection is between the series and the product, which is a generic form of the exponential function. The generic exponential is of the form

$$\prod(1+vX_j) = \sum v^n I_n(X_1, X_2, \ldots)$$

Recently a fairly complete exposition has been given by Gill and Johansen, and Gill [10][11] indicating the basic convergence theorems and showing applications to some statistical problems.

III. Abelian processes

First we consider the case where the variables X_i all commute. If X_i takes two values $\{\alpha, \beta\}$, then you can write

$$\prod(1+vX_j) = (1+\alpha v)^{(X-N\beta)/(\alpha-\beta)}(1+\beta v)^{(X-N\alpha)/(\beta-\alpha)}$$

If X_i takes values $\{\alpha_1, \ldots, \alpha_r\}$, use the Lagrange interpolation formula as follows. Write

$$p(X) = \prod(X-\alpha_i)$$

Denote by n_i the number of times the value α_i is taken, $\sum n_i = N$. Observe that

$$\sum_i \frac{p(X_j)}{p'(\alpha_i)(X_j-\alpha_i)} = 1$$

while

$$n_i = \sum_j \frac{p(X_j)}{p'(\alpha_i)(X_j-\alpha_i)}$$

Thus

$$\prod(1+vX_j) = \prod_i (1+\alpha_i v)^{\sum_j \frac{p(X_j)}{p'(\alpha_i)(X_j-\alpha_i)}}$$

If X_i take three values, $\{\alpha, \beta, \gamma\}$, then, e.g.,

$$n_\gamma = \frac{\sum X_j^2 - (\alpha+\beta)\sum X_j + \alpha\beta N}{\gamma^2 - (\alpha+\beta)\gamma + \alpha\beta}$$

which depends on the power sums $Y = \sum X_j^2$ and $X = \sum X_j$.

In general, we can write, a denoting the values taken by the X_i,

$$\begin{aligned}\prod(1+vX_j) &= \prod_a (1+va)^{\sum_j \delta(X_j - a)} = \prod_a (1+va)^{n_a} \\ &= e^{\sum_a n_a \log(1+va)} = e^{N\sum \frac{n_a}{N}\log(1+va)} \\ &\sim e^{N\int \log(1+va)\, f(a)\, da}\end{aligned}$$

where f is the density function for the distribution of X_j.

Another approach is to use power sums:

$$\prod(1+vX_j) = e^{\sum \log(1+vX_j)} = e^{\sum_j \sum_k (-1)^k X_j^k / k}$$

This form is convenient for evaluating asymptotic behavior.

Remark. This formulation is already interesting in the case of independent random variables, where it gives a class of basic orthogonal functionals of the process, see [7]. For general semimartingales, it gives the exponential martingale of Doleans-Dade ([11]).

IV. Operator calculus and formal series

Given a finite number of non-commuting indeterminates $\{X_1, \ldots, X_d\}$, one can consider formal series in monomials they generate. Assuming that multiplication is associative and linear with respect to an underlying set of scalars (possibly a commutative ring) one effectively has the tensor algebra. We consider the case where monomials of the form (n denoting the multi-index $(n_1, n_2, \ldots, n_d)$, $n_i \geq 0$)

$$\psi[n] = \psi[n_1, n_2, \ldots, n_d] = X_1^{n_1} \cdots X_d^{n_d}$$

are the basis for the associative algebra generated by the X_i, such as the case where they generate a finite-dimensional Lie algebra, according to the Poincaré-Birkhoff-Witt theorem (cf. [5]). A formal series of interest is of the form

$$\sum_n c[n]\, \psi[n]$$

Denote the basic multi-index having a single 1 in position i and zeros elsewhere by $\mathbf{e}_i$, so that $n = \sum n_i \mathbf{e}_i$.

