First steps towards Exact Algebraic Identification

Christiane HESPEL
I.N.S.A. Laboratoire L.A.N.S.
20, avenue des Buttes de Coësmes
35043 Rennes Cédex-FRANCE
e-mail : hespel@nimbus.irisa.fr

Gérard JACOB
L.I.F.L.(U.A. 369 C.N.R.S.)
Bat.M3, Université Lille I
59655 Villeneuve d'Ascq Cédex-FRANCE
e-mail : jacob@lifl.fr

Abstract

This paper presents the first step toward solving the problem of the Exact Algebraic Identification. This problem consists in computing the coefficients of a non commutative generating series when only the Taylor expansion of some inputs (at $t = 0$), and the Taylor expansion (at $t = 0$) of associated outputs are known. *

Cet article présente la première étape vers la résolution du problème de l'Identification Algébrique Exacte. Ce problème consiste à calculer les coefficients d'une série génératrice non commutative, connaissant les développements de Taylor (en $t = 0$) des entrées , et les développements de Taylor (en $t = 0$) des sorties correspondantes.

1 Introduction

The input/output behaviour of a dynamic system

$$(\Sigma) \left\{ \begin{array}{rcl} \dot{q} & = & g_0(q) + \sum_{i=1}^{m} a_i(t) g_i(q) \\ y(t) & = & h(q) \end{array} \right.$$

- where the state q belongs to a finite dimensional $\mathbb{R}$-analytic manifold Q,
- the vector fields f, g_i and the scalar observation function h are analytic and defined in a neighbourhood of $q(0)$.
- the inputs a_i are real and piecewise continuous and everywhere right continuous.

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The output of (Σ) may be obtained from its generating series [3]

$$G = \sum_{l \geq 0} \sum_{i_j=0}^{m} g_{i_1} \cdots g_{i_l} \circ h_{|_{q_0}} z_{i_1} \cdots z_{i_l}$$

where $g_i \circ h = \sum_{s=1}^{N} g_i^s(q) \dfrac{\partial h}{\partial q^s}$ denotes the Lie derivation of h by the vector field g_i.

This generating series G is a formal series in the non commutative variables taken in the encoding alphabet $Z = \{z_0, z_1, \dots, z_m\}$ corresponding to the system inputs $(a_0, a_1, \cdots, a_m)$ (we have to set $a_0(t) \equiv 1$).

In this paper, we are interested in solving the *"algebraic identification problem":* Is there an algorithm that enables to compute exactly all the coefficients of the generating series G, when only the Taylor expansions (at $t = 0$) of some finite set of inputs, and the Taylor expansions (at $t = 0$) of associated outputs are known?

A preliminary problem is the following: *is the generating series encoding a faithful representation?* In other words, if the output $y(t)$ is equal to zero whatever the input may be, is the generating series necessarily equal to zero? Independant positive proofs have been given by M.Fliess [5], C.Reutenauer [9],Y.Wang and E.D.Sontag [11]. The proof of M.Fliess is based on defining outputs for generalized inputs. The two other proofs are based on computing outputs for some parametrized inputs. The proof of C.Reutenauer uses analytic expansion of the output on parameters. The proof of Y.Wang and E.D.Sontag uses a noncommutative partial derivative computation towards parameters. Nevertheless, *none of these proofs provides an exact identification method.*

We present here a first step towards solving this problem of the algebraic identification: At once, *we compute the iterated derivatives of the output $y(t)$, as linear combinations of the input multiderivatives.* Hence, the formula of F.Lamnabhi and P.Crouch [8] is redemonstrated. Let us remark that an interpretation of this formula has been given by E.D.Sontag and Y.Wang [11]. Then, we provide an algorithm for the *computation of the coefficients of the input multiderivatives* appearing in the expressions of the iterated derivatives of the output $y(t)$. The problem of the Algebraic Identification will be solved when we are able to identify the coefficients of the generating series, from the coefficients of the input multiderivatives.

2 Preliminaries and notations

See [3, 4, 7, 10]. We associate with (Σ) the encoding alphabet $\mathcal{Z} = \{z_0, z_1, \cdots, z_m\}$. A *word* $w \in \mathcal{Z}^*$ is a finite sequence of letters in $\mathcal{Z}$. The *empty word* is denoted by ε. A *formal series* S (in the noncommuting variables z_i) with coefficients in $\mathbb{R}$ is any mapping of $\mathcal{Z}^*$ into $\mathbb{R}$:

$$S = \sum_{w \in Z^*} < S|w > w.$$

The operations of sum, and Cauchy product, of two series S and T are defined by

$$S + T = \sum_{w \in Z^*} (< S|w > + < T|w >)w \qquad S.T = \sum_{w \in Z^*} \sum_{uv=w} < S|u > . < T|v > w$$

In that way, the set of formal series on $\mathcal{Z}$ is an associative algebra denoted by $\mathbb{R} << \mathcal{Z} >>$. For each word $w \in Z^*$ we *define recursively* both a differential operator $\mathcal{Y}(w)$ and an iterated integral $\int_0^t \delta_a(w)$, as follows:

