New methods of constructing vertex-transitive non-Cayley graphs (An extended abstract)

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Abstract

A unifying approach to the problem of constructing vertex-transitive graphs that are not Cayley is presented. The construction is based on representing vertex-transitive graphs as coset graphs of groups, and on simple but powerful necessary arithmetic conditions for Cayley graphs. We present new constructions of infinite families of (finite as well as infinite) vertex-transitive non-Cayley graphs; many of these turn out to be even arc-transitive. The method is flexible enough for obtaining new results and reproving several older results concerning non-Cayley numbers (i.e., orders of vertex-transitive graphs that are not Cayley).

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Nous présentons une approche unifiée pour le problème de la construction de graphes qui sont transitifs par rapport aux sommets, mais qui ne sont pas des graphes de Cayley. La construction est basée sur une représentation de ces graphes comme graphes de classes à gauche de groupes, ainsi que sur une condition nécessaire arithmétique, simple mais puissante, pour les graphes de Cayley. Nons présentons des constructions nouvelles de familles infinies de graphes (finis on infinis) qui sont transitifs par rapport aux sommets mais qui ne sont pas des graphes de Cayley; il se trouve que beaucoup parmi eux sont transitifs par rapport aux arêtes aussi. La méthode est assez flexible pour nous permettre d' obtenir de nouveaux résultats ainsi que de redémontrer plusieurs résultats anciens concernant les nombres non-Cayley (nombres de sommets des graphes transitifs par rapport aux sommets qui ne sont pas des graphes de Cayley).

1 Introduction

Vertex-transitive graphs are interesting from both combinatorial as well as group-theoretical point of view, and have been studied extensively for more than a century. Although the well-known Cayley graphs have played a prominent role here, there has been an increasing interest in the other side of the fence - that is, in vertex-transitive graphs that are not Cayley graphs (we borrow the acronym VTNCG for such objects from [14]). By [11] , the problem of constructing VTNCG's is equivalent to the widely studied problem of the existence of certain permutation groups that do not have regular subgroups. And yet, only a few infinite families of VTNCG's have been described in the literature before 1980.

The situation, however, has dramatically changed in the last four years. Initiated originally by the problem of determining the so-called non-Cayley numbers, the orders of VTNCG's, posed by Marušič in [6], the search for VTNCG's has brought a wide range of different constructions ([3], [4], [6], [7], [8], [9], [10], and [14]). The general question of characterizing all VTNCG's is probably beyond our reach in the foreseeable future, but much progress has been done for orders that have only a few prime factors (see again [8] for references). New constructions of infinite families of VTNCG's are therefore of growing interest.

Basically, there seem to be two main approaches to the problem. The first assumes that we have enough information on the automorphism group of a given graph to show that it is transitive and cannot contain a regular subgroup. Examples with this property are mostly found among graphs that are related to some of the well-known families of finite groups, and most of the constructions listed or cited in [8] would fall in this category. The second approach consists in trying to reveal (without invoking the automorphism group) some structural conditions that a Cayley graph has to satisfy, and then show that these are not met by a particular class of vertex-transitive graphs. For such necessary conditons and corresponding constructions the reader is referred to [2, 3, 14].

We present new constructions of infinite families of (finite as well as infinite) VTNCG's. Moreover, imposing additional conditions we even obtain arc-transitive non-Cayley graphs (ATNCG's, for short). In a way, our method is a combination of the above ones. First, we represent a vertex-transitive graph by means of a suitable coset graph as in [5, 13, 12]. Then, we show that under some restrictions, our coset graphs do not satisfy simple but efficient necessary conditions for Cayley graphs. The results are constructions general enough for obtaining infinite families of vertex-transitive non-Cayley graphs as well as for reproving previously known results in a more efficient and systematic way. For the sake of brevity, proofs are not included in this extended abstract; some of them may be found in [4].

2 Preliminaries

Graphs considered may be finite or infinite, but are always locally finite (i.e., every vertex has finite valency), loopless and without multiple edges.

