ENUMERATING HOMOMORPHISMS AND SURFACE-COVERINGS

by Gareth A. Jones

Abstract. *Möbius inversion (in finite groups) and representation theory are used to enumerate homomorphisms from a finitely-generated group to a finite group, and hence to enumerate regular coverings of a surface with a given finite covering-group.*

Coverings. Let $\gamma : \tilde{\Sigma} \to \Sigma$ be an unbranched covering of a compact, connected, orientable surface $\Sigma = \Sigma_g$ of genus $g \geq 0$. The covering transformations of γ (the self-homeomorphisms α of $\tilde{\Sigma}$ such that $\gamma \circ \alpha = \gamma$) form a group $G = \operatorname{Aut}\gamma$ which preserves each fibre $\gamma^{-1}(p)$, $p \in \Sigma$. Suppose that G is finite, and that γ is *regular* (so G acts transitively on each fibre, or equivalently $\Sigma \approx \tilde{\Sigma}/G$ and γ is induced by the projection $\tilde{\Sigma} \to \tilde{\Sigma}/G$). Then $\deg\gamma = |G|$, that is, $|\gamma^{-1}(p)| = |G|$ for all $p \in \Sigma$, and $\tilde{\Sigma}$ is compact, of genus $\tilde{g} = 1 + |G|(g-1)$.

This situation can be described algebraically using the fundamental group $\Pi_g = \pi_1(\Sigma_g)$. The covering $\gamma : \Sigma_{\tilde{g}} \to \Sigma_g$ induces a monomorphism $\gamma_* : \Pi_{\tilde{g}} \to \Pi_g$, with image $N = \gamma_*(\Pi_{\tilde{g}}) \cong \Pi_{\tilde{g}}$ of index $\deg\gamma$ in Π_g. Two coverings γ_1 and $\gamma_2 : \Sigma_{\tilde{g}} \to \Sigma_g$ are *equivalent* if $\gamma_1 = \gamma_2 \circ \alpha$ for some self-homeomorphism α of $\Sigma_{\tilde{g}}$; this happens if and only if the corresponding subgroups $N_i = (\gamma_i)_*(\Pi_{\tilde{g}})$ are conjugate in Π_g. A covering γ is regular if and only if it corresponds to a normal subgroup N of Π_g, in which case $\Pi_g/N \cong G$. It follows that, for a given group G and genus g, the equivalence classes $[\gamma]$ of regular coverings γ of Σ_g, with covering group G, are in one-to-one correspondence with the elements of the set

$$\mathcal{N}_g(G) = \{\, N \triangleleft \Pi_g \mid \Pi_g/N \cong G \,\}$$

of normal subgroups of Π_g with quotient group G. My aim is to find the number $n_g(G) = |\mathcal{N}_g(G)|$ of such subgroups N (and hence the number of equivalence classes $[\gamma]$), as a function of g for each finite group G.

Counting normal subgroups. Hall [3] introduced a general technique for finding the number $n_\Gamma(G)$ of normal subgroups N of any finitely generated group Γ with a given finite quotient group $\Gamma/N \cong G$. Such subgroups are the kernels of the epimorphisms $\theta : \Gamma \to G$; the set $\operatorname{Epi}(\Gamma, G)$ of such epimorphisms is finite, since the generators of Γ can be mapped into G in only finitely many ways, so $n_\Gamma(G)$ is finite. If $\theta_1, \theta_2 \in \operatorname{Epi}(\Gamma, G)$, then $\ker\theta_1 = \ker\theta_2$ if and only if $\theta_2 = \theta_1 \circ \alpha$ for some automorphism α of G, so $n_\Gamma(G) = |\operatorname{Epi}(\Gamma, G)/\operatorname{Aut} G|$, the number of orbits of $\operatorname{Aut} G$ acting by composition on $\operatorname{Epi}(\Gamma, G)$. This action is fixed-point-free, so all orbits have length $|\operatorname{Aut} G|$ and hence

$$n_\Gamma(G) = \frac{|\operatorname{Epi}(\Gamma, G)|}{|\operatorname{Aut} G|}.$$

As an example, let Γ be the fundamental group Π_g, with presentation

$$\Pi_g = \langle A_1, B_1, \ldots, A_g, B_g \mid \prod_{i=1}^{g}[A_i, B_i] = 1 \rangle$$

(where $[A, B]$ denotes the commutator $A^{-1}B^{-1}AB$), and let $G = C_p$, a cyclic group of prime order p. The homomorphisms $\theta : \Pi_g \to G$ correspond to mappings of the generators A_i, B_i of Π_g to elements $a_i = A_i\theta, b_i = B_i\theta$ of G (since $G = C_p$ is abelian, $\prod[a_i, b_i] = 1$ and hence each such mapping extends to a homomorphism).

