Canonical Bases for the Birman-Wenzl Algebra

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Abstract

In this paper, which is a summary of work with Grojnowski [FG], we construct canonical bases for the Birman-Wenzl algebra BW_n, the q-analogue of the Brauer centralizer algebra, and so define left, right and two sided cells. We describe these objects combinatorially (generalizing the Robinson-Schensted algorithm for the symmetric group) and show that each left cell carries an irreducible representation of BW_n. In particular, we obtain canonical bases for each representation, defined over $\mathbf{Z}$. As a side effect of our techniques we give a particularly simple description of the representations of BW_n, which seems to be new. This description is independent of the results on the bases.

Résumé

Dans ce travail, nous construisons des bases de Kazhdan-Lusztig pour l'algèbre de Birman-Wenzl BW_n, et nous définissons ainsi des cellules gauches, droites, et bilatères. Nous donnons une description combinatoire de ces objets, au moyen d'une généralisation de l'algorithme de Robinson-Schensted (pour le groupe symétrique) et nous montrons que chaque cellule gauche est le support d'une représentation irréductible de BW_n. En particulier nous obtenons pour chaque représentation des bases canoniques, définies sur $\mathbf{Z}$.

1 Introduction

In the second section of this extended abstract, we define Brauer diagrams and statistics $l(d)$ and $h(d)$ on Brauer diagrams, tangles, and the Birman-Wenzl algebra BW_n. Also in the second section we define a standard basis for BW_n and an involution on BW_n. In the third section, we define the Kazhdan-Lusztig basis for BW_n. In Section 4 we define left, right, and two-sided cells, obtain the irreducible representations and characterize these irreducibles, and in Section 5 we describe a Schensted algorithm which characterizes them combinatorially. For full proofs, see [FG].

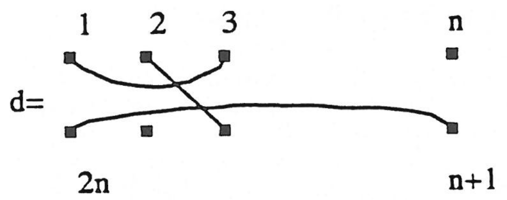
2 The Birman-Wenzl Algebra

2.1 Brauer diagrams

Let F be a finite set and R a ring. We write RF for the free R-module with basis F; so an element of RF is a map from F to R, usually denoted $\sum_{f \in F} n_f f$.

If $n \in \mathbb{N}$, write $2n!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$; and if S is a set, write $|S|$ for its cardinality.

A "Brauer diagram on n letters" is a partition of the set $\{1, \ldots, 2n\}$ into two element subsets. Write $B = B_n$ for the set of Brauer diagrams, so $|B| = 2n!!$. If $d \in B$, we represent d by a diagram in the plane where there are n dots



numbered $1, \ldots, n$ in the top row; n dots numbered $2n, \ldots, n+1$ in the bottom row, and the vertex i is joined to the vertex j if $\{i,j\} \in d$. We can draw this picture so two edges intersect at most once, there are no self-intersections, at most two edges intersect at any point, the only critical points of the functions representing the edges are the max (resp. min) of horizontal edges, etc. Call such a diagram *nice*.

If $d \in B$, write $\ell(d)$ for the number of pairs $\{i,j\},\{k,l\}$ in d such that $i < k < j < l$. In our nice diagram representing d, this is just the number of crossings of edges.

Also for $d \in B$, write $h(d)$ for the number of pairs $\{i,j\}$ in d with $i \leq n$ and $j \leq n$. This is just the number of horizontal edges in the top row of the diagram of d; clearly this is also the number of horizontal edges in the bottom row. The symmetric group on n letters, S_n, is isomorphic to the elements of $\{d \in B_n \mid h(d) = 0\}$.