Let G be a (finite or infinite) group and X be a unit-free symmetric subset of G, that is, $1 \notin X$ and $x^{-1} \in X$ whenever $x \in X$. The *Cayley graph* $\Gamma = C(G, X)$ has G as its vertex set, and two vertices $a, b \in G$ are adjacent if and only if $a^{-1}b \in X$. Note that we do not require the set X to be a generating set for G and therefore we allow also disconnected Cayley graphs. The graph Γ is locally finite if and only if the set X is finite. In any case, the group G acts regularly (as a subgroup of automorphisms) on the vertex set of $\Gamma = C(G, X)$ by left multiplication, which shows that every Cayley graph is vertex-transitive. This necessary condition is not sufficient, and it is therefore reasonable to ask for more conditions imposed by the Cayley graph structure.

The following generalizations of a result originally proved in [2] seem to be quite powerful in applications. They focus on oriented closed walks based at a fixed vertex, that is, on ordered sequences $(a_0, a_1, \ldots, a_n = a_0)$ of (not necessarily distinct) vertices such that a_{i-1} and a_i are adjacent for each i, $1 \leq i \leq n$.

Lemma 1 *Let $\Gamma = C(G, X)$ be a locally finite Cayley graph and p be an odd prime. Then the number of closed oriented walks of length p^n, $n \geq 1$, based at any fixed vertex of Γ, is congruent (mod p) to the number of elements in X for which $x^{p^n} = 1$.*

Lemma 2 *Let $\Gamma = C(G, X)$ be a locally finite Cayley graph, and p and q be two distinct odd primes. Let $n = pq$ and let j_n be the number of generators $x \in X$ for which $x^n = 1$. Then the number of closed oriented walks of length* *n, based at any fixed vertex of* Γ*, is congruent* $(mod\ p)$ *to* $j_n + kq$*, where* k *is a nonnegative integer.*

The central concept of this paper is the one of a coset graph. Such graphs have apparently been known as "folklore" for decades (see [12] , or [5] for a more recent treatment; they happen to be a special case of the two-sided coset graphs introduced in [13]). Let G be a group, H a subgroup of G and X a symmetric subset of elements of G such that $H \cap X = \emptyset$. The vertex set of the *coset graph* $Cos(G, H, X)$ is the set of all left cosets of H in G; two vertices (cosets) aH and bH are adjacent in $Cos(G, H, X)$ if and only if $a^{-1}b \in HXH = \{hxh';\ x \in X \text{ and } h, h' \in H\}$. It is easy to check that this definition is correct, i.e., it does not depend on the choice of coset representatives and it produces graphs without loops and parallel edges.

An alternate way to define the incidence relation on $Cos(G, H, X)$ is by referring to the *associated Cayley graph* $C(G, X)$: Two cosets aH, bH are adjacent in $Cos(G, H, X)$ provided that there exist $h, h' \in H$ such that ah and bh' are adjacent vertices in the associated Cayley graph $C(G, X)$. The coset graph $Cos(G, H, X)$ can therefore be viewed as a graph obtained by "factoring" the associated Cayley graph $C(G, X)$ by the subgroup H. It is an easy exercise to show that the coset graph $Cos(G, H, X)$ is connected if and only if the set HXH is a generating set for the group G. Observe that in the special case when $H = \{1\}$, the coset graph reduces to a Cayley graph. For more information on coset graphs we refer the reader to [5].

As in the case of Cayley graphs, the group G acts transitively as a group of automorphisms of the coset graph $Cos(G, H, X)$ by left multiplication, and therefore every coset graph is vertex-transitive. (However, the action is no longer regular in general.) The converse has been proved in [5, 13, 12] for finite graphs, but the same proof applies also to infinite graphs (possibly of infinite valency): Given a vertex-transitive graph Γ, take a transitive subgroup of its automorphisms for G, the G-stabilizer of a fixed vertex for H, and define X to be the subset of automorphisms of G that are sending the fixed vertex to its neighbours. Then Γ is isomorphic to $Cos(G, H, X)$.

Lemma 3 *A graph* Γ *is vertex-transitive if and only if it is isomorphic to some coset graph* $Cos(G, H, X)$.

In some cases we can guarantee even a higher degree of symmetry of the coset graphs, namely, their arc-transitivity. The following simple observation shows how the existence of suitable *group automorphisms* can be used in this context. For notational convenience, if $Cos(G, H, X)$ is a coset graph, let $Aut_{H;X}(G)$ be the group of all the automorphisms of G which fix both X and the subgroup H setwise.

