Conjectures and Constructions About Perpendicular Pairs — by Experiment

Extended Abstract and Preliminary Report
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Abstract

We have conducted extensive experiments with a novel software tool *Order Explorer* in order to advance the heretofore computationally inaccessible theme of *perpendicular orders* — orders on the same underlying set which have no common isotone self-map, besides the *constants* and the *identity*.

Isotone Self-Maps and Perpendicular Pairs

An *isotone self-map* of an ordered set P is a map f of P to P such that, for every $x \leq y, f(x) \leq f(y)$. Of course, the *identity* $id(x) = x$, for every $x \in P$ is a isotone self-map and so is every *constant* map : for every $c \in P$, $const_c(x) = c$, for every $x \in P$. There are many more : in fact, for any n-element ordered set P, $|P^P| \geq 2^{O(n)}$ isotone self-maps, where P^P stands for the set of all isotone self-maps of P to P (cf.. the survey [**Zaguia** (1993)]). Figure 1 illustrates some small examples each with its associated number of isotone self-maps.

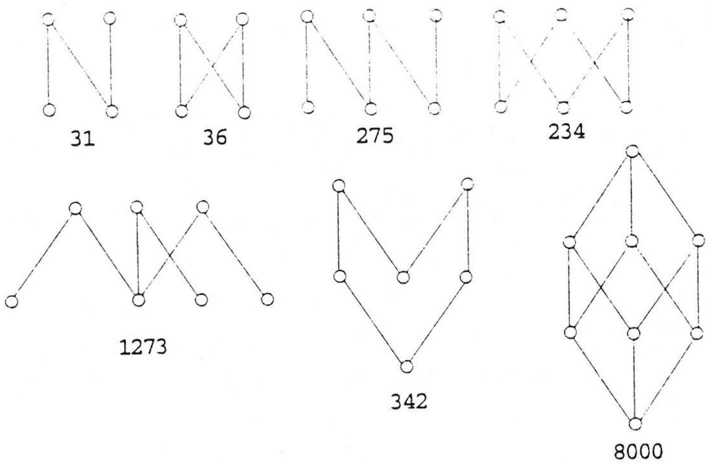

Figure 1: Enumeration of the isotone self-maps of an ordered set.

Let P and Q be ordered sets defined on the same n-element underlying set. Thus, the *identity* map on this underlying set is isotone for both P and Q. Every *constant* map, too, of either, is an isotone self-map of both. Call P and Q *perpendicular*, and write $P \perp Q$, if these are the only common isotone self-maps, that is, if $P^P \cap Q^Q$ consist just of the n *constant* self-maps and one *identity* self-map.

The concept of perpendicular orders has a quite recent origin, due to [**Demetrovics, Miyakawa, Rosenberg, Simovici,** and **Stojmenović** (1990)] who first showed that there exist (bipartite) perpendicular pairs on a set of size n, for each $n \geq 4$. They were motivated by the older problem to describe the structure of the lattice of "clones" on a set. (A *clone* on a set S is a composition-closed subset of the set of all maps of S^m to S, $m \geq 1$, which contains all of the projection maps $proj_i^m(a_1, a_2, \ldots, a_m) = a_i$, for all $a_1, a_2, \ldots, a_m \in S$.) Any perpendicular pair of orders yields a pair of isotone clones whose intersection is precisely the clone consisting of all projections and all constant maps (cf. [[**Länger** and **Pöschel** (1984)], [**Pálfy** (1984)]).

All evidence to date suggests that 'almost' every ordered set has a perpendicular, although it is easy to check that there are no perpendicular orders on a set of size $n \leq 3$. Moreover, any complete bipartite ordered set $K_{m,n}$ cannot have a perpendicular. There seem to be few other exceptions.

Conjecture 1 *Every ordered set without autonomous subset has a perpen-*

dicular.

A proper subset S, with at least two elements, of an order P, is *autonomous* if, for every $x, y \in S$ and for every $z \in P \setminus S, x \leq z, x \geq z$ if and only if $y \leq z, y \geq z$, respectively. Loosely speaking, the elements of an autonomous set "see" all outside elements the "same way". A perpendicular pair $P \perp Q$ cannot have a common autonomous subset at all. For, suppose, A were an autonomous subset of P and of Q. Let $a \in A$. Then the self-map defined by

$$f(x) = \begin{cases} a & \text{if } x \in A \\ x & \text{if } x \notin A \end{cases}$$

is a common isotone self-map of both which is neither a *constant* map nor the *identity*. Among the few positive results is this: *an order is perpendicular to any complementary order if and only if it has no autonomous subset* [**Rival** and **Zaguia** (1993)].