2.2 Tangles

A "tangle on n letters" is an equivalence class of certain pictures in the plane with $2n$ marked vertices $1, \ldots, 2n$ [Ka]. Denote $\mathcal{T}_n$ for the set of n-tangles. A picture t in the plane with lines between the vertices $1, \ldots, 2n$ (arranged as in a Brauer diagram), with over and undercrossings indicated and with some number of closed loops, represents a tangle t. If two such pictures differ only in the neighborhood of a crossing, where they are respectively of the form

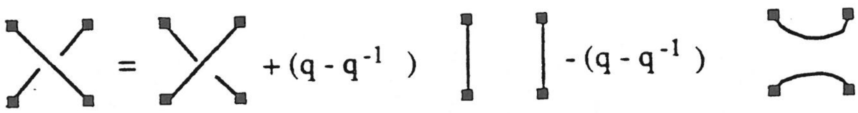
(or any diagram obtained by rotating these), then they represent the same element of $\mathcal{T}_n$; and the set of such pictures mod the equivalence relation generated by these two "Reidemeister moves" is $\mathcal{T}_n$.

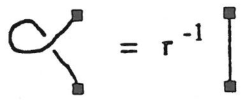
If $t_1, t_2 \in \mathcal{T}_n$, then we define $t_1 t_2$ to be the (equivalence class) of the tangle obtained by concatenating t_1 and t_2 (place t_1 above t_2 and join the dots). With this product, $\mathcal{T}_n$ is a monoid.

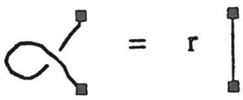
Let x, q, and r be indeterminants and let $\mathcal{A}$ be the ring

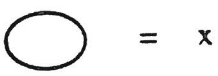
$$\mathcal{A} = \mathbf{Z}[r, r^{-1}, q, q^{-1}, x]/((1-x)(q-q^{-1}) + (r - r^{-1}))$$

and $\mathcal{A}' = \mathbf{Z}[x]$. Write $\widetilde{BW}_n$ for the quotient of $\mathcal{A}\mathcal{T}_n$ by the relations generated by

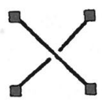
Q1 

Q2 

Q3 

Q4 

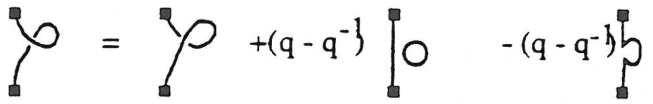
Here, by Q1 we mean that if t is a tangle with some crossing which looks like



and t' (resp. t'', t''') represents the same tangle with this crossing modified to

then $t = t' + (q - q^{-1})t'' - (q - q^{-1})t''$ in $\widetilde{BW}_n$. (These relations really do descend to $\mathcal{T}_n$). Similarly for Q2–Q4.

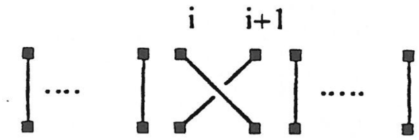
For example,



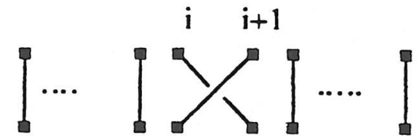
whence $(r - r^{-1}) = (q - q^{-1})(x - 1)$ by Q2,Q3, and Q4.

Define elements T_{s_i}, $T_{s_i}^{-1}$, and T_{e_i} in $\mathcal{T}_n$ by

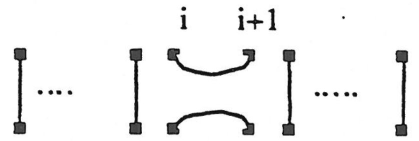
$T_{s_i} =$



$T_{s_i}^{-1} =$



$T_{e_i} =$



Define $BW = BW_n$ to be the submonoid of $\widetilde{BW_n}$ generated by T_{s_i}, $T_{s_i}^{-1}$, T_{e_i} for $1 \leq i < n$. This is a $\mathcal{A}$-algebra, the "Birman-Wenzl" algebra, and may be defined explicitly in terms of these generators and some relations. (See [BW]).

If $d \in B_n$, T_d is the picture obtained from a nice diagram for d by requiring $\{i, j\}$ to pass over $\{k, l\}$ if $i < k < j < l$. $\{T_d\}_{d \in B_n}$ is a basis for BW_n [HR], which we call the standard basis.

If t is a picture representing a tangle, write $\bar{t}$ for the picture obtained from t by interchanging every over and under crossing. It is clear that $^{-}$ respects Reidemeister moves, and so this operation on pictures descends to tangles.

