Relations among Lie Series Transformations and Isomorphisms between free Lie Algebras

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Abstract

We study the subgroup generated by the exponentials of formal Lie series. We show three different way to represent elements of this subgroup. These elements induce Lie series automorphisms. Relations among these family of transformations furnish algorithms of composition. Starting from the Lazard elimination theorem and the Witt's formula, we show isomorphisms between some submodules of free Lie algebras. Combining different results, we also show that the subalgebra generated by the homogeneous term of the Hausdorff series is free.

Résumé

Nous étudions le sous-groupe engendré par les exponentielles de séries formelles de Lie. Nous montrons trois manières de représenter les éléments de ce sous-groupe, lesquels fournissent des automorphismes de Lie. Les relations entre ces familles de transformations fournissent des algorithmes de composition. À partir du théorème d'élimination de M. Lazard et de la formule de Witt, nous mettons en évidence des isomorphismes entre sous-modules d'algèbres de Lie libres. En combinant ces résultats, on montre également que la sous-algèbre engendrée par les composantes homogènes de la série de Hausdorff est libre.

1. Introduction

Lie series automorphisms or Lie transformations play an important role in classical mechanics. They can be seen, for example, as time evolution in an hamiltonian system. The product of two such transformations may therefore be seen as the combined effects of two Hamiltonians.

The use of this formalism becomes efficient when it becomes easy to manipulate formal Lie series, to compute composition of Lie transformations or to express such transformations in several ways. They are universal identities in Lie algebras and we will work in a free Lie algebra. Instead of considering exponentials of Lie series, we will consider the group of Lie series automorphisms. Actually after having defined the Lie transformation, historically introduced by Deprit ([3]), the factored product transform introduced by Dragt and Finn ([4]) and the exponential of an inner derivation, we will show that these transformations are the same subgroup of the Lie series automorphisms close to identity. They can be seen as conjugation in the algebra of formal Lie series. All of them are defined by generating Lie series.

After having reminded some notations in free algebras in section **2.**, we will introduce formal Lie series on a weighted alphabet and define the Lie series transformations and their properties in section **3.**. In section **4.**, we will consider Lie series automorphisms they generate and their relations. In the last section, we will show several isomorphisms between free Lie algebras or subalgebras. We will prove, using combinatorial identities like the Witt's formula and a theorem of M. Lazard, that the subalgebra generated by the homogeneous terms of the Hausdorff series is a free Lie algebra.

2. Notations

In this paper X will denote a weighted alphabet, that is to say an ordered set (possibly endless), in which each letter x has a positive integer weight $\|x\|$.

R is a ring which contains the rational numbers $\mathbb{Q}$.

X^* is the free monoid generated by X. X^* is totally ordered with the lexicographic order.

$M(X)$ is the free magma generated by X.

$\mathcal{A}_R(X)$ is the associative algebra, that is to say the R-algebra of X^*.

$L_R(X)$ or $L(X)$ is the free Lie algebra on X. It is defined as the quotient of the R-algebra of $M(X)$ by the ideal generated by the elements (u,u) and $(u,(v,w)) + (v,(w,u)) + (w,(u,v))$. Its multiplication law $[,]$ is bilinear, alternate and satisfies the Jacobi identity

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0. \tag{1}$$

An element of $M(X)$ considered as element of $L(X)$ will be called a Lie monomial.

By posing for $x, y \in X$, $[x,y] = xy - yx$, we have $L(X) \subset \mathcal{A}(X)$.

On $L(X)$ so as on $\mathcal{A}(X)$, one considers the following gradations:

- Gradation by the length (the unique morphism that extends the function $x \mapsto 1$ on X). For $x \in X^*$ (resp. $M(X)$) $|x|$ denotes the length. $L_n(X)$ (resp. $\mathcal{A}_n(X)$) is the submodule generated by monomials of length n.
- One defines on X^* (resp. $M(X)$) the weight $x \mapsto \|x\|$ as the unique morphism that extends the weight on X. $\tilde{L}_n(X)$ (resp. $\tilde{\mathcal{A}}_n(X)$) is the submodule generated by monomials of weight n.
- The multi-degree is the unique morphism from X^* (resp. $M(X)$) onto $\mathbb{N}^{(X)}$ that extends $x \mapsto \mathbb{1}_x$. For a given α in $\mathbb{N}^{(X)}$, $L^\alpha(X)$ (resp. $\mathcal{A}^\alpha(\mathcal{X})$) denotes the submodule generated by monomials of degree α.