There are p^{2g} mappings, and since G is prime the only one which does not extend to an epimorphism is the trivial mapping given by $a_i = b_i = 1$ for all i, so $|\operatorname{Epi}(\Pi_g, C_p)| = p^{2g} - 1$. Now $|\operatorname{Aut} C_p| = \phi(p) = p - 1$, where ϕ is Euler's function on $\mathbf{N}$, so we obtain Mednykh's formula [7]

$$n_g(C_p) = \frac{p^{2g} - 1}{p - 1}.$$

In general, one can count epimorphisms $\Gamma \to G$ by first counting homomorphisms and then eliminating those which map Γ onto proper subgroups $K < G$. We have

$$|\mathrm{Hom}(\Gamma, G)| = \sum_{K \leq G} |\mathrm{Epi}(\Gamma, K)|,$$

and one can invert this equation, to count epimorphisms in terms of homomorphisms, by introducing the *Möbius function* for G. This assigns an integer $\mu(K)$ to each subgroup K of G by the recursive formula

$$\sum_{H \geq K} \mu(H) = \delta_{K,G} = \begin{cases} 1 & \text{if } K = G, \\ 0 & \text{if } K < G. \end{cases}$$

One then easily deduces Hall's formula

$$|\mathrm{Epi}(\Gamma, G)| = \sum_{H \leq G} \mu(H) |\mathrm{Hom}(\Gamma, H)|.$$

For many groups G, it is a routine task to find $|\mathrm{Aut}\, G|$ and $\mu(H)$ for all $H \leq G$, so one is left with the problem of counting homomorphisms $\Gamma \to H$.

When $\Gamma = \Pi_g$ the number

$$\sigma_g(H) = |\mathrm{Hom}(\Pi_g, H)|$$

of homomorphisms $\theta : \Pi_g \to H$ is equal to the number of solutions a_i, b_i in H of the equation $\prod_{i=1}^{g}[a_i, b_i] = 1$. One can evaluate this number by means of the following theorem, proved for $g = 1$ by Frobenius [1], and for $g > 1$ by Mednykh [6].

Theorem. *For any finite group H,*

$$\sigma_g(H) = |H|^{2g-1} \sum_{\rho} d_\rho^{2-2g},$$

where ρ ranges over the irreducible complex representations of H, and d_ρ denotes the degree of ρ.

Applying this to all $H \leq G$, one can now compute $n_g(G)$ for many finite groups G (see [4]). For example:

1) For a cyclic group C_n of order n we have

$$n_g(C_n) = \frac{1}{\phi(n)} \sum_{m|n} m^{2g} \mu(\frac{n}{m}),$$

where μ is the Möbius function on $\mathbf{N}$.

2) If $G = C_p \times \cdots \times C_p$ (k factors, p prime) is an elementary abelian p-group of rank k, then $n_g(G)$ is the Gaussian coefficient

$$\binom{2g}{k}_p = \frac{(2g)_p}{(k)_p.(2g-k)_p},$$

the number of k-dimensional subspaces in a $2g$-dimensional vector space over $GF(p)$, where

$$(n)_q = (q^n - 1)(q^{n-1} - 1) \ldots (q - 1).$$

3) The formula for $n_g(D_n)$ less pleasant, but for $n = p$ (an odd prime) it simplifies to

$$n_g(D_p) = \frac{(2^{2g} - 1)(p^{2g-2} - 1)}{(p - 1)}.$$

4)
$$n_g(A_4) = \frac{1}{6}(3^{2g} - 1)(4^{2g-2} - 1)$$

and
$$n_g(S_4) = \frac{1}{2}(2^{2g} - 1)(3^{2g-2} - 1)(4^{2g-2} - 1)\,.$$

Non-orientable surfaces. A similar theory applies to unbranched coverings of a *non-orientable* surface $\Sigma = \Sigma_g^-$ of genus $g \geq 1$. The only essential difference is that the fundamental group of Σ is now

$$\Pi_g^- = \langle R_1, \ldots, R_g \mid R_1^2 \ldots R_g^2 = 1 \rangle\,,$$

so in place of $\sigma_g(H)$ one needs

$\sigma_g^-(H) = |\mathrm{Hom}\,(\Pi_g^-, H)|$, the number of solutions r_i in H of the equation $r_1^2 \ldots r_g^2 = 1$. Again, one can count these using representation theory. Let $\chi : H \to \mathbf{C}, h \mapsto \mathrm{tr}(\rho(h))$ be the character of an irreducible complex representation ρ of H. The *Frobenius-Schur indicator* of ρ is

$$c_\rho = \begin{cases} 1 & \text{if } \rho \text{ is real,} \\ -1 & \text{if } \chi \text{ is real but } \rho \text{ is not,} \\ 0 & \text{if } \chi \text{ is not real.} \end{cases}$$

Frobenius and Schur [2] (with no apparent topological motivation) proved the following result.