Also write $^{-} : \mathcal{A} \to \mathcal{A}$ for the Z-linear ring homomorphism defined by

$$r \mapsto r^{-1}, \quad q \mapsto q^{-1}, \quad x \mapsto x.$$

This is an involution.

It is clear from Q1–Q4 that the $\mathcal{A}$-antilinear involution $^{-} : \mathcal{A}\mathcal{T}_n \to \mathcal{A}\mathcal{T}_n$, $\sum n_t t \mapsto \sum \bar{n}_t \bar{t}$ descends to an involution $^{-} : BW_n \to BW_n$. Further, $\overline{t_1 t_2} = \bar{t}_1 \bar{t}_2$ whenever we can concatenate tangles t_1 and t_2; i.e. $^{-}$ is an algebra homomorphism whenever this makes sense.

Observe that if d is a Brauer diagram,

$$\overline{T_d} = T_d + \sum_{d' : \ell(d') < \ell(d)} r_{d'd} T_{d'} \tag{1}$$

for certain $r_{d'd} \in \mathbf{Z}[(q - q^{-1})]$. This follows from Q1 by a straightforward induction.

3 Canonical Bases

We use the following lemma of [KL] to define our canonical basis for BW_n.

Lemma 1 *Let M be a free $\mathbf{Z}[q, q^{-1}]$-module, with a given basis $(e_i)_{i \in I}$, I some index set. Suppose also given a semilinear involution $^{-} : M \to M$ such that $\overline{qm} = q^{-1}\bar{m}$, $\overline{m + m'} = \bar{m} + \bar{m}'$, and a partial order $\leq$ on I such that $\{j \mid j \leq i\}$ is finite and*

$$\bar{e}_i = \sum_{j \leq i} r_{ji} e_j, \qquad r_{ji} \in \mathbf{Z}[q, q^{-1}] \text{ and } r_{ii} = 1.$$

Then there is a unique basis $(b_i)_{i \in I}$ of M such that i) $\bar{b}_i = b_i$, and

$$\text{ii) } b_i = \sum_{j \leq i} P_{ji} e_j, \qquad \text{with } P_{ii} = 1, \text{ and } P_{ji} \in q^{-1}\mathbf{Z}[q^{-1}] \text{ if } j < i.$$

This basis is called the "canonical" (or Kazhdan-Lusztig) basis of M.

We apply the lemma to BW_n, and to the involution $^{-}$, the standard basis T_d, and the partial order $d' \leq d$ if $\ell(d') < \ell(d)$ or $d = d'$. We may do this by (1). We denote the new basis by C_d

Observe that the polynomial $P_{d'd}$ are in $\mathbf{Z}[q^{-1}]$, that is they do not depend on r and x. For example, $C_{e_i} = T_{e_i}$, $C_1 = 1$, $C_{s_i} = T_{s_i} + q^{-1} - q^{-1}T_{e_i}$, $1 \leq i < n$.

4 Cells

Let h_{xyz} be the structure constants for multiplication in BW_n with respect to the canonical basis; i.e.

$$C_x C_y = \sum_{z \in B_n} h_{xyz} C_z \qquad \text{for } x, y \in B_n.$$

Let $\leq_L$ (resp. $\leq_R$) be the preorder on B_n generated by the relations $z \leq_L y$ (resp. $z \leq_R x$) if there exists an $x \in B_n$ (resp. $y \in B_n$) such that $h_{xyz} \neq 0$. Let $\leq_{LR}$ be the preorder generated by the relation $x \leq_{LR} y$ if $x \leq_L y$ or $x \leq_R y$. Write $x \sim_L y$ if $x \leq_L y$ and $y \leq_L x$; similarly for $\sim_R$, $\sim_{LR}$. The equivalence classes for $\sim_L$, $\sim_R$, $\sim_{LR}$ are called respectively left, right or two sided cells. Observe that if $x \sim_L y$, then $h(x) = h(y)$. If π_1 and π_2 are elements of S_k, write $\pi_1 \sim_{L_S} \pi_2$ if π_1 is left equivalent to π_2 as in [KL]. It turns out that if x and y are elements of B_n and $h(x) = h(y) = 0$, so that x and y may be considered elements of S_n, then $x \sim_L y$ if and only if $x \sim_{L_S} y$.