The operator calculus on these series is given on the basis $\psi[n]$ by the *boson operators*, which we denote by $\mathcal{R}_i$ and $\mathcal{V}_i$:

$$\mathcal{R}_i\psi[n] = \psi[n + \mathbf{e}_i]\,, \quad \mathcal{V}_i\psi[n] = n_i\psi[n - \mathbf{e}_i]$$

The vector space, say over $\mathbf{C}$, generated by the action of the operators $\mathcal{R}_i$ acting on $\psi[0]$ is called in physics terminology the *boson Fock space* (usually considered in the case where there are a countably infinite number of variables). These operators satisfy the commutation relations

$$[\mathcal{V}_i, \mathcal{R}_j] = \delta_{ij}I$$

where I denotes the identity operator. The idea is to use these operators to represent the action of left (or right) multiplication, in the associative algebra, by the basis elements X_i, and hence to write the algebra in terms of these operators acting on $\psi[0]$, often denoted by Ω, and called the vacuum state.

A basic fact is that any matrix Lie algebra has a boson realization in terms of the *Jordan map,* namely, we have the Lie isomorphism

$$A = (A_{ij}) \quad \leftrightarrow \quad \sum_{\alpha,\beta} \mathcal{R}_\alpha A_{\alpha\beta}\mathcal{V}_\beta \tag{4.1}$$

as is readily verified. Another way to interpret this is to use the natural correspondence

$$\mathcal{R}_i \leftrightarrow X_i\,, \qquad \mathcal{V}_i \leftrightarrow \frac{\partial}{\partial x_i}$$

acting on smooth functions $f(x_1,\ldots,x_d)$ with $X_if(x_1,\ldots,x_d) = x_if(x_1,\ldots,x_d)$. In this case, the Jordan map gives a realization of matrix algebra as an algebra of vector fields.

For finite-dimensional Lie algebras, using duality for the universal enveloping algebra. one can compute representations for the algebra. This is explained in detail in the next section. (One can find representations of quotients of the enveloping algebra and of the group as well. See [6].)

V. Dual representations

The "splitting technique" which is basic to the approach was developed from a different point of view by Wei and Norman [16][17] in the sixties. Here we start from a choice of basis for the Lie algebra considered as generators for the universal enveloping algebra. In general, a basic feature is a factorization of the Lie group into subgroups.

Remark. We denote partial derivatives by subscripting ∂, e.g., $\partial_A = \partial/\partial A$.

5.1 Representations on enveloping algebras

Let $\{\xi_1,\ldots,\xi_d\}$ be a basis for a Lie algebra. Let $\mathcal{A}$ be the corresponding universal enveloping algebra. Denote the d-tuple $(\xi_1,\ldots,\xi_d)$ by ξ. As basis for $\mathcal{A}$ and as basis for polynomials in commuting variables $A=(A_1,\ldots,A_d)$ we use

$$\psi_n(\xi)=\xi_1^{n_1}\cdots\xi_d^{n_d},\qquad c_n(A)=A_1^{n_1}\cdots A_d^{n_d}/(n_1!\cdots n_d!)$$

respectively. Note that products involving ξ_j are ordered.

The elements of the group near the identity may be expressed as products of one-parameter subgroups generated by the ξ_i. I.e., let

$$g(A,\xi)=e^{A_1\xi_1}\,e^{A_2\xi_2}\,\ldots\,e^{A_d\xi_d}$$

This may be expanded in the form

$$g(A,\xi)=\sum c_n(A)\,\psi_n(\xi)$$

and interpreted variously as:

1) a generating function for the $\{\psi_n\}$,

2) a generating function for the $\{c_n\}$ with non-commutative variables as coefficients, and

3) as a pairing $\langle\mathcal{C},\Psi\rangle$ of the sequences $\mathcal{C}=\{c_n\}$, $\Psi=\{\psi_n\}$.

