Compositions and q-rook Polynomials

by

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Abstract: Several identities involving compositions of vectors and q-rook polynomials are derived. Applications include some new results on Rawlings $(q-r)$ Simon Newcomb Problem, and a new recurrence relation for q-rook polynomials. A more general form of this recurrence occurs when studying a two variable rook polynomial, with connections to hypergeometric series.

Résumé: On établit plusieurs identités faisant intervenir des compositions vectorielles et les q-polynômes de tours. Les applications comprennent de nouveaux résultats sur le $(q-r)$ Problème de Simon Newcomb, de Rawlings, et une nouvelle relation de récurrence pour les q-polynômes de tours. Une forme plus générale de cette réurrence apparaît lorsqu'on étudie un polynôme de tours à deux variables, relié aux séries hypergéométriques.

1. Introduction.

For a given vector $\mathbf{v} \in \mathbf{N}^t$, let $f_k(\mathbf{v})$ be the number of *compositions* of $\mathbf{v}$ into k parts, i.e.

$$f_k(\mathbf{v}) := \sum_{\mathbf{w_1}+\ldots+\mathbf{w_k}=\mathbf{v}} 1 \qquad \mathbf{w_i} \in \mathbf{N}^t, \mathbf{w_i} \neq \mathbf{0}.$$

For example, $f_2(2,1) = 4$ since $(2,1) = (2,0)+(0,1) = (0,1)+(2,0) = (1,1)+(1,0) = (1,0)+(1,1)$. MacMahon showed that this function is closely related to *Simon Newcomb's Problem*, which asks for the number of permutations of a multiset with a specified number of descents. For the multiset where i occurs v_i times, let $N_k(\mathbf{v})$ denote the number of multiset permutations with exactly $k-1$ descents. MacMahon proved [Ma1]

$$\sum_k f_k(\mathbf{v}) z^{n-k} = \sum_k N_k(\mathbf{v})(z+1)^{n-k} \qquad n = v_1 + \ldots + v_t, \text{ and} \tag{1}$$

$$\sum_{k\geq 1} \binom{x}{k} f_k(\mathbf{v}) = \prod_i \binom{x+v_i-1}{v_i}. \tag{2}$$

In previous work the author showed that compositions can be studied using *rook theory*. A *board* B is a subset of an $n \times n$ chessboard of squares. Let $r_k(B)$ be the number of ways of placing k non-attacking rooks (no two in the same row or column) on B, and let $a_k(B)$ be the number of placements of n non-attacking rooks on the $n \times n$ chessboard, with exactly $n-k$ on B. Then

$$f_k(\mathbf{v}) = k! r_{n-k}(B_{\mathbf{v}})/\Pi_i v_i! \text{ and}$$

$$N_k(\mathbf{v}) = a_k(B_{\mathbf{v}})/\Pi_i v_i! \qquad \text{[Ha1], [Ha2],}$$

where $B_{\mathbf{v}}$ is a certain board, easily described in terms of the coordinates of $\mathbf{v}$ (in the notation of Figure 1, $B_{\mathbf{v}} = B(v_1-1, v_1; v_2, v_2; \ldots; v_t, v_t)$). Equations (1) and (2) can then be shown to follow from the two classical results

$$\sum_{k=0}^{n} r_k(B)(n-k)!(z-1)^k = \sum_{k=0}^{n} z^k a_{n-k}(B) \qquad \text{[K-R], and} \tag{3}$$

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k}(B) = \prod_{i=1}^{n}(x+c_i-i+1) \qquad \text{[GJW].} \tag{4}$$

In (4), it is assumed that B is a special type of board called a *Ferrers board*, with c_i squares in the i^{th} column.