Remark. — When $\|x\| = 1$ for each $x \in X$, then obviously $L_n(X) = \tilde{L}_n(X)$ (resp. $\mathcal{A}_n(X) = \tilde{\mathcal{A}}_n(X)$).

For $x \in L(X)$, we denote by L_x the inner derivation $y \mapsto [x, y]$. The set of inner derivations of X is the adjoint Lie algebra with commutator as Lie bracket and we have from the Jacobi identity (1)

$$L_{[x,y]} = [L_x, L_y] = L_x L_y - L_y L_x. \tag{2}$$

For $x_n \in L_n(X)$, (see [2]) let $Dx_n = nx_n$. For $x_n \in \tilde{L}_n(X)$, let $\tilde{D}x_n = nx_n$ We thus define two derivations D and $\tilde{D}$ on $L(X)$. They are not inner derivations.

We define the formal Lie series $\tilde{L}(X)$ and $\tilde{\mathcal{A}}(X)$ as

$$\tilde{L}(X) = \prod_{n\geq 0} \tilde{L}_n(X) \quad \text{and} \quad \tilde{\mathcal{A}}(X) = \prod_{n\geq 0} \tilde{\mathcal{A}}_n(X). \tag{3}$$

We will write $x \in \tilde{L}(X)$ as a series $\sum_{n\geq 0} x_n$. $\tilde{L}(X)$ is a complete Lie algebra with the Lie bracket

$$([x, y])_n = \sum_{p+q=n} [x_p, y_q]. \tag{4}$$

3. Lie series automorphisms

3.1. The exponential

Denoting by $\tilde{L}(X)^+$ (resp. $\tilde{\mathcal{A}}(X)^+$) the ideal of $\tilde{L}(X)$ (resp. $\tilde{\mathcal{A}}(X)$) generated by the elements of non-negative weight, one defines the exponential and the logarithm as

$$\begin{aligned} \exp : \tilde{\mathcal{A}}(X)^+ &\to 1 + \tilde{\mathcal{A}}(X)^+ \\ x &\mapsto \sum_{n\geq 0} \frac{x^n}{n!}, \end{aligned} \tag{5}$$

$$\begin{aligned} \log : 1 + \tilde{\mathcal{A}}(X)^+ &\to \tilde{\mathcal{A}}(X)^+ \\ x &\mapsto -\sum_{n\geq 1} \frac{(1-x)^n}{n}. \end{aligned} \tag{6}$$

They are mutually reciprocal functions and we have (see [1, Ch. II, §5]) the

THEOREM 1 (Campbell-Hausdorff). — *For $x, y \in \tilde{L}(X)^+$,*

$$H(x,y) = \log\left[\exp(x)\exp(y)\right] \in \tilde{L}(X)^+. \tag{7}$$

More precisely, we have the following

LEMMA 2. — *Given $x, y \in \tilde{L}(X)^+$, we have for $m \geq 0$*

$$H(x,y)_{m+1} - x_{m+1} - y_{m+1} \in \tilde{L}_{m+1}\left(x_1, \ldots, x_m, y_1, \ldots, y_m\right). \tag{8}$$

Given $x \in \tilde{L}(X)^+$, we consider $\exp(L_x)$ defined as

$$\exp(L_x)y = \sum_{i \geq 0} \frac{L_x^i}{i!} y. \tag{9}$$

THEOREM 3. — *For $x \in \tilde{L}(X)^+$, $\exp(L_x)$ is a Lie series automorphism (see [2]). We also have ([1, 6]) for $y \in \tilde{L}(X)$*

$$\exp(x) y \exp(x) = \exp(L_x)y, \tag{10}$$

$$\exp(x)\exp(y)\exp(x) = \exp(\exp(L_x)y). \tag{11}$$

Proof. — From the Jacobi identity (1), we have ([11]) by induction on $k \geq 0$, for any $f, g, h \in \tilde{L}(X)^+$,

$$L_f^k[g,h] = \sum_{i=0}^{k} \binom{k}{i} \left[L_f^i g, L_f^{k-i} h\right].$$

We therefore deduce that

$$\begin{aligned}
\exp(L_f)[g,h] &= \sum_{n \geq 0} \frac{1}{n!} \sum_{p=0}^{n} \binom{n}{p} \left[L_f^p g, L_f^{n-p} h\right] \\
&= \sum_{p+q \geq 0} \frac{1}{(p+q)!} \frac{(p+q)!}{p!q!} \left[L_f^p g, L_f^q h\right] \\
&= [\exp(L_f)g, \exp(L_f)h]. \quad \square
\end{aligned} \tag{12}$$

From the Campbell-Hausdorff theorem (1), the set of all $\exp(L_x)$ is a group G that we will call the Lie transformations group.