By duality we have a Lie homomorphism $\xi_i\to\xi_i^*$ which is determined on the basis elements via

$$\langle\mathcal{C},\Psi\xi_j\rangle=\langle\xi_j^*\mathcal{C},\Psi\rangle$$

where $\Psi\xi_j$ denotes the sequence with components $\{\psi_n\xi_j\}$. The right action by multiplication of ξ_j on a basis element ψ_n is re-expressed in terms of the ψ's. The action may be calculated using the generating functions $g(A,\xi)$:

$$g(A,\xi)\xi_j=\xi_j^*g(A,\xi)$$

where ξ_j^* is a differential operator acting on functions of A. To see this, denote $e^{A_i\xi_i}$ by E_i. Then we use the relations

$$[E_kE_{k+1}\cdots E_l,\xi_j]=\sum_r E_kE_{k+1}\cdots[E_r,\xi_j]E_{r+1}\cdots E_l$$

$$E_k\xi_j=(e^{A_k\operatorname{ad}\xi_k}\,\xi_j)\,E_k$$

where $(\operatorname{ad}\xi_k)\xi_j=[\xi_k,\xi_j]$. The idea is to commute ξ_j next to the factor E_j in $E_1\cdots E_d$. Each appearance of ξ_iE_i is replaced by the differential operator $\partial_{A_i}=\partial/\partial A_i$. This effectively computes ξ_j^*, giving a realization of the Lie algebra as a Lie algebra of vector fields, which we call the *right dual* representation.

Similarly, we have a Lie anti-homomorphism, the *left dual*, $\xi_j\to\xi_j^\dagger$, by acting on the left

$$\langle\mathcal{C},\xi_j\Psi\rangle=\langle\xi_j^\dagger\mathcal{C},\Psi\rangle$$

5.2 Coordinates of the second kind

Group elements in a neighborhood of the identity can be expressed as

$$g(\alpha) = e^{\alpha_\mu \xi_\mu} = g(A) = e^{A_1\xi_1}\, e^{A_2\xi_2} \cdots e^{A_d\xi_d} \tag{5.2.1}$$

The α_i are called *coordinates of the first kind* and the A_i, *coordinates of the second kind*. The *dual representations* denote realizations of the Lie algebra as vector fields in terms of the coordinates of the second kind acting on the left or right respectively. I.e., define the left (respectively, right) *principal matrices*, $\pi^\dagger(A)$ (respectively, $\pi^*(A)$) according to:

$$\xi_j\, g(A) = \pi^\dagger_{j\mu}(A)\partial_\mu g(A)\,, \qquad g(A)\xi_j = \pi^*_{j\mu}(A)\partial_\mu g(A)$$

where here and in the following $\partial_\mu = \partial/\partial A_\mu$. We write the *dual representations:*

$$\xi^\dagger_j = \pi^\dagger_{j\mu}(A)\partial_\mu\,, \qquad \xi^*_j = \pi^*_{j\mu}(A)\partial_\mu$$

If A depends on a parameter s, then we have, for any function $f(A)$, the flow

$$\dot f = \dot A_\mu \partial_\mu f$$

So, let $X = \alpha_\mu\xi_\mu$ and consider group elements

$$g(A(s)) = e^{sX} = e^{s\alpha_\mu\xi_\mu} \tag{5.2.2}$$

These form a one-parameter abelian subgroup. First, we have, acting by X on the left,

$$\dot g = Xg = \alpha_\mu\xi_\mu\, g = \alpha_\mu\xi^\dagger_\mu g = \alpha_\mu\pi^\dagger_{j\mu}\partial_\mu g$$

And from the right,

$$\dot g = gX = g\,\alpha_\mu\xi_\mu = \alpha_\mu\xi^*_\mu g = \alpha_\mu\pi^*_{j\mu}\partial_\mu g$$

Since, as remarked above, we have in general $\dot g = \dot A_\mu\partial_\mu g$, we see the result:

5.2.1 Lemma. *Splitting Lemma*
Denote by $\pi(A)$ either the left or the right principal matrices. Then we have

$$\dot A_k = \alpha_\lambda \pi_{\lambda k}(A)$$

with initial values $A_k(0) = 0$.

(These equations are a constant-coefficient version of the basic equations studied by Fliess [8] [9].) In particular, evaluating at $s = 1$ gives the coordinate transformation $A = A(\alpha)$ corresponding to (5.2.1). With nonzero initial conditions, this yields the group law, equivalently, the matrix elements for the group, which we have shown how to calculate recursively in [6].