• The *differential operators* $\mathcal{Y}(w)$ are defined as follows, starting with $\mathcal{Y}(\varepsilon) = Id$:

$$\begin{array}{rcll} \mathcal{Y}(z_i) & = & \sum_{s=1}^{N} g_i^s(q)\frac{\partial}{\partial q^s} & \forall i \quad 0 \le i \le m \\ \mathcal{Y}(vz_j) & = & \mathcal{Y}(v) \circ \mathcal{Y}(z_j) & \forall v \in Z^* \qquad \forall z_j \in Z. \end{array}$$

Any word $w = z_{i_1} \cdots z_{i_p}$ and any analytical function h give rise to the iterated Lie derivative:

$$\mathcal{Y}(w) \circ h = g_{i_1} \cdots g_{i_p} \circ h$$

• Similarly, the *iterated integrals* $\int_0^t \delta_a(w)$ are defined as follows:

$$\left\{ \begin{array}{rcll} \int_0^t \delta_a(\varepsilon) & = & 1 & \forall t \ge 0 \\ \int_0^t \delta_a(vz_i) & = & \int_0^t \left(\int_0^\tau \delta_a(v)\right) a_i(\tau) d\tau & \forall v \in Z^* \quad \forall z_j \in Z. \end{array} \right.$$

We introduce naturally the two following formal series:

• $G = \sum_{w \in Z^*} \left[\mathcal{Y}(w) o h_{|q_0}\right] w$ is the *generating series*, or *'Fliess series'* of (Σ).

• $\mathcal{C}_a(t) = \sum_{w \in Z^*} \left[\int_0^t \delta_a(w)\right] w$ is the *'Chen series'* of the input a (see [1]).

According to Chen's notation [1], (the order used here being the *reverse of Fliess' one* [4]), the Fliess fundamental formula can be alternatively written as:

$$\begin{array}{rcl} y(t) & = & \sum_{w \in Z^*} \left[\mathcal{Y}(w) \circ h_{|q_0}\right] \int_0^t \delta_a(w) \\ y(t) & = & \sum_{w \in Z^*} < G|w >< \mathcal{C}_a(t)|w > \; = < G||\mathcal{C}_a(t) > \end{array}$$

(where $||$ means infinite sum). More generally, *any power series H over $\mathcal{Z}$*, the coefficients of which satisfy the following *convergence condition*: "$\exists K, L \in \mathbb{R}^+$ such that any $w \in Z^*$ satisfies $|< H|w>| < K|w|!L^{|w|}$" defines the causal functional defining the output:

$$y_H(t) = \sum_{w \in Z^*} < H|w >< C(t)|w >$$

We define *the concatenated control* $a\sharp v$ of the two inputs a defined in $[0, t_1[$ and v defined in $[0, t_2[$, as being the input defined on $[0, t_1 + t_2[$ by:

$$a\sharp v(t) = \left\{ \begin{array}{lll} a(t) & if & 0 \le t < t_1 \\ v(t - t_1) & if & t_1 \le t < t_1 + t_2 \end{array} \right.$$

Then we have the relation between the Chen series [12]: $\mathcal{C}_{a\sharp v}(t_1 + \tau) = \mathcal{C}_a(t_1).\mathcal{C}_v(\tau)$.

Lemma: *The derivative of a Chen series* C_a *is* $\frac{d}{dt}C_a = C_a.\mathcal{L}_a$, *where* $\mathcal{L}_a = \sum_{0\leq i\leq m} a_i.z_i$.

Proof : Let us call *left remainder by some polynomial* P, noted "$P\triangleleft$", the linear map defined on the power series $G \in \mathbb{R} << \mathcal{Z} >>$ by the property:

$$\forall w \in \mathcal{Z}^*, \qquad < P \triangleleft G|w >=< G|w.P >$$

Hence for any words u and v we have $u \triangleleft w = v$ if $w = vu$, else 0. We obtain successively:

$$\begin{aligned}\frac{d}{dt} < C_a|w > &= \sum_{z\in\mathcal{Z}} \left[\int_0^t \delta(z \triangleleft w)\right] a_z(t) \\ &= \sum_{z\in\mathcal{Z}} < C_a|z \triangleleft w > a_z(t) = \sum_{z\in\mathcal{Z}} < C_a.z|w > a_z(t) \qquad \square\end{aligned}$$

In other words, $\qquad y^{(1)}(t) =< \mathcal{L}_a(t) \triangleleft G||C_a(t) >=< G|C_a(t).\mathcal{L}_a(t) >$.

(Note that this lemma has been presented by Y.Wang and E.D.Sontag [13] in the input-output functional derivative form, by using concatenated inputs).