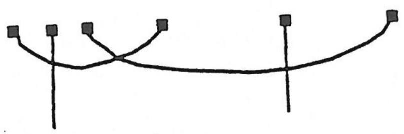
If Γ is a left cell in B_n, then if we set

$$F^\Gamma = \mathcal{A}\{C_x \mid x \leq_L \Gamma\}$$

F^Γ is a left ideal in BW_n. Write $F^{<\Gamma}$ for the sum of the $F^{\Gamma'}$ such that $\Gamma' \leq_L \Gamma$, $\Gamma \neq \Gamma'$; and write $gr^\Gamma = F^\Gamma / F^{<\Gamma}$. This is a left BW_n module. Similarly, for Γ a right or two sided cell, the analogously defined F^Γ are right (resp. two sided ideals), and gr^Γ is a right module (resp. $BW_n \times BW_n^\circ$ module).

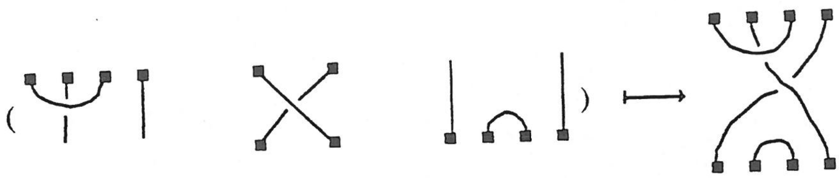
Our main result is an explicit description of the equivalence classes $\sim_L$, and hence an explicit construction of bases in the irreducible modules for BW_n with structure constants in $\mathcal{A}$. In order to describe these classes, we need to decompose tangles into *dangles* and elements of the symmetric group.

A "flat (n, k) dangle" is a subset of $\{1, \ldots, n\}$ of size $2k$, which is partitioned into k 2-element subsets. Write $D^k = D^k_n$ for the set of flat (n, k)-dangles, so $|D^k_n| = \binom{n}{2k} k!!$. If $d \in D^k$, we can represent d by a diagram in the plane



such that i is joined to j if $\{i, j\} \in d$ and there is a vertical line from i if $i \notin d$. We can insist that two edges intersect at most once, and no vertical edges intersect, etc. Elements of D^k_n represent the tops of Brauer diagrams. Define ${}^\circ D^k_n$ to be D^k_n, but draw the pictures dangling upward rather than down, and label the vertices $2n, \ldots, n+1$. These represent the bottom part of Brauer diagrams.

We define a map $D_n^k \times S_{n-2k} \times {}^{\circ}D_n^k \to B_n$, by concatenation, e.g.



Note that $D^k \times S_{n-2k} \times {}^{\circ}D^k$ bijects to $\{d \in B_n \mid h(d) = k\}$. Write $d \mapsto (\tau(d), \pi(d), \beta(d))$ for the inverse map.

Theorem 1 *We have $d \sim_L d'$ if and only if $h(d) = h(d')$, $\beta(d) = \beta(d')$, and $\pi(d) \sim_{L_S} \pi(d')$ in $S_{n-2h(d)}$. Further, if Γ and Γ' are two left cells in the same two sided cell, then gr^{Γ} is isomorphic to $gr^{\Gamma'}$ as a BW_n-module with basis. Finally, let F be a field, $\alpha : \mathcal{A} \to \mathcal{F}$ a homomorphism of rings, and suppose $BW_n \otimes_{\mathcal{A}} F$ is semisimple. Then each representation $gr^{\Gamma} \otimes_{\mathcal{A}} F$ is irreducible.*

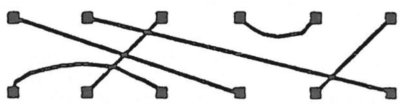
In the course of the proof of Theorem 1, we observed the following simple result, which seems to be new.

Let V be an irreducible representation of S_{n-2k}. Then one can give $\mathcal{A}D_n^k \otimes V$ the structure of an irreducible representation of BW_n in a unique way. Further, representations constructed in this way are distinct, and exhaust the representations of BW_n. For the proof of these results, see [FG].