3.2. Factored Product Transform

Using the preceding lemmas we deduce (see [11]) the

PROPOSITION 4 (Factored product expansion). — *For $k \in \tilde{L}(X)^+$, there is a unique $g \in \tilde{L}(X)^+$ such that*

$$\exp(\textstyle\sum_{n \geq 1} k_n) = \cdots \exp(g_n) \cdots \exp(g_1) = (Mg)^{-1}. \tag{13}$$

Proof. — The above proposition is proved by induction, constructing $g \in \tilde{L}(X)$ and $k^{(p)} \in \prod_{n > p} \tilde{L}_n(X)$ such that, for each $p \geq 1$,

$$\exp(k) = \exp(k^{(p)}) \exp(g_p) \cdots \exp(g_1). \quad \square \tag{14}$$

Remark. — This fact is also a variant of the Zassenhaus formula (see [9]).

3.3. The Lie transform

We also define the Lie transform $T : \tilde{A}(X)^+ \to 1 + \tilde{A}(X)^+$ by

$$(Tx)_0 = 1, \quad (Tx)_n = -\sum_{p=1}^{n} \tfrac{p}{n}(Tx)_{n-p}x_p, \; n \geq 1. \tag{15}$$

We therefore deduce that $\tilde{D}(Tx) = -Tx\tilde{D}x$.

Conversely, the series y in $1 + \tilde{A}(X)^+$ given by

$$y_0 = 1, \quad y_n = -\sum_{p=1}^{n} \tfrac{p}{n} y_{n-p}x_p, \; n \geq 1, \tag{16}$$

is the unique solution of $\tilde{D}y = -y\tilde{D}x$. From $Tx(Tx)^{-1} = \mathbb{1}$, we deduce that

$$\tilde{D}\left(Tx(Tx)^{-1}\right) = -Tx\tilde{D}x(Tx)^{-1} + Tx\tilde{D}(Tx)^{-1} = 0, \tag{17}$$

that is to say $\tilde{D}(Tx)^{-1} = \tilde{D}x(Tx)^{-1}$. We thus have

$$((Tx)^{-1})_0 = 1, \quad ((Tx)^{-1})_n = \sum_{p=1}^{n} \tfrac{p}{n} x_p((Tx)^{-1})_{n-p}, \; n \geq 1. \tag{18}$$

Remark. — If $\left[x, \tilde{D}x\right] = 0$, then $Tx = \exp(x)$. The Lie transform appears as a generalized exponential.

3.4. Relations between transformations

PROPOSITION 5. — *Let $g \in \tilde{L}(X)^+$, there is a unique series $w \in \tilde{L}(X)^+$ such that*

$$Tw = Mg = \exp(-g_1)\cdots\exp(-g_n)\cdots. \tag{19}$$

Proof. — Let $x = x_n \in \tilde{L}_n(X)$, we have

$$\tilde{D}\exp(x_n) = \sum_{p\geq 0} \frac{1}{p!}\tilde{D}x_n^p = \sum_{p\geq 0}\frac{1}{p!}pnx_n^p = nx_n\exp(x_n). \tag{20}$$

We have therefore

$$\begin{aligned}
\tilde{D}(Mg)^{-1} &= \sum_{n\geq 1}[\cdots\exp(g_{n+1})]\left[\tilde{D}\left[\exp(g_n)\right]\right][\exp(g_{n-1})\cdots\exp(g_1)] \\
&= \sum_{n\geq 1}[\cdots\exp(g_{n+1})][ng_n\exp(g_n)][\exp(g_{n-1})\cdots\exp(g_1)] \\
&= \sum_{n\geq 1}[\cdots\exp(g_{n+1})][ng_n][\exp(-g_{n+1})\cdots](Mg)^{-1} \\
&= \left[\sum_{n\geq 1} n\left[\cdots\exp(L_{g_{n+1}})g_n\right]\right](Mg)^{-1}.
\end{aligned}$$