Let $a\sharp\mu$ the *concatenated input* of an input a defined in $[0, t_1[$, and of an input μ defined in $[0, t_2[$. According to our formalism, we obtain the corresponding output derivative:

$$\begin{array}{lrcl} \text{for } 0 < t < t_1 & \dot{y}(t) & = & < G||C_a(t).\mathcal{L}_a(t) > \\ \text{for } t_1 < t_1 + t < t_1 + t_2 & \dot{y}(t) & = & < G||C_a(t_1).C_\mu(t).\mathcal{L}_\mu(t) > \\ \text{and then} & lim_{t\to 0^+}[\dot{y}(t_1 + t)] & = & < G||C_a(t_1).\mathcal{L}_\mu(0^+) > \end{array}$$

3 Three proofs about undistinguishability

We recall briefly three proofs of the following proposition:

Proposition: *The generating series representation of the input/output functionals is* faithful. *In other words, any series undistinguishable from* 0 *is identically null.*

3.1 The proof of C. Reutenauer

(See [9]). For $k = 1 \cdots p$, let $i_k \in [1 \cdots m]$ and let $a_k(t)$ the input defined on $[0, t_k[$ by:

$$a_{k,0}(t) = 1, \qquad a_{k,i_k}(t) = \frac{\alpha_k}{t_k}, \qquad a_{k,i} = 0 \quad if \ \ i \neq i_k$$

The Chen series of the input a_k at time t_k is $C_{a_k}(t_k) = e^{t_k z_0 + \alpha_k z_{i_k}}$.

Then the Chen series associated to the input $b = a_1\sharp a_2\sharp \cdots \sharp a_p$ at time $t_1 + \cdots t_p$ is

$$C_{a_1}(t_1)\cdots C_{a_p}(t_p) = e^{t_1 z_0 + \alpha_1 z_{i_1}} \cdots e^{t_p z_0 + \alpha_p z_{i_p}}$$

The power series expansion of this exponential product can be reordered toward the $2p$ variables $t_1, \cdots, t_p, \alpha_1, \cdots, \alpha_p$. So, using multi-indices $\mathbf{j} = (j_1, \cdots ..., j_p)$ and $\mathbf{n} = (n_1, \cdots, n_p)$, and standard multi-index power notation, we can write:

$$y(t) =< G||\sum_{\mathbf{j},\mathbf{n}} t^{\mathbf{j}}\alpha^{\mathbf{n}}C_{\mathbf{j},\mathbf{n}} >= \sum_{\mathbf{j},\mathbf{n}} t^{\mathbf{j}}\alpha^{\mathbf{n}} < G||C_{\mathbf{j},\mathbf{n}} >$$

The last expression defines an analytical function of the $2p$ variables t_i and α_k in a neighbourhood of zero. But by assumption, it must be equal to zero whatever $t_i(>0)$ and α_k may be. Thus it is identically equal to zero. But the coefficient of the monomial $t_1^{j_1}\alpha_1\cdots t_{p-1}^{j_{p-1}}\alpha_{p-1}t_p^{j_p}$ in $y(t)$ is exactly reduced to the term: $<G|z_0^{j_0}z_1\cdots z_0^{j_{p-1}}z_{i_{p-1}}z_0^{j_p}>$. Since every word w can be written in the form $w=z_0^{j_0}z_1\cdots z_0^{j_{p-1}}z_{i_p}$, the series G is identically null.

3.2 The proof of M. Fliess

(See [5]). For $\varepsilon>0$, ε small, there are inputs a and b the Chen series of which are

$$\mathcal{C}_a = e^{\varepsilon z_0} \quad ; \quad \mathcal{C}_b = e^{2\varepsilon z_0}$$

And then, $\mathcal{C}_b-\mathcal{C}_a = e^{2\varepsilon z_0}-e^{\varepsilon z_0}=\varepsilon z_0+O(\varepsilon^2)$ and $z_0 = lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\mathcal{C}_b-\mathcal{C}_a]$.

If $z_i\neq z_0$ an input a_i may be defined, the Chen series of which is

$$\mathcal{C}_{a_i} = e^{\varepsilon z_0+\varepsilon^2 z_i}$$

And then, $\mathcal{C}_{a_i}-\mathcal{C}_a = e^{\varepsilon z_0+\varepsilon^2 z_i}-e^{\varepsilon z_0}=\varepsilon^2 z_i+O(\varepsilon^3)$ and $z_i = lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}[\mathcal{C}_{a_i}-\mathcal{C}_a]$

Now, G being undistinguishable from zero, we proof that its left remainders by the letters have the same property. Indeed, we have for any Chen series $\mathcal{C}$:

$$\begin{aligned} <z_0\triangleleft G\|\mathcal{C}> = <G\|\mathcal{C}z_0> &= lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon}[<G\|\mathcal{C}.\mathcal{C}_b> - <G\|\mathcal{C}.\mathcal{C}_b>]=0 \\ \forall i\in[1,m],\quad <z_i\triangleleft G\|\mathcal{C}> = <G\|\mathcal{C}z_i> &= lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon^2}[<G\|\mathcal{C}.\mathcal{C}_{a_i}> - <G\|\mathcal{C}.\mathcal{C}_a>]=0 \end{aligned}$$

Then, the left remainder power series $z_0\triangleleft G$ and $z_i\triangleleft G$ are undistinguishable from zero. Recursively, we deduce that it is true also for the left remainder of G by any word w. From what follows that G is identically null:

$$\forall w\in Z^* \; <G|w> = <w\triangleleft G|\varepsilon> = 0$$

3.3 The proof of E.D. Sontag and Y. Wang

(See[11]). This proof is based on computation of some noncommutative parameters partial derivatives of the input-output map. With our formalism, let us consider a concatenation of constant inputs:

$$b=\mu_1\sharp\mu_2\sharp\cdots\sharp\mu_p$$

$$\text{the input } \mu_k=(\mu_{k,1},\mu_{k,2},\cdots,\mu_{k,m}) \quad \text{is defined on} \quad [0,t_k[$$

