Noncommutative Symmetric Functions and the Chromatic Polynomial

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Abstract

In a recent preprint, R. P. Stanley defines a symmetric function in commuting indeterminates, X_G where G is a finite graph, which generalizes the chromatic polynomial of G. In my work, I consider a *noncommutative* analogue, Y_G, which becomes X_G when the variables are allowed to commute. The advantage of Y_G is that it satisfies a version of the deletion-contraction rule, while X_G does not. Using this property and induction, we can express Y_G in terms of various bases for the ring of non-commutative symmetric functions. Letting the indeterminates commute, one recovers the corresponding results of Stanley in a uniform manner. An example for the power sum symmetric functions is provided.

Résumé

Stanley a défini une fonction symétrique en indéterminants commutatifs, $X(G)$, où G est un graphe, qui généralise le polynôme chromatique de G. Nous étudions ici un analogue noncommutatif, $Y(G)$, se réduisant à $X(G)$, lorsque les variables commutent. L'avantage de $Y(G)$ est que cette fonction vérifie une loi de contraction-suppression, contrairement à $X(G)$. Utilisant cette propriété et par induction nous pouvons exprimer $Y(G)$ en termes de bases diverses de l'anneau des fonctions symétriques noncommutatives. En laissant les indéterminants commuter, on retrouve les résultats correspondants de Stanley d'une manière plus uniforme. Nous donnons un example pour la base des fonctions symétriques sommes de puissance.

In "A Symmetric Function Generalization of the Chromatic Polynomial of a Graph" [2] (see also [1]), R. P. Stanley introduces a symmetric function, X_G, associated with a labelled (loopless) graph on d vertices as follows: Let G have vertex set $V(G) = \{v_1, v_2, \ldots, v_d\}$. Then for the (commuting) indeterminates $x_1, x_2, \ldots$ define a homogeneous function of degree d,

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_d)} = \sum_{\kappa} x_{\kappa}$$

where the sum ranges over all proper colorings, $\kappa : V(G) \to \mathbf{Z}^+$. It is clear from the definition that X_G is a symmetric funtion. It is also a generalization of the chromatic polynomial, $\mathcal{X}_G(n)$, since setting $x_1 = x_2 = \ldots = x_n = 1$ and $x_i = 0$ for all $i > n$ in X_G yields $\mathcal{X}_G(n)$. Stanley then proceeds, by various arguments, to study the expansion of X_G in terms of several of the standard symmetric function bases. He also connects X_G with acyclic orientations and computes X_G for various specific graphs.

Since the symmetric function X_G is a generalization of $\mathcal{X}_G$, several of Stanley's results for X_G closely parallel those of Whitney [4] for the chromatic polynomial. For example:

Theorem 1 *For a finite graph, G,*

$$\mathcal{X}_G(n) = \sum_{S \subseteq E(G)} (-1)^{|S|} n^{c(S)}$$

where $c(S)$ is the number of connected components of the spanning subgraph of G with edge set S. □

Stanley's extension of this for X_G is [2, Theorem 2.5]:

Theorem 2 *For a finite graph, G,*

$$X_G = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda(S)}.$$

Here $\lambda(S)$ is the partition of d with parts equal to the sizes of the connected components of the spanning subgraph of G with edge set S. Also $p_{\lambda(S)}$ is the power sum symmetric function for $\lambda(S)$. □

Whitney's Thoerem can be proven easily using induction and the deletion-contraction property of the chromatic polynomial. To recall this rule, let $e \in E(G)$. Denoting G with e deleted by $G \setminus e$, and G with e contracted to a point by G/e. Then

$$\mathcal{X}_G(n) = \mathcal{X}_{G\setminus e}(n) - \mathcal{X}_{G/e}(n)$$

Unfortunately, Stanley's symmetric function has no such deletion-contraction property, which deprives him of induction as a tool for his proofs.

In my work, I define an analogue of X_G which is a symmetric function in *noncommutative* variables. That is, we fix a graph G and a labelling of its vertex set $V(G) = \{v_1, v_2, \ldots, v_d\}$ and define the analogue of X_G as

$$Y_G = \sum_{\kappa} x_{\kappa(v_1)}, \ldots, x_{\kappa(v_d)}$$

where again the sum is over all proper colorings of G, but the x_i are now *noncommuting* variables. If we distinguish an edge $e \in E(G)$ then we choose the labelling of the vertex set so that $e = v_{d-1}v_d$. I also define an operation $\uparrow$ on noncommutative symmetric functions which simply raises the power of the last variable in each term by one. More formally, for a monomial $x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_k}^{j_k}$, define

$$\uparrow (x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_k}^{j_k}) = x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_{k-1}}^{j_{k-1}} x_{i_k}^{j_k+1}$$

and extend linearly. With these definitions, Y_G satisfies a deletio! n-contraction relationship simila

Lemma 1 $Y_G = Y_{G\setminus e} - \uparrow Y_{G/e}$

Proof. The proof is very similar to that for the deletion-contraction property of $\mathcal{X}_G$. Consider proper colorings of $G \setminus e$. They can be split disjointly into two types:

1. proper colorings of $G \setminus e$ with vertices v_{d-1} and v_d different colors;
2. proper colorings of $G \setminus e$ with vertices v_{d-1} and v_d the same color.