Differentiating (5.2.1) directly, we have

$$Xg = \sum_i e^{A_1\xi_1} \cdots e^{A_{i-1}\xi_{i-1}} \dot{A}_i \xi_i e^{A_i\xi_i} \cdots e^{A_d\xi_d} \tag{5.3.1}$$

Evaluating at $s = 0$ in (5.3.1) yields, with $g(0)$ the identity,

$$X = \alpha_\mu \xi_\mu = \dot{A}_\mu(0)\xi_\mu$$

Thus, $\dot{A}_k(0) = \alpha_k$, i.e.,

$$\dot{A}(0) = \alpha$$

And letting $s = 0$ in Lemma 5.2.1, we thus have

$$\pi(0) = \text{identity}$$

The right dual mapping $\xi \to \xi^*$ gives a Lie homomorphism, i.e., $[\xi_i, \xi_j]^* = [\xi_i^*, \xi_j^*]$, while the action on the left reverses the order of operations, giving a Lie antihomomorphism $[\xi_i, \xi_j]^\dagger = [\xi_j^\dagger, \xi_i^\dagger]$. In terms of the adjoint representation, we thus have, for the Lie bracket of the corresponding vector fields:

Right dual Taking commutators and then evaluating at $A = 0$:

$$\begin{aligned} \pi^*_{i\mu}\partial_\mu \pi^*_{jk} - \pi^*_{j\mu}\partial_\mu \pi^*_{ik} &= c^\mu_{ij}\pi^*_{\mu k} \\ \partial_i \pi^*_{jk}(0) - \partial_j \pi^*_{ik}(0) &= c^k_{ij} \end{aligned} \tag{5.3.2}$$

Left dual Similarly, we have

$$\begin{aligned} \pi^\dagger_{i\mu}\partial_\mu \pi^\dagger_{jk} - \pi^\dagger_{j\mu}\partial_\mu \pi^\dagger_{ik} &= c^\mu_{ji}\pi^\dagger_{\mu k} \\ \partial_i \pi^\dagger_{jk}(0) - \partial_j \pi^\dagger_{ik}(0) &= c^k_{ji} \end{aligned} \tag{5.3.3}$$

As well, we see that by construction, the left and right actions commute, so that we have

Combined Commuting the left and right yields

$$\pi^*_{i\mu}\partial_\mu \pi^\dagger_{jk} = \pi^\dagger_{j\mu}\partial_\mu \pi^*_{ik}$$

$$\partial_i \pi^*_{jk}(0) = \partial_j \pi^\dagger_{ik}(0)$$

Combining this last relation with (5.3.2), we have

$$\partial_i \pi^*_{jk}(0) - \partial_i \pi^\dagger_{jk}(0) = c^k_{ij}$$

the transpose of the adjoint representation. This can be summarized in the phrase: the transposed adjoint representation is the linearization of the difference between the right and left duals. We thus define the *extended adjoint representation* as the difference of the right and left duals:

$$\tilde{\xi}_j = \xi^*_j - \xi^\dagger_j$$

which gives a representation of the Lie algebra, since $\xi_j \to -\xi^\dagger_j$ gives a Lie homomorphism, the minus sign reversing the commutators. And we set the corresponding $\tilde{\pi}$:

$$\tilde{\pi}(A) = \pi^*(A) - \pi^\dagger(A) \tag{5.3.4}$$

Expanding out equation (5.2.1) in series, we have

$$g(A) = \sum_{n_1,\ldots,n_d} \frac{A_1^{n_1}\cdots A_d^{n_d}}{n_1!\cdots n_d!}\,\xi_1^{n_1}\cdots\xi_d^{n_d} = \sum_n \frac{A^n}{n!}\,\psi_n \tag{5.4.1}$$

using multi-index notation, $n = (n_1, \ldots, n_d)$. I.e., the group element is the generating function for the monomials ψ_n. Now we observe the action of the operators of multiplication by A_j and ∂_j, differentiation with respect to A_j on $g(A)$ dualized to acting on the basis ψ_n: $A_j g(A)$ is the same as mapping $\psi_n \to n_j\psi_{n-\mathbf{e}_j}$, and $\partial_j g(A)$ is the same as mapping $\psi_n \to \psi_{n+\mathbf{e}_j}$. As in §IV, we define operators $\mathcal{R}_j$, $\mathcal{V}_j$, on ψ_n, thus

$$\mathcal{R}_j\psi_n = \psi_{n+\mathbf{e}_j}\,, \qquad \mathcal{V}_j\psi_n = n_j\psi_{n-\mathbf{e}_j}$$