Theorem. *For any finite group H,*

$$\sigma_g^-(H) = |H|^{g-1} \sum_\rho c_\rho^g d_\rho^{2-g},$$

where ρ ranges over the irreducible complex representations of H.

Using this, one can find the number $n_g^-(G)$ of equivalence classes of regular unbranched coverings of Σ_g^-, with covering group G, for many finite groups G (see [4]). For example:

1)
$$n_g^-(C_n) = \frac{1}{\phi(n)} \sum_{m|n} \mu(\frac{n}{m}) \eta_m m^{g-1}\,,$$

where $\eta_m = 1$ or 2 as m is odd or even.

2)
$$n_g^-(D_n) = \frac{1}{\phi(n)} \sum_{m|n} \mu(\frac{n}{m}) m^{g-2}(m + \eta_m(2^g - 2))\,.$$

3)
$$n_g^-(A_4) = \frac{1}{6}(3^{g-1} - 1)(4^{g-2} - 1)\,.$$

Punctured surfaces and branched coverings. If $\Sigma_{g,r}$ and $\Sigma_{g,r}^-$ are formed by removing r points from Σ_g and Σ_g^- respectively, where $0 < r < \infty$, then one has to add r generators $X_1, \ldots, X_r$ (corresponding to loops around the punctures) to the fundamental groups Π_g and Π_g^-, and change their defining relations to $[A_1, B_1] \ldots [A_g, B_g] X_1 \ldots X_r = 1$ and $R_1^2 \ldots R_g^2 X_1 \ldots X_r = 1$ respectively.

Using representation theory, one can calculate the numbers of solutions of the corresponding equations in any finite group H (with $x_i = X_i\theta$ restricted to various unions of conjugacy classes, such as the elements of a given order, if necessary). From this, one can count the equivalence classes of regular unbranched coverings of these punctured surfaces. Similarly, one can count regular *branched* coverings by removing branch-points to create unbranched coverings of punctured surfaces. For example the number of equivalence classes of regular branched coverings of Σ_g, with r given branch-points and with covering group C_p (p prime), is $p^{2g-1}((p-1)^{r-1} + (-1)^r)$. In [5] this method is extended to count normal subgroups of a non-euclidean crystallographic group without reflections; these correspond to regular coverings of an orbifold whose underlying surface is without boundary.

Self-homeomorphisms of Σ. Finally, one can consider the effect of self-homeomorphisms of $\Sigma = \Sigma_g$ (in addition to those of $\Sigma_{\bar{g}}$) in counting regular coverings. By the Dehn-Nielsen Theorem [8, 10], two such coverings are equivalent under self-homeomorphisms of Σ_g if and only if the corresponding normal subgroups N of Π_g are equivalent under $\mathrm{Aut}\,\Pi_g$, so the number $\overline{n}_g(G)$ of such equivalence classes is equal to $|\mathcal{N}_g(G)/\mathrm{Aut}\,\Pi_g|$. When $G = C_p \times \cdots \times C_p$, an elementary abelian p-group of rank k (p prime), one can calculate this number by using the facts that all such subgroups N contain the commutator subgroup Π_g', and that $\mathrm{Aut}\,\Pi_g$ acts on the first homology group

$$H_1(\Sigma_g, \mathbf{Z}) = \Pi_g^{\mathrm{ab}} = \Pi_g/\Pi_g'$$

as the general symplectic group $\mathrm{GSp}(2g, \mathbf{Z})$ corresponding to the bilinear intersection form on Σ_g. Using Witt's Theorem [9] on equivalence of subspaces in a symplectic space, one finds that

$$\overline{n}_g(G) = 1 + \lfloor \frac{1}{2} \min(k, 2g-k) \rfloor ,$$

where $\lfloor x \rfloor$ denotes the integer part of x. (See [4] for details.)

For instance, by putting $k = 1$ we have $\overline{n}_g(C_p) = 1$, so all $n_g(C_p) = (p^{2g}-1)/(p-1)$ coverings are equivalent in this sense. Putting $k = 2$, we see that $\overline{n}_g(C_p \times C_p) = 1$ or 2 as $g = 1$ or $g \geq 2$, and in the case $g = 2$ one can easily illustrate two inequivalent actions of $C_p \times C_p$ on $\Sigma_{\bar{g}} = \Sigma_{1+p^2}$.

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Department of Mathematics
University of Southampton
Southampton SO17 1BJ
United Kingdom