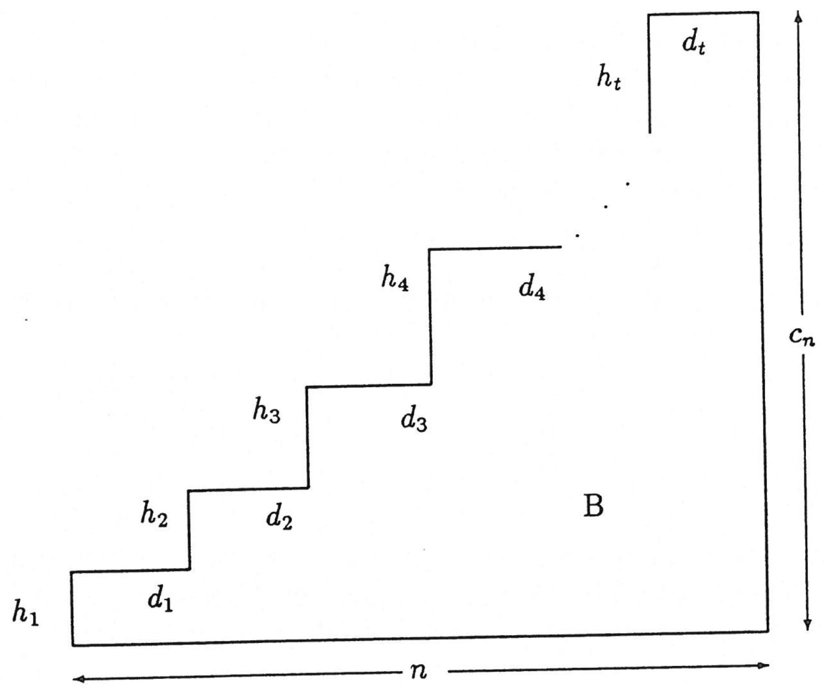

Figure 1: The Ferrers board $B = B(h_1, d_1; h_2, d_2; \ldots; h_t, d_t)$.
The first d_1 columns have height h_1, the next d_2 columns have height $h_1 + h_2$, etc.

2. q-versions.

For Ferrers boards, Garsia and Remmel [G-R] found q-versions of (3) and (4), namely

$$\sum_{k=0}^{n}[k]!R_{n-k}(B)z^k \prod_{i=k+1}^{n}(1-zq^i) = \sum_{k=0}^{n} z^k A_k(B), \text{ and} \tag{5}$$

$$\sum_{k=0}^{n}[x][x-1]\cdots[x-k+1]R_{n-k}(B) = \prod_{i=1}^{n}[x+c_i-i+1]. \tag{6}$$

Here $[x] := (1-q^x)/(1-q)$ for any real x, $[k]! := [1][2]\cdots[k]$, and $R_k(B) := \sum_C q^{inv(C)}$, with the sum over all placements C of k non-attacking rooks on B, and inv(C) a statistic associated to C. The polynomials $A_k(B)$ reduce to $a_k(B)$ when $q=1$. Garsia and Remmel proved these polynomials have nonnegative integral coefficients, and in [Hal] their proof was extended to show $A_k(B)$ is also symmetric and unimodal.

The author originally noticed that if we define $f_k[\mathbf{v}]$ by taking q-versions of the identity $f_k(\mathbf{v}) = k!r_{n-k}(B_{\mathbf{v}})/\Pi_i v_i!$, i.e.

$$f_k[\mathbf{v}] := [k]!R_{n-k}(B_{\mathbf{v}})/\Pi_i[v_i]!,$$

then $f_k[\mathbf{v}]$ appeared to be a polynomial in q. The question naturally arose as to whether or not $f_k[\mathbf{v}]$ can be written as a sum over compositions as follows;

$$f_k[\mathbf{v}] = \sum_{\mathbf{w_1}+\ldots+\mathbf{w_k}=\mathbf{v}} q^{\beta(\mathbf{w_1},\ldots,\mathbf{w_k})}$$

for some statistic β. The solution to this question builds on a construction originally due to Cheema and Motzkin [C-M] which in modified form has previously found application to questions involving partitions of vectors [G-G], [Gor]. Given a sequence of vectors $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}$, Cheema and Motzkin construct a sequence of permutations $\pi_1, \pi_2, \ldots, \pi_t$ as follows; let M be the matrix whose i^{th} row, j^{th} column contains w_{ij}. Let π_1 be the permutation of the rows of M needed to sort the first column of M into non-increasing order, with two given rows not permuted with respect to each other if they have the same first column entry. Call this new matrix M_1. Now do the same procedure to the second column, letting π_2 denote the permutation of the rows of M_1 needed to put the second column in non-increasing order, where two rows with the same second column entry are not permuted with each other. If our vectors have t coordinates, we end up in this way with t permutations $\pi_1, \ldots, \pi_t$. Letting $\mathrm{inv}\pi_i(\mathbf{w_1}, \ldots, \mathbf{w_k})$ denote the number of inversions of the i^{th} permutation so obtained, the q-version of $f_k(\mathbf{v})$ we seek is

$$f_k[\mathbf{v}] = \sum_{\mathbf{w_1}+\ldots+\mathbf{w_k}=\mathbf{v}} q^{\Sigma_i inv\pi_i(\mathbf{w_1},\ldots,\mathbf{w_k})+2\eta(\mathbf{w_1},\ldots,\mathbf{w_k})}.$$