Now we can take the right and left duals acting on functions of A and convert them to operators acting on the enveloping algebra, and 'functions of ξ,' e.g., exponential functions, and by extension to Fourier integrals. Thus, the left and right *double duals*

$$\hat{\xi}_j = \mathcal{R}_\mu \pi^\dagger_{j\mu}(\mathcal{V})\,, \quad \hat{\xi}^*_j = \mathcal{R}_\mu \pi^*_{j\mu}(\mathcal{V})$$

where we drop the dagger for the left double dual and just call it the 'double dual.' For the left action, the double dual thus gives a Lie homomorphism. These give the left and right multiplication by the basis elements ξ_j on the enveloping algebra in terms of the basis ψ_n. I.e.,

$$\xi_j\psi_n = \hat{\xi}_j\psi_n, \qquad \psi_n\xi_j = \hat{\xi}^*_j\psi_n$$

We set

$$\hat{\pi} = (\pi^\dagger)^t, \qquad \hat{\pi}^* = (\pi^*)^t$$

so that

$$\hat{\xi}_j = \mathcal{R}_\mu\hat{\pi}_{\mu j}(\mathcal{V}), \qquad \hat{\xi}^*_j = R_\mu\hat{\pi}^*_{\mu j}(\mathcal{V})$$

We can formulate this in terms of vector fields. We apply the algebraic version of Fourier transform, interchanging variables A with their derivatives ∂_A, cf., the duality $A \leftrightarrow \mathcal{V}$, $\partial_A \leftrightarrow \mathcal{R}$, to the left dual. We use $(y_1, \ldots, y_d)$ as variables with the corresponding meaning $\partial_j = \partial/\partial y_j$. For polynomials f, we have, as well as $[\partial_j, f(y_j)] = f'(y_j)$,

$$[f(\partial_j), y_j] = f'(\partial_j)$$

And similarly for $f(\partial_1, \ldots, \partial_d)$. These extend directly to smooth functions. Now, we form dual to a vector field $X = a_\mu(y)\partial_\mu$

$$\hat{X} = y_\mu a_\mu(\partial)$$

We have, with $Y = y_\mu b_\mu(\partial)$, subscripts preceded by a comma denoting partial derivatives,

$$\begin{aligned}[\hat{Y}, \hat{X}] &= [y_\mu b_\mu(\partial), y_\mu a_\mu(\partial)] \\ &= y_\mu(a_\lambda(\partial)b_{\mu,\lambda}(\partial) - b_\lambda(\partial)a_{\mu,\lambda}(\partial)) \\ &= [X, Y]^\wedge\end{aligned}$$

For $\xi^\dagger_j = \pi^\dagger_{j\mu}\partial_\mu$, we have, with $\mathbf{y}$ as a row vector multiplying from the left,

$$\hat{\xi} = \mathbf{y}\hat{\pi}(\partial)$$

5.4.1 *Orbits for general Hamiltonians*

We can use the double duals to find 'Hamilton's equations' for the Lie algebra. Let $H(\xi)$ be a function on $\mathcal{G}$, given as a formal series of monomials ψ_n. We want to solve

$$\frac{\partial u}{\partial t} = [H, u]$$

for functions of ξ. Consider $u(0) = \xi_j$. We have

$$\begin{aligned}\dot{\xi}_j = [H, \xi_j] = H\xi_j - \xi_j H &= (\hat{\xi}_j^* - \hat{\xi}_j)H \\ &= \mathcal{R}_\mu \tilde{\pi}_{j\mu}(\mathcal{V})H\end{aligned}$$

cf., equation (5.3.4). Note that this involves exponentiation of the difference between the right and left duals, which *commute*, so that one can exponentiate them separately, then multiply the results together.