The Chen series of b is $\mathcal{C}_b=\mathcal{C}_{\mu_1}.\mathcal{C}_{\mu_2}\cdots\mathcal{C}_{\mu_k}$. The output $y(t)$ is a function of the t_j, and we compute the iterated partial noncommuting derivatives as follows:

$$\begin{aligned} \frac{\partial}{\partial t_k}(<G\|\mathcal{C}_{\mu_1}\cdots\mathcal{C}_{\mu_k}>)_{t_k=0+} &= <G\|\mathcal{C}_{\mu_1}\cdots\mathcal{C}_{\mu_{k-1}}.\mathcal{L}_{\mu_k}(0^+)> \\ \frac{\partial}{\partial t_{k-1}}\frac{\partial}{\partial t_k}(<G\|\mathcal{C}_{\mu_1}\cdots\mathcal{C}_{\mu_k}>)_{t_{k-1}=0+} &= <G\|\mathcal{C}_{\mu_1}\cdots\mathcal{C}_{\mu_{k-2}}.\mathcal{L}_{\mu_{k-1}}(0^+).\mathcal{L}_{\mu_k}(0^+)> \end{aligned}$$

By iterating, we obtain:

$$\begin{aligned} \left(\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_k}(y)\right)_{t_1=0+} &= <G\|\mathcal{L}_{\mu_1}(0^+)\cdots\mathcal{L}_{\mu_k}(0^+)> \\ &= \sum_{l_1,\cdots,l_k} <G|z_{l_1}\cdots z_{l_k}> \mu_{1,l_1}\mu_{2,l_2}\cdots\mu_{k,l_k} \end{aligned}$$

By assumption, this expression is equal to zero for any value of the parameters μ_{i,j_i}. Then we recover the coefficients of G as follows:

$$< G||z_{j_1}\cdots z_{j_k} >= \frac{\partial^k}{\partial_{\mu_{1,j_1}}\cdots\partial_{\mu_{k,j_k}}}\left(\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_k}(y)\right)_{t_1=0+} = 0$$

Thus G is identically null.

3.4 Conclusion

All these proofs require the knowledge of $y(t)$ and of its iterated derivatives everywhere in some neighbourhood of 0. None of these proofs provides an exact identification method. They do not give even a way for stable numeric computation. A fortiori, they do not solve the algebraic identification in our meaning.

4 Iterated derivatives of the output.

(See [6]). For any power series G with constant coefficients (satisfying the convergence condition) the i^{th} time derivative of $y_G(t) =< G||C(t) >$ is given by the following expansion $y_G{}^{(i)}(t) =< G||C^{(i)}(t) >$, convergent in some neighborhood of zero. Then we compute first the iterated time derivatives of the Chen's series.

4.1 Derivatives of Chen series.

Since $\frac{dC_a}{dt} = C_a.\mathcal{L}_a$, we obtain by iterating (denoting by D_t the usual time derivation operator):

$$\left.\begin{array}{l} C_a^{(i)} = C_a A_i \\ \text{where } \left\{\begin{array}{lcl} A_1 & = & \mathcal{L}_a \\ A_{i+1} & = & \mathcal{L}_a A_i + D_t A_i \end{array}\right. \end{array}\right\} \quad (R)$$

It should be noticed that $C_a(0) = 1$ and then $C_a^{(i)}(0) = A_i(0)$. Consequently

$$y^{(i)}(0) = \sum_{w\in Z^*} < G|w >< C_a^{(i)}(0)|w > =< G||A_i(0) >$$

4.2 Explicit computation of A_i

Let us define the derivative $\mathcal{L}_a^{[\rho]}$ of $\mathcal{L}_a$ by a multi-index $\rho = (\rho_1, \cdots, \rho_p)$ (with $\rho_i \in \mathbb{N}$):

$$\mathcal{L}_a^{(\rho_k)} = \sum_i a_i^{(\rho_k)}\, z_i \qquad \mathcal{L}_a^{[\rho]} = \mathcal{L}_a^{(\rho_1)}\cdots\mathcal{L}_a^{(\rho_p)} = \sum_{i_1,\cdots i_p} a_{i_1}^{(\rho_1)}\cdots a_{i_p}^{(\rho_p)}\, z_{i_1}\cdots z_{i_p}.$$

The *degree* and the *weight* of ρ are given by:

$$\deg(\rho) = p, \qquad \mathrm{wgt}(\rho) = \sum_{j=1}^{p}(1+\rho_j) = \deg(\rho) + \sum_{j=1}^{p}\rho_j.$$