Those of the first type clearly correspond to proper colorings of G. If κ is a coloring of $G \setminus e$ of the second type then, since the vertices v_{d-1} and v_d are the same color, we have

$$x_{\kappa(v_1)} \cdots x_{\kappa(v_{d-1})} x_{\kappa(v_d)} = \uparrow (x_{\kappa(v_1)} \cdots x_{\kappa(v_{d-1})}) = \uparrow x_{\tilde{\kappa}}$$

where $\tilde{\kappa}$ is a proper coloring of G/e. So those of the second type are exactly the terms of $\uparrow Y_{G/e}$, and we have that $Y_{G\setminus e} = Y_G + \uparrow Y_{G/e}$. Equivalently, $Y_G = Y_{G\setminus e} - \uparrow Y_{G/e}$. $\square$

This deletion-contraction lemma for Y_G gives us a tool for using induction. Define $\gamma = \gamma(n) = (\gamma_1/\gamma_2/\dots/\gamma_k)$ to be a partition of $\{1, 2, \dots, n\}$ if the γ_i are disjoint subsets of $\{1, 2, \dots, n\}$ whose union is $\{1, 2, \dots, n\}$. Define $p_{\gamma(n)}$ to be the symmetric function in noncommutative variables which is the sum of all monomials $x_{i_1}x_{i_2}\cdots x_{i_n}$ with the condition that if l and m are in the same block of $\gamma(n)$ then $i_l = i_m$. Note that the set $\{p_{\gamma(n)} : n \in \mathbf{Z}^+\}$ forms a basis for these symmetric functions. Further, for $S \subseteq E(G)$ let $\gamma(S)$ denote the partition of $\{1, 2, \dots, d\}$ with blocks corresponding to the connected components in the spanning subgraph of G with edge set S.

Proposition 1 *With the notation above,*

$$Y_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\gamma(S)}.$$

Proof. Induct on $|E(G)|$. If $|E(G)| = 0$, then S=ϕ and so we see $\gamma(S) = (1/2/\dots/n)$. Thus

$$\sum_{S \subseteq E} (-1)^{|S|} p_{\gamma(S)} = p_1^d,$$

which is clearly Y_G for the totally disconnected graph on d vertices. From the lemma we know that $Y_G = Y_{G\setminus e} - \uparrow Y_{G/e}$, and since we allow multiple edges when contracting e,

$$|E(G \setminus e)| = |E(G/e)| = |E(G)| - 1.$$

So we can apply induction to $Y_{G\setminus e}$ and $Y_{G/e}$. Thus

$$Y_G = \sum_{S \subseteq E(G\setminus e)} (-1)^{|S|} p_{\gamma(S)} - \uparrow \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|} p_{\gamma(\tilde{S})}.$$

But if $e \notin S$,

$$\sum_{S \subseteq E(G\setminus e)} (-1)^{|S|} p_{\gamma(S)} = \sum_{\substack{S \subseteq E(G) \\ e \notin S}} (-1)^{|S|} p_{\gamma(S)}.$$

Hence it suffices to show that if $e \in S$

$$- \uparrow \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|} p_{\gamma(\tilde{S})} = \sum_{\substack{S \subseteq E(G) \\ e \in S}} (-1)^{|S|} p_{\gamma(S)}.$$

Define a map

$$\Theta : \{\tilde{S} \subseteq E(G/e)\} \to \{S \subseteq E(G) : e \in S\} \text{ by :}$$
$$\Theta(\tilde{S}) = \tilde{S} \bigcup \{e\} = S.$$

Then Θ is a bijection, since we allow multiple edges to occur when we contract e to v_{d-1} . Also, $|\tilde{S}| + 1 = |S|$, and

$$\gamma(S) = (\gamma_1(S)/\ldots/\gamma_{k-1}(S)/\gamma_k(S)) = (\gamma_1(\tilde{S})/\ldots/\gamma_{k-1}(\tilde{S})/\gamma_k(\tilde{S}) \cup \{d\})$$

letting γ_k contain $\{d-1\}$. Hence for $e \in S$

$$\begin{aligned} -\uparrow \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|} p_{\gamma(\tilde{S})} &= \uparrow \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|+1} p_{\gamma(\tilde{S})} \\ &= \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|+1} p_{(\gamma_1(\tilde{S})/\ldots/\gamma_k(\tilde{S}) \cup \{d\})} \\ &= \sum_{\substack{S \subseteq E(G) \\ e \in S}} (-1)^{|S|} p_{\gamma(S)}. \end{aligned}$$

This completes the proof. □

By letting the x_i commute, we then obtain Stanley's theorem as an easy corollary. There are other expansions for Y_G in the standard bases which may also be obtained by applying induction and the deletion-contraction property. This provides a uniform approach to obtaining some of Stanley's results for X_G.

References

[1] R. P. Stanley, Graph Colorings and Related Symmetric Functions: Ideas and Applications, preprint.

[2] R. P. Stanley, A Symmetric Function Generalization of the Chromatic Polynomial of a Graph, preprint.

[3] W. T. Tutte, A Contribution to the Theory of Chromatic Polynomials, *Canad. J. Math.* **6** (1953), 80–91.

[4] H. Whitney, A Logical Expansion in Mathematics, *Bull. Amer. Mth. Soc.* **38** (1932), 572–579.