5.4.2 *Coadjoint orbits*

For calculating the coadjoint orbits, or effectively what is the same, to calculate the exponential of the adjoint representation, the matrices π are sufficient. Denote the matrices of the adjoint representation in the basis ξ_i by $\check{\xi}_i$. Define $\check{\pi}$ to be the matrix of the group element g given by exponentiating the adjoint representation. Then we have,

5.4.2.1 Theorem. *The exponential of the adjoint representation, $g(A, \check{\xi})$ is given by the relation*

$$\hat{\pi}^* = \hat{\pi} g(A, \check{\xi})$$

i.e.,

$$\check{\pi} = \hat{\pi}^{-1}\hat{\pi}^*$$

Proof: Start with

$$g\xi_j = \xi_j^* g = \gamma_{\mu j}\xi_\mu g$$

for some matrix γ. We know that such exists, since

$$\gamma_{\mu j}\xi_\mu = g\xi_j g^{-1} = g(A, \operatorname{ad}\xi)(\xi_j)$$

On the other hand

$$\gamma_{\mu j}\xi_\mu g = \gamma_{\mu j}\xi_\mu^\dagger g$$

so that $\gamma_{\mu j}\xi_\mu^\dagger = \xi_j^*$. Or,

$$\gamma_{\mu j}\pi_{\mu\varepsilon}^\dagger \partial_\varepsilon = \pi_{j\varepsilon}^* \partial_\varepsilon$$

i.e., $\pi^* = \gamma^t \pi^\dagger$ and the result follows upon taking transposes. ∎

Using the double dual representation, we have the form of the product $\prod(1+vX_j)$ acting on the enveloping algebra as

$$\left[\prod(1+v\hat{X}_j)\right]\psi_n$$

and similarly for acting on a formal series as in §IV. One can consider limit theorems for increments by suitably scaling the X_j. For example, if there are dilation automorphisms of the Lie algebra, then these can be implemented and then a limit of the corresponding product taken.

VI. Example of representations on enveloping algebras

Consider the 3-step nilpotent Lie algebra (cf. [9]) generated by the operators d/dx and $X^2/2$, acting on smooth functions $f(x)$. Identifying these two operators as ξ_4 and ξ_3 respectively, we can formulate the corresponding abstract Lie algebra with commutation relations

$$[\xi_4,\xi_3]=\xi_2\,,\quad [\xi_4,\xi_2]=\xi_1$$

with other commutators (among the basis elements) zero. A matrix realization is given by

$$X=\alpha\xi_1+\beta\xi_2+\gamma\xi_3+\delta\xi_4=\begin{pmatrix}0&\delta&0&\alpha\\0&0&\delta&\beta\\0&0&0&\gamma\\0&0&0&0\end{pmatrix}$$

with corresponding Jordan map, cf. (4.1),

$$\xi_1=\mathcal{R}_1\mathcal{V}_4\,,\quad \xi_2=\mathcal{R}_2\mathcal{V}_4\,,\quad \xi_3=\mathcal{R}_3\mathcal{V}_4\,,\quad \xi_4=\mathcal{R}_1\mathcal{V}_2+\mathcal{R}_2\mathcal{V}_3$$

At this point one can compute directly with matrices. Calculate $g(A,B,C,D;\xi)$ as the product of exponentials of the corresponding matrices ξ_i as follows

$$g=\begin{pmatrix}1&D&D^2/2&A\\0&1&D&B\\0&0&1&C\\0&0&0&1\end{pmatrix}$$

A direct exponentiation of X gives

$$\begin{pmatrix}1&\delta&\delta^2/2&\delta^2\gamma/6+\delta\beta/2+\alpha\\0&1&\delta&\delta\gamma/2+\beta\\0&0&1&\gamma\\0&0&0&1\end{pmatrix}$$

which yields the coordinates of the second kind directly:

$$A=\alpha+\tfrac{1}{2}\beta\delta+\tfrac{1}{6}\gamma\delta^2,\quad B=\beta+\tfrac{1}{2}\gamma\delta,\quad C=\gamma,\quad D=\delta$$