The polynomials A_i can be written as: $A_i = \sum_{\mathrm{wgt}(\rho)=i} \alpha_\rho \mathcal{L}_a^{[\rho]}$. We set $A_0 = 1$. Let us denote by $\mathcal{A}$ the formal sum of the A_i. Then by a combinatorial analysis of the identity:

$$\sum_{i\in\mathbb{N}} A_i = \mathcal{A} = 1 + \mathcal{L}_a\mathcal{A} + D_t\mathcal{A} = \sum_\rho \alpha_\rho \mathcal{L}_a^{[\rho]}$$

we obtain easily the following formula:

$$\alpha_\rho = \prod_{i=1}^{p} \begin{pmatrix} \sum_{j=1}^{i} \rho_j + i - 1 \\ \rho_i \end{pmatrix} = \binom{\rho_1}{\rho_1}\binom{\rho_1+\rho_2+1}{\rho_2}\cdots\binom{\rho_1+\cdots\rho_k+k-1}{\rho_k} \qquad (E).$$

4.3 Expression of the output derivatives.

$$\begin{aligned} y^{(n)}(t) &= \sum_{\mathrm{wgt}(\rho)=n} \alpha(\rho) \sum_{i_j} a_{i_1}^{(\rho_1)}(t)\cdots a_{i_k}^{(\rho_k)}(t) < z_{i_1}\cdots z_{i_k} \triangleleft G \| \mathcal{C}_a(t) > \\ &= \sum_{\mathrm{wgt}(\rho)=n} \alpha(\rho) \sum_{i_j} a_{i_1}^{(\rho_1)}(t)\cdots a_{i_k}^{(\rho_k)}(t) < G \| \mathcal{C}_a(t) z_{i_1}\cdots z_{i_k} > \end{aligned}$$

For an *analytical system* (Σ), this formula may be interpreted by two means.

- Initializing (Σ) at time t (and by setting $\mathcal{C}_a(t) = 0$), we get the *local expression* of $y^{(n)}(t)$ in the state $q(t)$ (formula of Lamnabhi-Lagarrigue and Crouch [8]):

$$y^{(n)}(t) = \sum\sum \alpha(\rho) a_{i_1}^{(\rho_1)}(t)\cdots a_{i_k}^{(\rho_k)}(t)(g_{i_1}\cdots g_{i_k} oh)_{|_{q(t)}}.$$

- Initializing (Σ) at time 0, we get a *global expression* of $y^{(n)}(t)$:

$$y^{(n)}(t) = \sum\sum \alpha(\rho) a_{i_1}^{(\rho_1)}\cdots a_{i_k}^{(\rho_k)} y_{i_1 i_2\cdots i_k}(t)$$

where $y_{i_1 i_2\cdots i_k}$ is the output of (Σ) initialized in $q(0)$, for the observation $g_{i_1}g_{i_2}\cdots g_{i_k} \circ h$. For $t=0$, according to $\mathcal{C}_a(0) = 1$, we obtain the formula:

$$\left.\begin{aligned} y^{(n)}(0) &= \sum_{k=0}^{n}\sum_{i_j} < G | z_{i_1}\cdots z_{i_k} > . \\ &\left[\sum_{\rho_1+\cdots+\rho_k=n-k} \binom{\rho_1}{\rho_1}\binom{\rho_1+\rho_2+1}{\rho_2}\cdots\binom{\rho_1+\cdots\rho_k+k-1}{\rho_k} a_{i_1}^{(\rho_1)}(0)\cdots a_{i_k}^{(\rho_k)}(0)\right] \end{aligned}\right\} \quad (F_n)$$

5 First step for computing the algebraic identification

This problem consists in computing, *by effective formulae,* the coefficients of a generating series when we know the Taylor expansions at $t = 0$ of some inputs and the Taylor expansions at $t = 0$ of the corresponding outputs. This problem remains unsolved. We provide here an algorithm for identifying the coefficients of *the input mutiderivatives a^μ (as defined below),* by solving the following system of equations in order to compute the unknown coefficients $< G \| l_\mu >$:

$$y^{(k)}(0) = \sum_\mu a^\mu < G \| l_\mu >$$

5.1 Single input without drift

We denote the single input by $a(t)$ and determine it polynomial: $a(t) = \sum_{i=0}^{k} \frac{c_i}{i!} t^i$. We have:

$$a^{(\rho_1)} \cdots a^{(\rho_p)} = a^{\mu_1} \cdots (a^{(k-1)})^{\mu_k} = a^{\mu}$$

So we obtain: $$y^{(n)}(0) = \sum_{\mathrm{wgt}=n} c_0^{\mu_1} \cdots c_{k-1}^{\mu_k} < G|z_1^{\mu_1+\cdots+\mu_k} > \lambda_\mu$$

We get a triangular infinite system of linear equations:

$$\left\{ \begin{array}{lcl} y(0) & = & < G|\varepsilon > \\ y^{(1)}(0) & = & c_0 < G|z_1 > \\ y^{(2)}(0) & = & a_0^{(1)} < G|z_1 > + c_0^2 < G|z_1^2 > \\ \vdots & & \\ y^{(n)}(0) & = & \sum_{\substack{\mathrm{wgt}(\mu) = n \\ i = \deg(\mu) < n}} a^{\mu}(0)\lambda_\mu < G|z_1^i > + c_0^n < G|z_1^n > \\ & \vdots & \end{array} \right.$$

The computation of the coefficients $< G|z_1^k >$ is then obvious for a selected input that satisfies: $a(0) = c_0 \neq 0$, if the resulting output derivatives $y^{(p)}(0)$ are known for $0 \leq p \leq k$.