Lemma 4 *Let $\Gamma = Cos(G, H, X)$ be a coset graph. Assume that the group $Aut_{H;X}(G)$ contains a subgroup that acts transitively on X. Then Γ is an arc-transitive graph.*

3 Main results

The coset graph construction is general enough to yield all vertex-transitive graphs. In order to obtain VTNCG's, we need to impose certain restrictions on the triple (G, H, X). Applying Lemma 1, we then are able to show that the resulting graphs are not Cayley; the proof itself is not trivial.

Theorem 1 *Let G be a group, let H be a finite subgroup of G, and let X be a finite symmetric unit-free subset of G such that $XHX \cap H = \{1\}$. Further, suppose that there are at least $|X| + 1$ distinct ordered pairs $(x, h) \in X \times H$ such that $(xh)^p = 1$ for some fixed prime $p > |X||H|^2$. Then the coset graph $\Gamma = Cos(G, H, X)$ is a vertex-transitive non-Cayley graph.*

Combining Theorem 1 with Lemma 4 we obtain a means of constructing not only vertex-transitive, but even *arc-transitive* non-Cayley graphs:

Theorem 2 *Let a group G, a subgroup $H < G$, and a subset $X \subset G$ satisfy all assumptions of Theorem 1. Moreover, suppose that the group $Aut_{H;X}(G)$ contains a subgroup that acts transitively on X. Then the coset graph $\Gamma = Cos(G, H, X)$ is an arc-transitive non-Cayley graph.*

The following examples illustrate the use of Theorems 1 and 2. We start with the simplest case when the set X contains only one element (which is necessarily an involution). Note that the coset graphs built with a one-element set X are automatically arc-transitive (see Lemma 4).

Example 1. Let G be a non-trivial (finite or infinite) quotient of the *triangle group* $(2, r, p)$, that is, $G = \langle x, y | x^2 = y^r = (xy)^p = \ldots = 1\rangle$.

Assume that the presentation of G contains no relation of type $xy^ix = y^j$. Further, let $r \geq 3$ and let p be a prime greater than r^2. Then, Theorem 2 implies that the graph $Cos(G, <y>, \{x\})$ is a (connected) arc-transitive non-Cayley graph. Indeed, let $H = <y>$ and $X = \{x\}$. It is easy to see that the set HXH generates G. The absence of relations of the above type guarantees that $XHX \cap H = \{1\}$. Now, since $x^2 = 1$, $(xy)^p = 1$ implies that also $(xy^{-1})^p = 1$. Hence if $r \geq 3$ then there are at least 2 $(= |X| + 1)$ pairs $(x, h) \in X \times H$ such that $(xh)^p = 1$ for the prime $p > r^2$ $(= |X||H|^2)$. The rest follows from Theorem 2. □

Note that any graph constructed in this way is an underlying graph of a (finite or infinite) regular map. We thus obtained a special case of a more general result of [3], where it is shown that the underlying graph of any r-valent p-covalent regular map is an ATNCG provided that $r \geq 3$ and p is a prime greater than $r(r-1)$.

Example 2. Let $r \geq 2$ and $s \geq 2$ be such that $r+s$ is odd. Let S_p be the full symmetric group acting on the set $\{1, 2, \ldots, p\}$ where $p \geq r+s+2$ is a prime. Let $H = <y, z>$ be the subgroup of S_p generated by the permutations $y = (1, 2, \ldots, r)$ and $z = (r+1, r+2, \ldots, r+s)$. Obviously, $H \simeq Z_r \times Z_s$, and so $|H| = rs$. Further, let $x = (1, 2, \ldots, p)$ be a cyclic permutation of the entire underlying set and let $X = \{x, x^{-1}\}$. Let us consider the coset graph $\Gamma_p = Cos(S_p, H, X)$.