As well, one can calculate with the adjoint representation. Or one can compare Xg with $\dot{g}$ and use the splitting lemma. The resulting dual representations are given by the following table

	$\dagger$	$*$	$\hat{}$
ξ_1	∂_A	∂_A	y_1
ξ_2	∂_B	$\partial_B + D\partial_A$	y_2
ξ_3	∂_C	$\partial_C + D\partial_B + (D^2/2)\partial_A$	y_3
ξ_4	$\partial_D + B\partial_A + C\partial_B$	∂_D	$y_4 + y_1\partial_2 + y_2\partial_3$

Thus the π matrices are given by

$$\pi^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ B & C & 0 & 1 \end{pmatrix}, \pi^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ D & 1 & 0 & 0 \\ D^2/2 & D & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \check{\pi} = \begin{pmatrix} 1 & D & D^2/2 & -B \\ 0 & 1 & D & -C \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

VII. Concluding remarks

This approach gives a theoretical basis for explicit computation of representations of Lie algebras and Lie groups. At the same time, the operator calculus presented in this paper is well-suited for for symbolic computations. Besides applications in control theory, we are employing these methods in probability theory and stochastic analysis.

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References

1. K.T. Chen, *Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula*, Ann. of Math. 65, 1, 1957, 163–178.

2. K.T. Chen, *Iterated path integrals*, Bull. Amer. Math. Soc., 83, 1977, 831–879.

3. D. Cheng, W.P. Dayawansa and C.F. Martin, *Observability of systems on Lie groups and coset spaces*, SIAM J. Control and Optimisation, 28, 3, 1990, 570–581.

4. P.E. Crouch and M. Irving, *On finite Volterra series which admit hamiltonian realizations*, Math. Systems Theory 17, 1984, 293–318.

5. G. Duchamp and D. Krob, *Computing with P.B.W. in enveloping algebras*, report LITP 91.11, 1991.

6. P. Feinsilver and R. Schott, *Special functions and infinite-dimensional representations of Lie groups*, Math. Zeit., 203, 1990, 173–191.

7. P. Feinsilver and R. Schott, *Krawtchouk polynomials and finite probability theory*, Conference Proceedings: Probability Measures on Groups X, 129–136, Plenum Press, 1991.

8. M. Fliess, *Fonctionnelles causales non linéaires et indéterminées non commutatives*, Bull. Soc. Math. France, 109, 1981, 3–40.

9. M. Fliess and D. Normand-Cyrot, *Algèbres de Lie nilpotentes, formule de Baker-Campbell-Hausdorff et intègrales itérées de K.T. Chen*, Séminaire de Probabilités, XVI, Lect. Notes in Math., Springer-Verlag, 1982.

10. R.D. Gill and S. Johansen, *A survey of product integration with a view toward applications in survival analysis*, Ann. Stat., 18, no. 4, 1990, 1501–1555.

11. R.D. Gill, *Lectures on survival analysis*, St. Flour Lectures on Probability Theory, Lect. Notes in Math., 1581, 1994, 115–236.

12. H. Hermes, *Nilpotent and high-order approximations of vector fields systems*, SIAM REVIEW, 33, 2, 1991, 238–264.

13. B. Jakubczyk and E.D. Sontag, *Controllability of nonlinear discrete-time systems: a Lie-algebraic approach*, SIAM J. Control and Optim., 28, no. 1, 1990, 1–33.

14. H.J. Sussmann, *A general theorem on local controllability*, SIAM J. Control Optim., 25, 1987, 158–194.

15. X.G. Viennot, *Algèbres de Lie libres et monoïdes libres*, Lect. Notes in Math., 691, Springer-Verlag, 1978.

16. J. Wei and E. Norman, *On global representation of the solutions of linear differential equations as a product of exponentials*, Proc. A.M.S., 15, 1964, 327–334.

17. J. Wei and E. Norman, *Lie algebraic solution of linear differential equations*, J. Math. Phys., 4, 1963, 575–581.