5 Combinatorial Description of the Cells

We now describe an algorithm, due to Sundaram [S], for bijecting Brauer diagrams B_n onto pairs (p, q) of up-down paths of length n in Young's lattice. The paths p and q begin at the same shape, end in the empty partition, and each partition differs from its predecessor by one square. In this language, if $d_1, d_2 \in B_n$ and $d_1 \to (p_1, q_1)$ and $d_2 \to (p_2, q_2)$, then the first sentence of Theorem 1 translates to $d_1 \sim_L d_2$ if and only if $p_1 = p_2$. This is a generalization of the relationship between tableaux and cells for the symmetric group. [KL], [K].

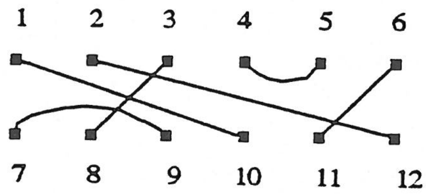
Throughout this section, let

$d_1 =$ 

Here are the steps, for $d \in B_n$ with $h(d) = k$.

1. In this section, number the top row 1 to n and the bottom row from $n+1$ to $2n$, both from left to right.

d_1 =



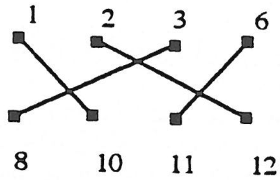
2. Let $\pi = \pi(d)$ be the permutation defined by the vertical edges of d, as in Section 4. Define two $2 \times k$ arrays L_τ and L_β. For each horizontal edge in the top row $\{i, j\}$, $1 \leq i < j \leq n$, add the column

$$\begin{pmatrix} j \\ i \end{pmatrix}$$

to L_τ. For each horizontal edge in the bottom row $\{i, j\}$, $n+1 \leq i < j \leq 2n$, add the same column to L_β

$$L_\tau(d_1) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, L_\beta(d_1) = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

$\pi(d_1)$ =



3. Use the Robinson-Schensted correspondence to obtain a pair of tableaux (P, Q) from π. The labels in P will be from the bottom row of D and the labels in Q will be from the top row of d.

$$(P(d_1), Q(d_1)) = \left(\begin{array}{|c|c|}\hline 8 & 11 \\ \hline 10 & 12 \\ \hline \end{array} \, , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 6 \\ \hline \end{array} \right)$$

4. Using $P(d)$ and L_β, we proceed inductively as follows to build a path $p(d) = (\lambda^n, \lambda^{n-1}, \ldots, \lambda^0)$ in Young's lattice. Let $P_n = P(d)$ and let λ^n be the shape of $P(d)$. Suppose we have the standard Young tableau P_j of shape λ^j at the jth step. If $n+j$ is a label of a square in P_j, delete that square. We now have the standard Young tableau P_{j-1} of shape λ^{j-1}. If $n+j$ is not a label in P_j, then it appears in the top row of $L_\beta(d)$, in a column

$$\binom{j}{i}$$

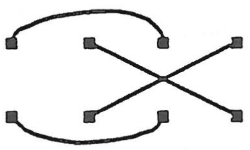
with $i < j$. In this case, column insert $n+i$ into P_j to obtain P_{j-1} of shape λ^{j-1}.

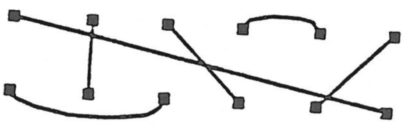
i	6	5	4	3	2	1	0
P_i	8 11 / 10 12	8 11 / 10	8 / 10	8	7 8	7	
λ^i	(2,2)	(2,1)	(1,1)	(1)	(2)	(1)	()

5. Using $Q(d)$ and $L_\tau(d)$, follow the same procedure as above, replacing P with Q, β with τ, $n+j$ with j, and $n+i$ with i.

$$q(d_1) = ((2,2),(2,1),(2,1,1),(2,1),(2),(1),\emptyset)$$

If

$d_2 =$  and

$d_3 =$ 

then,

$$p(d_2) \quad = \quad p(d_1)$$

$$q(d_2) = ((2,2),(2,1),(1,1),(2),(1),\emptyset)$$
$$p(d_3) = ((3,1),(3),(2),(1),(2),(1),\emptyset)$$
$$q(d_3) = ((3,1),(2,1),(2,1,1),(2,1),(1,1),(1),\emptyset)$$

So we see that $d_1 \sim_L d_2$ and that d_3 is in a different two-sided cell from d_1 and d_2.

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