It is easy to see that if $r+s$ is odd then $<x, y, z> = S_p$. For instance, if r is even, then the permutation $w = (xy^{-1}x^{-1}yxy^{-1})^{p-r}$ is just a transposition of the elements 1 and r (the composition is to be read from the right to the left). Since p is a prime, the p-cycle x together with the transposition w are sufficient to generate S_p. Consequently, the coset graph Γ_p is connected. A routine checking of the conditions of Theorem 1 shows that Γ is also a vertex-transitive graph that is not Cayley. □

Eample 3. Again, let S_p be the symmetric group on the set $\{1, 2, \ldots, p\}$, where $p \geq 2(r+s)+1$ is a prime and $r+s$ is odd $(r, s \geq 2)$. We consider the same $X = \{x, x^{-1}\}$ where $x = (1, 2, \ldots, p)$. However, this time we pick a larger subgroup of S_p: Let $H' = <y, y', z, z'>$ where $y = (1, 2, \ldots, r)$, $y' = (p-1, p-2, \ldots, p-r)$, $z = (r+1, r+2, \ldots, r+s)$, and finally, $z' = (p-r-1, p-r-2, \ldots, p-r-s)$. Now, $H' \simeq Z_r \times Z_r \times Z_s \times Z_s$, and $|H'| = (rs)^2$.

Again, the coset graph $\Gamma'_p = Cos(S_p, H', X)$ is a (connected) VTNCG if $p > |X||H'|^2 = 2(rs)^4$. But we can show more. Let $p = 2k+1$ and let $\sigma \in S_p$ be the involution $(1, p-1)(2, p-2)\dots(k, k+1)$. Denote by ξ_σ the inner automorphism of S_p defined by σ, that is, $\xi_\sigma(w) = \sigma w \sigma$ for every $w \in S_p$. It is easy to verify that the subgroup $K = \{id, \xi_\sigma\} \subset Aut(S_p)$ fixes both H' and X and (obviously) acts transitively on X. It follows from Theorem 2 that Γ'_p is an ATNCG if $p > 2(rs)^4$; it has order $p!/(rs)^2$ and valency $2(rs)^2$. □

We have seen how to construct coset graphs that are VTNCG's (and also ATNCG's) using a cyclic group or some products of cyclic groups in place of H. Our next example presents a sufficiently general principle that can be adopted to construct finite as well as infinite VTNCG's by means of fairly arbitrary (abstract) groups H, and with arbitrarily large sets X.

Example 4. Let $m \geq 1$ and let M_i $(-m \leq i \leq m)$ be a system of pairwise disjoint sets of equal cardinality $|M_i| = q$ where $q \geq 3$ is an odd number. Let $p \geq (2m+1)q$ be a prime number. Take a finite set M' disjoint from all M_i such that $|M'| = p - (2m+1)q$. Let $L = (\cup_{-m \leq i \leq m} M_i) \cup M'$; clearly, $|L| = p$. Further, let M'' be an arbitrary (finite or infinite) set disjoint from L and let $\Omega = L \cup M''$.

Denote by S_Ω the (full) symmetric group on the set Ω. For $1 \leq i \leq m$ let $x_i \in S_\Omega$ be a permutation of order p (i.e., $x_i^p = id$) such that its restriction to the set L is a cyclic permutation of L with the property that $x_i(M_{-i}) = M_0$ and $x_i(M_0) = M_i$ (that is, the images of the set M_0 under x_i and x_i^{-1} are M_i and M_{-i}, respectively). Consider next the action of the permutation x_1 on the set M_0. Let $M_0 = \{a_1, a_2, \dots, a_q\}$ and let the restriction of x_1 on L be the cyclic permutation $(a_{j_1}, \dots, a_{j_2}, \dots, \dots, a_{j_q}, \dots)$, where the dots represent the remaining $p - q$ elements of the set L. This way, x_1 defines a unique permutation $x_0 \in S_\Omega$ whose restriction to M_0 is the cyclic permutation $x_0 = (a_{j_1}, a_{j_2}, \dots a_{j_q})$ of M_0, and such that x_0 fixes every element in $\Omega \setminus M_0$. The important fact to observe is that $(x_1 x_0)^p = id$ (this is the place where we use the fact that q is odd).