The purpose of this paper is to illustrate the advances that we have made on this conjecture using *Order Explorer*.

Order Explorer

Wherever and whenever data is presented for decision-making, it is common to display it pictorially, or, at least, graphically. Interactivity and visualization are key features driving the current graph drawing enterprise.

Although there is a widespread recognition of the importance of upward drawings of ordered sets, there is still little practical progress on the problem to display them in a conceptually transparent manner. And there is little evidence of any graph editor that serves as a substantial research tool. The major stumbling block is the *Initial Value Problem*

This is the problem to construct any upward drawing at all — subject to the customary constraints, and subject, especially, to the tradeoff between space limitations and unwanted collineations.

What are the customary constraints?

The elements of the ordered set P are drawn on a surface, traditionally a plane, as disjoint small circles, arranged in such a way that, for $a, b \in P$, the circle corresponding to a is higher than the circle corresponding to b whenever $a > b$ and an arc, monotonic with respect to a fixed direction, usually south to north, and straight, is drawn to join them just if *a covers b* (that is, for each $x \in P$, $a > x \geq b$ implies $x = b$). We say that a is an *upper cover* of b or b is a *lower cover* of a, and write $a \succ b$ or $b \prec a$.

What are unwanted collineations?

Apart from simple blackboard illustrations, much more need be said. With limited space (e.g., a standard 14" computer screen) layout of the vertices is crucial. However, as anyone who has ever drawn more than a few upward drawings will appreciate, constricting the space on which it is drawn risks unwanted coincidences of vertices with edges.

While there are various artifacts to resolve the problem of unwanted comparabilities they all result in sacrifice of considerable space, thereby limiting the size of the orders whose upward drawings may effectively be illustrated. (For instance, we may locate the vertices, according to some linear extension of the ordered set, monotonically along the arc of a circle. All covering edges will lie properly in the interior of the circle, thus avoiding unwanted coincidences. Not a helpful upward drawing, however!) Indeed, except for the apparently exceptional case of ordered sets which are embeddable on a two-dimensional grid, there seems little[1] (even theoretical) hope.

The *Initial Value Problem* for upward drawings is the problem to exhibit, automatically, an upward drawing of an ordered set.

Order Explorer[2] is a novel software tool which, already in the laboratory setting, has made substantial inroads both as a visual display of hierarchical structures as well as a research tool. In brief, *Order Explorer* replaces the cumbersome tradition of *graph edges* by the flexibility and ease of *interactivity*. Simple "mouse" clicks reveal *comparabilities, adjacencies, upper* and *lower covers*, etc. *Order Explorer* draws on a small (but growing) library of algorithmic tools from the classical order-theoretic literature. It is ironic that, while *Order Explorer* would seem to complicate the graph drawing enterprise by concealing what is, heretofore, commonly regarded as the graph's essence, it thereby removes the source of much graphical reading and writing

[1] A common industrial approach is to add subdivision (dummy) vertices until the ordered is *graded* and draw the ordered set according to an antichain decomposition. Finally, the dummy vertices may be removed by replacing paths on which they lie by a spline approximation.

[2] *Order Explorer* is written in *Visual Basic*. It easily manipulates orders of size 75 – 100 elements, having an essentially instantaneous response time for such algorithms as *chain decomposition, antichain decomposition, dimension two testing* and others.

difficulties!

Computer technology is largely of use in experimental mathematics where, typically, the computational advantage is dedicated to complicated calculations. On the other hand, in the tradition of modern combinatorial theory, it is common, in the *ordered sets* literature, to find numerous pictures as aids to reasoning, to communicate and explain the main ideas, and, moreover, to motivate and illustrate conjectures. Ideal would be a visualization tool driven by a sophisticated computational engine.