5.2 Single input with drift

We set the input equal to a polynomial: $a_1(t) = \sum_{i=0}^{k} \frac{c_i}{i!} t^i$ and we set $a_0(t) \equiv 1$. We get a linear equation system that must be satisfied for any choice of the coefficients c_i:

$$\left. y^{(n)}(0) = \sum_{\mathrm{wgt}(\mu) \leq n} c_0^{\mu_1} \cdots c_{k-1}^{\mu_k} \cdot \sum_{\substack{w \\ |w|_{z_1} = \mu_k + \mu_{k-1} + \cdots + \mu_1 \\ |w|_{z_0} = n - |w|_{z_1} - (k-1)\mu_k - \cdots - \mu_2}} < G|l_\mu > \right\} \quad (\mathrm{FD})_n$$

We take only one equation $(\mathrm{FD})_n$ and we prove that we can identify every coefficient of the multiderivatives appearing in this equation, by choosing suitable inputs:

Consider the equation $(\mathrm{FD})_n$; there is a determinant T_n which is not zero, the columns of which are indexed by the multiderivatives of the input and the rows of which are indexed by several choices of the input and its derivatives at $t = 0$.

1. **The matrix T_n has a lower triangular block form.**
   - **Definition of an order of the columns:**
     ◇ We order the multiderivatives (the columns) by increasing order of the number q of the different derivation orders, appearing in these multiderivatives.
     ◇ For the same order q, we put together the multiderivatives containing the common set of derivatives $\{a_1^{(i_1)}, a_1^{(i_2)} \cdots, a_1^{(i_q)}\}$.
   - **Choice of the values of the associated c_i**
     For every set of derivatives $\{a_1^{(i_1)}, a_1^{(i_2)} \cdots, a_1^{(i_q)}\}$, appearing in the multiderivatives indexing the columns, we choose a corresponding set of inputs values such that every c_i

is equal to zero except $\{c_{i_1}, c_{i_2} \cdots, c_{i_q}\}$. So, we obtain T_n in a lower triangular block form. Let us denote these blocks by $D_1, \cdots, D_q, \cdots, D_p$.

2. **Choice of the inputs values .**
Select these values in order to obtain every diagonal block determinant $D_q \neq 0$:
•First, let us prove the feasibility:
◇ It is obvious that D_1 may be constructed different from 0 for some values $(c_{i,j})_{1 \leq j \leq r}$ of $a_1^{(i)}$, since D_1 is a Vandermonde determinant.
◇ We prove that D_{q+1} associated to the set of the derivatives $\{a_1^{(i_1)}, \cdots, a_1^{(i_{q+1})}\}$ may be constructed $\neq 0$, if D_q associated to the set $\{a_1^{(i_1)}, \cdots, a_1^{(i_q)}\}$ has been constructed $\neq 0$: D_{q+1} can be written in a lower triangular block form, by executing some suitable linear combinations of the columns, for some suitable choices of the values $c_{i_{q+1},j}$ of the $a_1^{(i_{q+1})}$.

Proof : First, we order the columns $((a_1^{(i_1)})^{e_1} \cdots (a_1^{(i_{q+1})})^{e_{q+1}})_{1 \leq e_j \leq sup_j}$ of D_{q+1} according to the increasing e_{q+1} and we select only the values of the $a_1^{(i_{q+1})}$:

Let $c_{i_{q+1},1}$ the value associated to $e_{q+1} = 1$, etc $\cdots$ $c_{i_{q+1},sup_{q+1}}$ the value associated to $e_{q+1} = sup_{q+1}$ where $c_{i_{q+1},j} \neq c_{i_{q+1},k}$ for $j \neq k$.
We remark that every column $((a_1^{(i_1)})^{e_1} \cdots (a_1^{(i_{q+1})})^{e_{q+1}})$ where $2 \leq e_{q+1}$, has a 'column predecessor' $((a_1^{(i_1)})^{e_1} \cdots (a_1^{(i_{q+1})})^{e_{q+1}-1})$ in the same block D_{q+1}. So, we can deduce that D_{q+1} may be written as a lower triangular block determinant, by executing :

for $2 \leq e_{q+1}$,
$column((a_1^{(i_1)})^{e_1} \cdots (a_1^{(i_{q+1})})^{e_{q+1}}) \mapsto itself - c_{i_{q+1},1} * column((a_1^{(i_1)})^{e_1} \cdots (a_1^{(i_{q+1})})^{e_{q+1}-1})$
etc $\cdots$
for $s \leq e_{q+1}$,
$column((a_1^{(i_1)})^{e_1} \cdots (a_1^{(i_{q+1})})^{e_{q+1}}) \mapsto itself - c_{i_{q+1},s} * column((a_1^{(i_1)})^{e_1} \cdots (a_1^{(i_{q+1})})^{e_{q+1}-1})$

where $s = sup_{q+1} - 2$. If $1 = sup_{q+1}$, we need not to transform some column, but we need only to factorize the coefficient associated to $a_1^{(i_{q+1})}$. So we obtain:

$e_{q+1} = 1$	$e_{q+1} = 2$	$\cdots$	$e_{q+1} = sup_{q+1}$
$(c_{i_{q+1},1})^{o_1}$ $D_{q,1}$	0	$\cdots$ $\cdots$	0
α	$(c_{i_{q+1},2})^{o_2}$ $(c_{i_{q+1},2} - c_{i_{q+1},1})^{o_2}$ $D_{q,2}$	$0 \cdots$	0
. .	. .	. .	. .
β	γ	$\cdots$	$(c_{i_{q+1},sup_{q+1}})^{o_{sup_{q+1}}}$ $\Pi_t (c_{i_{q+1},sup_{q+1}} - c_{i_{q+1},t})^{o_{sup_{q+1}}}$ $D_{q,sup_{q+1}}$

with o_j denoting the dimension of the i^{th} block and $D_{q,j}$ denoting the determinant associated with q different orders of derivation for the j^{th} block. □

• Effective construction :
◇ order on the columns of D_q:

For the columns $((a_1^{(i_1)})^{e_1}\cdots(a_1^{(i_q)})^{e_q})_{1\leq e_k\leq sup_k}$, order of the increasing e_q
For e_q constant, order of the increasing e_{q-1}
...
For $e_q,\cdots,e_2$ constant, order of the increasing e_1

◇ **We propose the following choices of the $(a_1^{(j)})^e$:**

○ **values of the $a_1^{(i_q)}$:**

Let be o_1, the dimension of the block $((a_1^{(i_1)})^{e_1}\cdots(a_1^{(i_{q-1})})^{e_{q-1}}a_1^{(i_q)})_{1\leq e_1\cdots 1\leq e_{q-1}}$.
We choose $a_1^{(i_q)} = c_{i_q,1} \neq 0$ for the o_1^{th} first rows of D_q, denoted by $D_{q;1}$.
etc $\cdots$
Let be o_{sup_q}, the dimension of the block $((a_1^{(i_1)})^{e_1}\cdots(a_1^{(i_{q-1})})^{e_{q-1}}(a_1^{(i_q)})^{sup_q})_{1\leq e_1\cdots 1\leq e_{q-1}}$.
We choose $a_1^{(i_q)} = c_{i_q,sup_q} \neq 0$ where $c_{i_q,sup_q} \neq c_{i_q,j}$ $\forall j < sup_q$ for the $o_{sup_q}^{th}$ last rows of D_q, denoted by $D_{q;sup_q}$.
...

○ **values of the $a_1^{(i_1)}$:**

Let be $o_{e_q,e_{q-1},\cdots,e_2,1}$, the dimension of the block $((a_1{}^{(i_1)})\cdots(a_1{}^{(i_{q-1})})^{e_{q-1}}a_1{}^{(i_q)e_q})$.
We choose $a_1^{(i_1)} = c_{i_1,1} \neq 0$ for the $o_{e_q,e_{q-1},\cdots,e_2,1}^{th}$ first rows of $D_{q;e_q,\cdots,e_2}$, denoted by $D_{q;e_q,\cdots,e_2,1}$.
etc $\cdots$
Let be $o_{e_q,e_{q-1},\cdots,e_2,sup_1}$, the dimension of the block $((a_1{}^{(i_1)})^{sup_1}\cdots(a_1{}^{(i_{q-1})})^{e_{q-1}}a_1{}^{(i_q)e_q})$.
We choose $a_1^{(i_1)} = c_{i_1,sup_1} \neq 0$ where $c_{i_1,sup_1} \neq c_{i_1,j}$ $\forall j \leq sup_1$ for the $o_{e_q,e_{q-1},\cdots,e_2,sup_1}^{th}$ last rows of $D_{q;e_q,\cdots,e_2}$, denoted by $D_{q;e_q,\cdots,e_2,sup_1}$.

example: *for T_{13}, let D_3 be the block containing the derivatives $a_1^{(1)}, a_1^{(2)}, a_1^{(3)}$:*