Now, let $X = \{x_i, x_i^{-1};\ 1 \leq i \leq m\}$; note that $|X| = 2m$. Let H be an arbitrary subgroup of S_Ω fixing the set $\Omega \setminus M_0$ pointwise and such that $x_0 \in H$. Let $G_{H,X}$ be the subgroup of S_Ω generated by the elements in $H \cup X$. Since $X^{-1} = X$, the coset graph $\Gamma_{H,X} = Cos(G_{H,X}, H, X)$ is well defined and connected. Moreover, if the set M'' (and hence Ω) is infinite, the group $G_{H,X}$ may be infinite as well (observe that we did not restrict the action of the permutations in X on the set $\Omega \setminus L$ in any other way except for the requirement that $x_i^p = id$). But in any case, the group H and the set X are finite, and so our coset graph is always a locally finite VTNCG. □

A close examination of the proof of Theorem 1 in [4] reveals a possibility for an immediate improvement of the original lower bound on p. Let l_p denote the number of distinct pairs (x, h) in $X \times H$ such that $(xh)^p = 1$. For obvious reasons, $l_p \leq |X||H|$, and, in general, l_p is considerably smaller than $|X||H|$. Without any alteration of the proof, the original bound $p > |X||H|^2$ can be replaced by $p > l_p|H|$. This is usually a considerable improvement of the size of p used in applications. Consider, for instance, the case of the triangle group $(2, r, p)$ (Example 1 of [4]). The original lower bound $p > r^2$ can be improved by using the fact that $(x, 1)$ obviously does not satisfy the identity $(x \cdot 1)^p = 1$. Thus l_p is not bigger than $r - 1$ and therefore it is enough to require $p > r(r-1)$. This lower bound matches the one in [3] obtained by much more subtle methods. A number of other improvements finally lead to the following result.

Theorem 3 *Let G be a group, let H be a finite subgroup of G, and let X be a finite symmetric unit-free subset of G such that $XHX \cap H = \{1\}$. Let m be an odd positive integer, $m = p_1^{k_1} \ldots p_r^{k_r}$ be its prime factorization, and let $l_i, 1 \leq i \leq r$, denote the number of distinct pairs (x, h) in $X \times H$ such that $(xh)^{p_i^{k_i}} = 1$. Suppose that $\sum_{i=1}^r l_i > |X|$, and, for all i, $p_i > l_i|H|$. Then the coset graph $\Gamma = Cos(G, H, X)$ is a vertex-transitive non-Cayley graph.*

We have stated Theorem 3 in a very general setting with m being quite arbitrary. For practical applications we would like to make the following remark. Suppose that G is finite and p is an odd prime that *does not* divide the order of G. Then p *does not* divide the size of the vertex set of $\Gamma = Cos(G, H, X)$ either, and therefore Γ, even if it happens to be Cayley, *cannot* possibly have generators of order p. On the other hand, since G contains no elements of order p, the number of pairs (x, h) in $X \times H$, satisfying the equality $(xh)^p = 1$, is zero as well. Thus, p contributes 0 to both sides of our inequality and therefore carries no information of whether the obtained graph is Cayley or not. Consequently, to construct finite VTNCG's, we are only interested in numbers m that divide the order of G. Obviously, there are no limits on the choice of m for infinite G's.

The obvious advantage of Theorem 3 against Theorem 1 is well illustrated in the following simple example.

Example 5. Let $p > q$ be two odd primes such that $2(p-q)(p-q+1) < q$. Take $y = (1\ 2 \ldots p-q+1)$ and $x = (p-q+1 \ldots p)$, two permutations of the set $\{1, 2, \ldots, p\}$, and consider the permutation group $G = <x, y>$, generated by x and y. Let $H = <y>$ and $X = \{x, x^{-1}\}$. Obviously, $XHX \cap H = \{1\}$. Furthermore, $l_q \geq 2$, since x and x^{-1} are both of order q, and $l_p \geq 2$, since $(xy)^p = (x^{-1}y)^p = 1$. Thus, $l_q + l_p \geq 4 > 2 = |X|$. Also, $p > q > 2(p-q)(p-q+1)$, where $2(p-q)(p-q+1)$ is an upper bound for both $l_q|H|$ and $l_p|H|$. Theorem 2 implies that $Cos(G, H, X)$ is a vertex-transitive non-Cayley graph. □

One of the basic questions related to VTNCG's is the problem of characterizing the positive integers n, for which there exists a VTNCG of order n, the so-called *non-Cayley numbers* ([6]). Since any of the multiples of a non-Cayley number is also non-Cayley, most of the work in the area is devoted to products of small powers of prime factors ([7], [8], [9], [10]).

The previous constructions are easy to use and yield a large number of possible alterations, and, eventually, of new VTNCG's. The sizes of the obtained graphs, however, are usually close to factorials. It is also hard to have control over the sizes of the obtained graphs. This makes this construction unsuitable for finding non-Cayley numbers. The following theorem gives rise to several other constructions, more feasible for constructing VTNCG's of a prechosen order. (Recall that $Cos(G, H, X)$ is connected iff the set HXH generates the group G.)