Order Explorer as Research Tool

To verify that a particular pair P, Q of orders is perpendicular — even small ones of size ten elements — takes many hours of manual work. On the other hand, the trouble with a direct computational check, using matrix input, (incidence matrix) say, is that, once a negative outcome obtains, how are we to modify Q, to obtain Q' and then check whether $P \perp Q'$? That is, how to decide what is the obstruction to the perpendicularity of P and Q? Indeed, if the outcome is negative, it is because there are nontrivial common isotone self-maps, which, therefore, must be understood, with a view to modifying Q appropriately. *Order Explorer* provides a tool.

Given a comparable pair $a > b$ in an ordered set P, it is clear that

$$f(x) = \begin{cases} a & \text{if } x \geq a \\ b & \text{if } x \ngeq a \end{cases}$$

is an isotone self-map. Moreover, although it is not always the case, one typical obstruction[3] to $P \perp Q$ is the existence of a *common* comparable pair, that is, a pair a, b of elements such that $a > b$ in P and either $a > b$ or $a < b$ in Q. Our opening strategy to construct a perpendicular to an order P (without autonomous subset) is to construct an order Q, on the same set as P, and with no comparabilities in common with P. To avoid a trivial construction for Q (e.g. Q an antichain) we intend to construct Q without autonomous subset, too. *Order Explorer* has the facility to easily display upward drawings as well as to visualize a common nontrivial isotone self-map — if one exists.

[3]There exist perpendicular pairs $P \perp Q$ which contain common comparable pairs, e.g., $P = \{a < b < c < d\} \perp \{c < a < d < b\} = Q$.

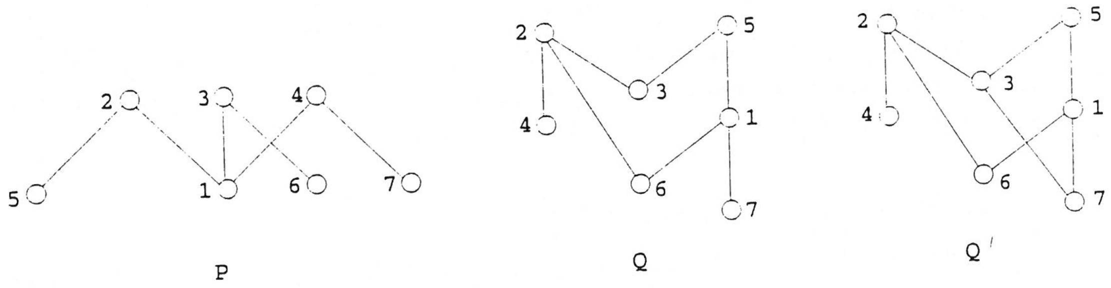

Figure 2: Three orders P, Q, and Q′. No pair has a common comparability, Q is not perpendicular to P, but Q′ is perpendicular to P.

Question 1 *If orders P and Q on the same set have no common comparability is $P \perp Q$?*

With *Order Explorer* it is possible, for an order P, to construct, on-line, an order Q with no common comparability. Despite the fact that a given order P has many isotone self-maps, *Order Explorer* is able to detect common isotone self-maps quickly. Indeed, for a given order P, there are many orders Q, without common comparabilities, and we uncovered many which are not perpendicular to P at all[4]. Ironically, the relative *ease* of this computation led us to this conjecture.

Conjecture 2 *Orders, on the same underlying set, which have no common comparability and no autonomous subset, have "few" common isotone self-maps.*

It is evident that progress on the question of perpendicular orders relies much on the availability of examples, to nourish conjectures and to disprove them. Is there a construction which furnishes new perpendicular pairs from existing ones? *Order Explorer* made it easy to consider this natural question.

[4]In laboratory trials with ordered sets of size up to thirteen, in which there is no autonomous subset, we were able to use *Order Explorer* to construct, after a few iterations, a perpendicular — typically within fifteen minutes of interactivity.

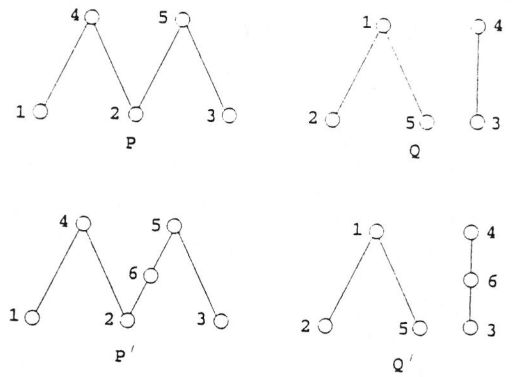

Figure 3: P and Q are perpendicular, but P′ and Q′ are not.