			$e_3 = 1$				$e_3 = 2$
			$a_1^{(1)}a_1^{(2)}a_1^{(3)}$	$(a_1^{(1)})^2a_1^{(2)}a_1^{(3)}$	$(a_1^{(1)})^3a_1^{(2)}a_1^{(3)}$	$a_1^{(1)}(a_1^{(2)})^2a_1^{(3)}$	$a_1^{(1)}a_1^{(2)}(a_1^{(3)})^2$
$a_1^{(3)} = c_{3,1}$	$a_1^{(2)} = c_{2,1}$	$a_1^{(1)} = c_{1,1}$	$c_{1,1}c_{2,1}c_{3,1}$	$c_{1,1}^2c_{2,1}c_{3,1}$	$c_{1,1}^3c_{2,1}c_{3,1}$	$c_{1,1}c_{2,1}^2c_{3,1}$	$c_{1,1}c_{2,1}c_{3,1}^2$
		$a_1^{(1)} = c_{1,2}$	$c_{1,2}c_{2,1}c_{3,1}$	$c_{1,2}^2c_{2,1}c_{3,1}$	$c_{1,2}^3c_{2,1}c_{3,1}$	$c_{1,2}c_{2,1}^2c_{3,1}$	$c_{1,2}c_{2,1}c_{3,1}^2$
		$a_1^{(1)} = c_{1,3}$	$c_{1,3}c_{2,1}c_{3,1}$	$c_{1,3}^2c_{2,1}c_{3,1}$	$c_{1,3}^3c_{2,1}c_{3,1}$	$c_{1,3}c_{2,1}^2c_{3,1}$	$c_{1,3}c_{2,1}c_{3,1}^2$
	$a_1^{(2)} = c_{2,2}$	$a_1^{(1)} = c_{1,1}$	$c_{1,1}c_{2,2}c_{3,1}$	$c_{1,1}^2c_{2,2}c_{3,1}$	$c_{1,1}^3c_{2,2}c_{3,1}$	$c_{1,1}c_{2,2}^2c_{3,1}$	$c_{1,1}c_{2,2}c_{3,1}^2$
$a_1^{(3)} = c_{3,2}$	$a_1^{(2)} = c_{2,1}$	$a_1^{(1)} = c_{1,1}$	$c_{1,1}c_{2,1}c_{3,2}$	$c_{1,1}^2c_{2,1}c_{3,2}$	$c_{1,1}^3c_{2,1}c_{3,2}$	$c_{1,1}c_{2,1}^2c_{3,2}$	$c_{1,1}c_{2,1}c_{3,2}^2$

By executing the following processings on the columns:

$$column(a_1^{(1)}a_1^{(2)}(a_1^{(3)})^2) \mapsto itself - c_{3,1} * column(a_1^{(1)}a_1^{(2)}a_1^{(3)})$$

by factorizing $c_{3,1}^4$ in the block associated to $e_3 = 1$ and executing

$$column(a_1^{(1)}(a_1^{(2)})^2 a_1^{(3)}) \mapsto itself - c_{2,1} * column(a_1^{(1)} a_1^{(2)} a_1^{(3)})$$

and by factorizing $c_{2,1}^3$ in the block associated to $e_3 = 1, e_2 = 1$, we obtain:

$$\begin{array}{rcl} D_3 & = & (c_{3,2} - c_{3,1}) * c_{1,1} * c_{2,1} * c_{3,2} \\ & & *(c_{2,2} - c_{2,1}) * c_{1,1} * c_{2,2} * c_{3,1}^4 \\ & & *c_{2,1}^3 * \end{array} \begin{vmatrix} c_{1,1} & c_{1,1}^2 & c_{1,1}^3 \\ c_{1,2} & c_{1,2}^2 & c_{1,2}^3 \\ c_{1,3} & c_{1,3}^2 & c_{1,3}^3 \end{vmatrix}$$

5.3 Several inputs with drift

We consider the case of 2 inputs $a_1(t)$ and $a_2(t)$. We choose them polynomials:

$$a_1(t) = \sum_{i=0}^{l_1} \frac{c_i}{i!} t^i, \quad a_2(t) = \sum_{i=0}^{l_2} \frac{d_i}{i!} t^i.$$

We obtain a linear equations system, which must be satisfied for every coefficients c_i and d_i:

$$y^{(n)}(0) = \sum_{wgt \leq n} c_0^{\mu_{1,1}} \cdots c_{k-1}^{\mu_{1,k}} d_0^{\mu_{2,1}} \cdots d_{n-1}^{\mu_{2,n}} \sum_w < G|w > \lambda_\mu^w$$

$$\mu = ((\mu_{1,1}, \cdots, \mu_{1,k}); (\mu_{2,1}, \cdots, \mu_{2,n})) \qquad wgt(\mu) = wgt(\mu_1) + wgt(\mu_2)$$

The identification of the multiderivatives coefficients appearing in the previous equation is solved by the same method as for a single input system:

1. We order the multiderivatives according to the increasing order of the number q of the different derivation orders of $a_1(t)$ or $a_2(t)$. And then, for the same q, we put together the multiderivatives containing the same q derivatives of the sets $\{a_1^{(i_1)} a_2^{(j_1)}, \cdots, a_1^{(i_q)} a_2^{(j_q)}\}$.

   Then, for every set of derivatives $\{a_1^{(i_1)} a_2^{(j_1)}, \cdots, a_1^{(i_q)} a_2^{(j_q)}\}$, appearing in the multiderivatives indexing the columns, we select a corresponding set of inputs values such that every c_i and d_i are equal to zero except $c_{i_1} d_{j_1}, \cdots, c_{i_q} d_{j_q}$. So, we obtain T_n in a lower triangular form.

2. We select some inputs values in order to get every diagonal block D_r of T_n different from zero. We order the columns by increasing order of the exponent of the maximal order derivative of a_1 or a_2. For some suitable choices of the inputs values, we prove that every D_r may be written as a non degenerated scalar multiple of some non singular Vandermonde determinant.

5.4 Conclusion

A first step has been taken towards the algebraic identification. It remains to separate the coefficients of the words identical except for the order of the letters composing them.

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