Theorem 4 *Let* $\Gamma = Cos(G, H, X)$ *be a* connected *coset graph and* p *be an odd integer which satisfy the following conditions:*

- $XHX \cap H = \{1\}$,
- *no finite group* G' *of order* $|G|/|H|$ *can be generated by a set of elements all of which are of order divisible by* p,
- *the number* l_p, *of pairs* $(x, h) \in X \times H$ *for which* $(xh)^p = 1$, *is greater than or equal to* $|X|$, *and*

- $p > l_p|H|$.

Then Γ is a vertex-transitive non-Cayley graph.

We illustrate the above result in our last example.

Example 6. Let $p > q$ be two primes and $n < p/2$ be a positive integer. Suppose that p *does not* divide any of the numbers $q^i - 1$, $1 \leq i \leq n$. Then any group of order $(pq)^n$ contains a normal Sylow p-group and *cannot* be generated by elements of order divisible by p alone. Once more, this conclusion allows us to construct a coset graph satisfying the conditions of Theorem 3. Let G be the *wreath product* of the group $\mathcal{Z}_p \times \mathcal{Z}_q$ with $\mathcal{Z}_n$ acting on $\{1, 2, \dots, n\}$ in the usual cyclic way. Then $|G| = (pq)^n n$. Let $H = \langle((0,0), \dots, (0,0); (12\dots n))\rangle$ be the isomorphic copy of $\mathcal{Z}_n$ in G, and let $X = \{((1,0), (0,0), (0,0), \dots, (0,0); id), ((p-1,0), (0,0), (0,0), \dots, (0,0); id)\}$. Then $XHX \cap H = \{((0,0), \dots, (0,0); id)\}$, $l_p = 2$ (since $p > n$, H cannot contain elements of order divisible by p), and $p > l_p|H| = 2n$, by assumption. All this together proves that $Cos(G, H, X)$ is a VTNCG of order $(pq)^n$. $\square$

References

[1] B. Alspach and T.D.Parsons, A construction for vertex-transitive graphs, *Can. J. Math.* **34** (1982), 307-318.

[2] D. Fronček, A. Rosa and J. Širáň, The existence of selfcomplementary circulant graphs, *European J. Combin.* (in press).

[3] P. Gvozdjak and J. Širáň, A topological construction of arc-transitive non-Cayley graphs, *Preprint* (submitted).

[4] R. Jajcay and J. Širáň, A construction of vertex-transitive non-Cayley graphs, *Australasian J. Combin.* (to appear).

[5] P. Lorimer, Vertex-transitive graphs: Symmetric graphs of prime valency, *J. Graph Theory* **8** (1984), 55-68.

[6] D. Marušič, Cayley properties of vertex-symmetric graphs, *Ars Combinatoria* **16B** (1983), 297-302.

[7] D. Marušič and R. Scapellato, Classifying vertex-transitive graphs whose order is a product of two primes, *Combinatorica* **14** (2) (1994), 187-201.

[8] B. D. McKay and Ch. E. Praeger, Vertex-transitive graphs which are not Cayley graphs, I and II, *Preprints* (submitted).

[9] A. A. Miller and C. E. Praeger, Non-Cayley vertex-transitive graphs of order twice the product of two odd primes, Research Report at The University of Western Australia.

[10] C. E. Praeger and M.-Y. Xu, Vertex primitive graphs of order a product of two distinct primes, *J. Combin. Theory* Ser. B. (in press).

[11] G. Sabidussi, On a class of fixed-point free graphs, *Proc. Amer. Math. Soc.* **9** (1958), 800-804.

[12] H. H. Teh and S. C. Shee, *Algebraic Theory of Graphs*, Lee Kong Chian Institute of Mathematics and Computer Science, Singapore (1976).

[13] A. Vince, Locally homogeneous graphs from groups, *J. Graph Theory* **5** (1981), 417-422.

[14] M. E. Watkins, Vertex-transitive graphs that are not Cayley graphs, *in: Cycles and Rays* (G.Hahn et al. (eds.), Kluwer, Netherlands, 1990), 243-256.