The statistic $\eta(\lambda)$ equals $\Sigma_i(i-1)\lambda_i$ if λ is an integer partition; for a sequence of vectors adding to $\mathbf{v}$, associate the t partitions $\zeta_1, \ldots, \zeta_t$, where ζ_i is the i^{th} column of the matrix after it has been sorted by the permutation π_i, and let $\eta(\mathbf{w_1}, \ldots, \mathbf{w_k})$ be the sum of $\eta(\zeta_i)$ for i in the range $1 \le i \le$t.

Proving that this definition of $f_k[\mathbf{v}]$ works is rather complicated [Hal]. The hard part is to establish the identity

$$\sum_{k\geq 1} \begin{bmatrix} x \\ k \end{bmatrix} f_k[\mathbf{v}] = \prod_i \begin{bmatrix} x+v_i-1 \\ v_i \end{bmatrix}$$

after which (6) is applied. As usual, $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the q-binomial coefficient.

MacMahon also studied *unitary compositions* of a vector $\mathbf{v}$. A composition is unitary if all the coordinates w_{ij} of all the parts $\mathbf{w_i}$ are 0 or 1. Defining

$$g_k[\mathbf{v}] = \sum_{\mathbf{w_1}+\ldots+\mathbf{w_k}=\mathbf{v}} q^{\Sigma_i inv\pi_i(\mathbf{w_1},\ldots,\mathbf{w_k})},$$

$g_k[\mathbf{v}]$ can be shown to satisfy

$$\sum_{k\geq 1} \begin{bmatrix} x \\ k \end{bmatrix} g_k[\mathbf{v}] = \prod_i \begin{bmatrix} x \\ v_i \end{bmatrix}, \tag{7}$$

which implies $g_k[\mathbf{v}] = [k]!R_{n-k}(G_{\mathbf{v}})/\Pi_i[v_i]!$ for a certain Ferrers board $G_{\mathbf{v}}$ (the boards $G_{\mathbf{v}}$ originally occurred in the work of Kaplansky and Riordan, who showed $N_k(\mathbf{v}) = a_{n-k+1}(G_{\mathbf{v}})/\Pi_i v_i!$). Using the mathematics underlying juggling patterns, a bijective proof of (7) has recently been discovered by Ehrenborg and Readdy [E-R].

A q-version of the function $N_k(\mathbf{v})$ was already introduced by MacMahon [Ma2]; set

$$N_k[\mathbf{v}] := \sum_{\substack{\sigma \\ k-1 \text{ descents}}} q^{maj\sigma},$$

where $maj\sigma$ is the sum of the places where σ has descents, namely $\Sigma_{\sigma_i>\sigma_{i+1}} i$. This q-version turns out to be exactly what we need to extend our theorems connecting $N_k(\mathbf{v})$ to $A_j(B_{\mathbf{v}})$ and $A_j(G_{\mathbf{v}})$; we end up with the four identities

$$f_k[\mathbf{v}] = [k]!R_{n-k}(B_{\mathbf{v}})/\Pi_i[v_i]! \qquad N_k[\mathbf{v}] = A_k(B_{\mathbf{v}})/\Pi_i[v_i]!$$

$$g_k[\mathbf{v}] = [k]!R_{n-k}(G_{\mathbf{v}})/\Pi_i[v_i]! \qquad N_k[\mathbf{v}] = q^{E(k,v)}A_{n-k+1}(G_{\mathbf{v}})/\Pi_i[v_i]!$$

where $E(k,\mathbf{v}) = (k-1)n - \sum_{i=1}^{t} v_i(v_1+\ldots+v_{i-1})$. Formulas like

$$\sum_{k=0}^{n} x^k f_k[\mathbf{v}] \prod_{i=k+1}^{n} (1-xq^i) = \sum_{k=0}^{n} x^k N_k[\mathbf{v}]$$

now follow as consequences.