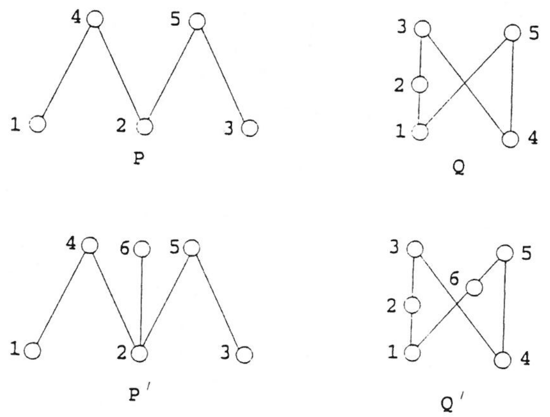

Figure 4: P and Q are perpendicular, but P′ and Q′ are not.

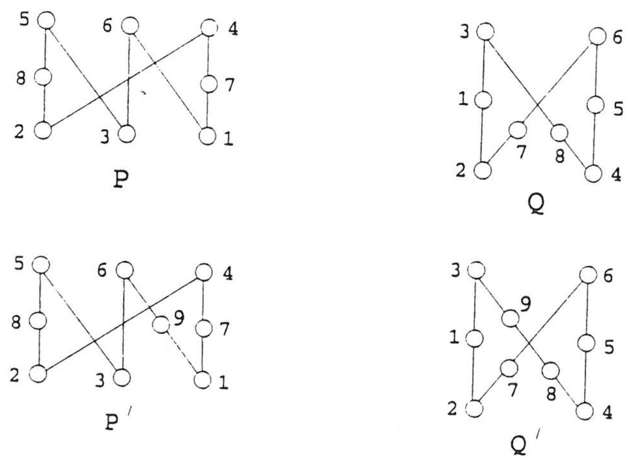

Figure 5: P and Q are perpendicular, P′ and Q′ are perpendicular,

Question 2 *Let $P \perp Q$. If $P' = P \cup \{a\}$ and $Q' = Q \cup \{a\}$ are both obtained by subdividing an edge such that P' and Q' have no common compar bility, is $P' \perp Q'$?*

Given a covering edge $x \succ y$ in P (that is, a comparability $x > y$ such that, for any $z \in P$, $x > z \geq y$ in P implies $z = y$), the ordered set $P' = P \cup \{a\}$ whose order is the transitive closure of P with $x \succ a \succ y$, is called a *subdivision* of P. Evidently, an isotone self-map f of P is readily extended to an isotone self-map of P' by $f(a) = x$ or $f(a) = y$. Thus, there would seem to be little difference between P^P and $P'^{P'}$.

Again, using *Order Explorer* we were able to find examples of orders satisfying the conditions of **Question 2** which, however, are not perpendicular, and that has led to this conjecture.

Conjecture 3 *Let $P \perp Q$, neither containing an autonomous subset. Then $P' \perp Q'$ where P', Q' have no common comparability and are obtained from P, Q, respectively, by subdivision.*

Finally, here is an instance of a theorem which was inspired largely by experiments with numerous such examples.

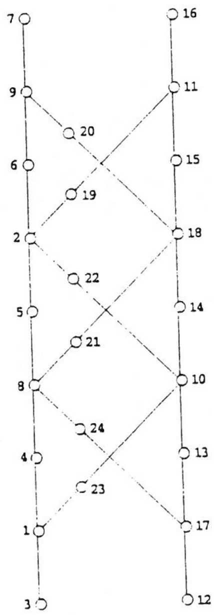

Figure 6: P is perpendicular to the multipartite order with minimals 1, 2, 10, 11, maximals 8, 9, 17, 18, and middle level the rest.

Theorem 1 (Zaguia (1994)) *Let P be a complete multipartite order and let C be a chain on the same underlying set. The $P \perp C$ if and only if P and C have neither a common convex subset containing at least one level of P nor a common autnomous subset.*

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