3. The r parameter.

Rawlings has introduced a more general version of the q-Simon Newcomb Problem which also depends on a parameter r [Raw]. He sets

$$N_k[\mathbf{v},r] := \sum_{\substack{\sigma \\ k-1\ r-\text{descents}}} q^{r-maj\sigma}$$

where an r-descent is a value of i for which $\sigma_i - \sigma_{i+1} \geq r$, and $r - maj\sigma$ equals the sum over all these i (where r-descents occur) plus the cardinality of the set $(i,j) : 1 \leq i \leq j \leq n$

and $\sigma_i > \sigma_j > \sigma_{i-r}$. This reduces to *maj*$\sigma$ when $r = 1$, and to *inv*σ when $r = t$. This also connects nicely with q-rook theory; one can define boards $B_{\mathbf{v},r}$ and $G_{\mathbf{v},r}$ so that

$$N_k[\mathbf{v},r] = A_k(B_{\mathbf{v},r})/\Pi_i[v_i]!, \qquad \text{and}$$

$$N_k[\mathbf{v},r] = q^{E(k,v,r)}A_{n-k+1}(G_{\mathbf{v},r})/\Pi_i[v_i]!$$

with $E(k,\mathbf{v},r) = (k-1)n - \sum_{i=1}^{t} v_i(v_1 + \ldots + v_{i-r})$. In the notation of Figure 1,

$$B_{\mathbf{v},r} = B(V_t - V_{t-r} - 1, v_t; v_{t-r}, v_{t-1}; v_{t-r-1}, v_{t-2}; \ldots; v_2, v_{r+1}; v_1, V_r), \text{and}$$

$$G_{\mathbf{v},r} = B(0, V_r; v_1, v_{r+1}; v_2, v_{r+2}; \ldots; v_{t-r}, v_t),$$

with $V_i = v_1 + v_2 + \ldots + v_i$. One interesting corollary is a generalized version of Worpitsky's identity;

$$\prod_{i=1}^{t} \begin{bmatrix} z + v_{i-r+1} + v_{i-r+2} + \ldots + v_i - 1 \\ v_i \end{bmatrix} = \sum_{j=0}^{n} \begin{bmatrix} z + n - j \\ n \end{bmatrix} N_j[\mathbf{v},r]$$

(Worpitsky proved the case $v = 1^n, q = r = 1$ of the above). Another result obtained is that the polynomials $N_k[\mathbf{v},r]$ are all symmetric and unimodal. This gives rise to the question of whether or not the functions

$$f_k[\mathbf{v},r] := [k]!R_{n-k}(B_{\mathbf{v},r})/\Pi_i[v_i]!, \text{ and}$$

$$g_k[\mathbf{v},r] := [k]!R_{n-k}(G_{\mathbf{v},r})/\Pi_i[v_i]!,$$

can be written as sums over compositions for some appropriately defined statistics. For $g_k[\mathbf{v},r]$, the answer is yes ([Ha3], p.20; for an equivalent result formulated in terms of juggling see [E-R], Theorem 8.5). The question remains unanswered in general for $f_k[\mathbf{v},r]$, although the special case $\mathbf{v} = 1^n$ can be dealt with by material in [EHR].

4. Recurrence relations.

Let $\mathbf{v}' = (v_1, \ldots, v_{t-1})$. It is easy to derive a recurrence relation for $R_k(B)$ [G-R] which in turn implies the recurrence

$$f_k[\mathbf{v}] = \sum_{j=0}^{v_t} f_{k-j}[\mathbf{v}'] \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} k - 1 + v_t - j \\ v_t - j \end{bmatrix} q^{(k-1)j}.$$

By applying induction to a result of Rawlings one can show that

$$N_k[\mathbf{v},r] = \sum_{j=0}^{v_t} N_{k-j}[\mathbf{v}',r] \begin{bmatrix} n + k - 1 - V_{t-r} - j \\ v_t - j \end{bmatrix} \begin{bmatrix} V_{t-r} - k + 1 + j \\ j \end{bmatrix} q^{j(k-1+V_{t-1}-V_{t-r})}.$$

Here $V_j = v_1 + \ldots + v_j$. Since the polynomials $N_k[\mathbf{v},r]$ are special cases of the $A_k(B)$, one would suspect the A_k satisfy some kind of recurrence as well, which led to the following result:

Theorem 1 Let $B = B(h_1, d_1; \ldots; h_t, d_t)$ be the Ferrers board of Figure 1. Let $B' = B(h_1, d_1; \ldots; h_{t-1}, d_{t-1})$ be the board obtained from B by truncating the last d_t columns. Then

$$A_k(B) = [d_t]! \sum_{k-d_t \le s \le k} A_s(B') \begin{bmatrix} c_n - n + d_t + s \\ d_t - k + s \end{bmatrix} \begin{bmatrix} 2n - d_t - c_n - s \\ k - s \end{bmatrix} q^{(k-s)(c_n+k-n)}. \quad (8)$$

Proof: A (seven page) combinatorial proof for the $q = 1$ case, for some B, is given in [Ha1,pp.73-80]. The general case is proven algebraically; let

$$PROD(x, B) = \prod_{i=1}^{n} [x + c_i - i + 1]$$

where c_i = the height of the i^{th} column, and start with the identity

$$A_k(B) = \sum_{j=0}^{k} \begin{bmatrix} n+1 \\ k-j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PROD(j, B)$$

which can be derived from (5) using the q-Vandermonde convolution. Now replace $PROD(j, B)$ by $[j + c_n - n + 1]PROD(j, B^*)$, where $B^* = B(h_1, d_1; \ldots; h_{t-1}, d_{t-1}; h_t - 1, d_t - 1)$. Using

$$\begin{bmatrix} n+1 \\ k-j \end{bmatrix} = \begin{bmatrix} n \\ k-j \end{bmatrix} + q^{n+1-k+j} \begin{bmatrix} n \\ k-j-1 \end{bmatrix}$$

we get, after some rearrangement,

$$A_k(B) = [k + c_n - n + d_t]A_k(B^*) + [2n + 1 - c_n - d_t - k]A_{k-1}(B^*)q^{k-1+c_n-n+d_t}.$$

Iterating this d_t times yields (8). ∎

5. The x parameter.

Recently the author has been studying the function

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k}(B)(-1)^k(z-1)^{n-k} := \sum_{k=0}^{n} z^k a_{n-k}(x, B)$$

and its q-version

$$\sum_{k=0}^{n} [-x][-x+1]\cdots[-x+k-1]R_{n-k}z^k \prod_{i=k+1}^{n} (1 - zq^{i-x-1}) := \sum_{k=0}^{n} z^k A_k(x, B). \quad (10)$$

The motivation for introducing this two-variable polynomial is that if $x = -1$, (10) reduces to (5), while the coefficient of z^n in the left hand side of (10) equals $(-1)^n q^{\binom{n}{2}-xn}$ times the left hand side of (6).

Using the methods outlined in section 4, $A_k(x, B)$ can be expressed explicitly;

$$A_k(x,B) = \sum_{j=0}^{k} \begin{bmatrix} n-x \\ k-j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} -x+j-1 \\ j \end{bmatrix} PROD(j,B), \tag{11}$$

or recursively;

$$A_k(x,B) = [d_t]! \sum_{k-d_t \le s \le k} A_s(x,B')$$

$$\begin{bmatrix} c_n - n + d_t + s \\ d_t - k + s \end{bmatrix} \begin{bmatrix} 2n - d_t - c_n - s - x - 1 \\ k - s \end{bmatrix} q^{(k-s)(c_n+k-n)}. \tag{12}$$

Equation (12) can be viewed as a result in *basic hypergeometric series*. In the standard notation, ${}_{t+1}\phi_t\left(\begin{matrix} a_1, a_2, \ldots, a_{t+1} \\ b_1, b_2, \ldots, b_t \end{matrix}\right)$ stands for the sum

$$\sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{t+1})_n}{(q)_n (b_1)_n \cdots (b_t)_n}$$

where $(w)_n = (1-w)(1-wq)\cdots(1-wq^{n-1})$. The right hand side of (11) can be expressed as a ${}_{t+2}\phi_{t+1}$ using the simple identity

$$PROD(j,B) = PROD(0,B) \prod_{i=1}^{t} \frac{(q^{H_i - D_{i-1}+1})_j}{(q^{H_i - D_i + 1})_j}$$

(for $H_i \ge D_i$ with $H_i = h_1 + \ldots + h_i$, $D_i = d_1 + \ldots + d_i$, and B the board of Figure 1). In the case $t = 2$, the right hand side of (12) can also be expressed as a ${}_4\phi_3$ (by iterating the recurrence, then converting the q-binomial coefficients to q-rising factorials). Comparing (11) and (12) we get one ${}_4\phi_3$ equals another ${}_4\phi_3$, which is equivalent to Sears transformation [GaR,p.41]. Full details will be included in [Ha4].

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