Let $\tilde{D}w = \sum_{n\geq 1} n\left[\cdots \exp(L_{g_{n+1}})\right] g_n$, that is to say

$$w_n = \sum_{k=1}^{n} \frac{k}{n} \sum_{\substack{(k+1)m_{k+1}+\cdots \\ +(n-k)m_{n-k}=n-k}} \frac{L_{g_{n-k}}^{m_{n-k}} \cdots L_{g_{k+1}}^{m_{k+1}}}{m_{k+1}! \cdots m_{n-k}!} g_k, \tag{21}$$

we have

$$\tilde{D}(Mg)^{-1} = \tilde{D}w(Mg)^{-1}, \quad \tilde{D}(Tw)^{-1} = \tilde{D}w(Tw)^{-1}. \tag{22}$$

We thus deduce that $Mg = Tw$. □

Combining propositions 5 and 4, we deduce the

PROPOSITION 6. — *Given $w, k, g \in \tilde{L}(X)^+$, there exist*

— $k' \in \tilde{L}(X)^+$ *with* $k'_n - w_n \in L(w_1, \ldots w_{n-1})$ *such that* $\exp(k') = (Tw)^{-1}$,

— $g' \in \tilde{L}(X)^+$ *with* $g'_n - k_n \in L(k_1, \ldots k_{n-1})$ *such that* $(Mg')^{-1} = \exp(k)$,

— $w' \in \tilde{L}(X)^+$ *with* $w'_n - g_n \in L(g_1, \cdots g_{n-1})$ *such that* $Tw' = Mg$.

4. Lie Transformations

For any Lie series automorphism T, we have by definition $[Tf, Tg] = T[f, g]$. We call Lie transformation a Lie automorphism close to the identity, that is to say, which satisfies for each $x \in X$,

$$Tx - x \in \prod_{n>||x||} \tilde{L}_n(X, R). \tag{23}$$

The Lie series automorphisms act on the adjoint Lie algebra by

$$TL_fT^{-1} = L_{Tf}. \tag{24}$$

Using preceding lemmas and proposition (6), we deduce that for each $x \in \tilde{L}(X)^+$,

$$\begin{aligned} \exp(L_x): y &\mapsto \exp(x)y\exp(-x), && (25)\\ T_x: y &\mapsto (Tx)y(Tx)^{-1}, && (26)\\ M_x: y &\mapsto (Mx)y(Mx)^{-1} && (27) \end{aligned}$$

are conjugations in $\tilde{L}(X)$ and therefore Lie series automorphisms.

4.1. The Lie transform

Given $w \in \tilde{L}(X)^+$, T_w is defined by (26). For $y \in \tilde{L}(X)$, we have

$$\begin{aligned} \tilde{D}(T_w^{-1}y) = \tilde{D}\left[(Tw)^{-1}yTw\right] &= \tilde{D}w(Tw)^{-1}yTw + (Tw)^{-1}\tilde{D}yTw - (Tw)^{-1}yTw\tilde{D}w \\ &= \left[\tilde{D}w, (Tw)^{-1}yTw\right] + (Tw)^{-1}\tilde{D}yTw && (28)\\ &= L_{\tilde{D}w}T_w^{-1}y + T_w^{-1}\tilde{D}y. && (29) \end{aligned}$$

Let $G = T_w^{-1} g = \sum_{n,m \geq 0} G_{n,m}$ where

$$G_{n,m} = \left(T_w^{-1} g_m\right)_{n+m} = \sum_{p+q=n} ((Tw)^{-1})_p g_m (Tw)_q \in \tilde{L}_{n+m}(X). \tag{30}$$

Using (29), we get

$$\tilde{D} G_{n,m} = (n+m) G_{n,m} = \sum_{p=1}^{n} p L_{w_p} G_{n-p,m} + m G_{n,m}, \tag{31}$$

so

$$G_{n,m} = \sum_{p=1}^{n} \tfrac{p}{n} L_{w_p} G_{n-p,m}. \tag{32}$$

Using this algorithm, we show that $G_r = \sum_{n+m=r} G_{n,m}$ may be calculated in $\mathcal{O}(r^2)$ Lie brackets evaluations, by an iterative way.

4.2. Composition

Let $w_1, w_2 \in L(X)^+$ and $T = T_{w_1} T_{w_2}$. From (29) we deduce that

$$\begin{aligned} \tilde{D}(T^{-1} y) &= L_{\tilde{D} w_2} T^{-1} y + T_{w_2}^{-1} \tilde{D}(T_{w_1}^{-1} y) \\ &= (L_{\tilde{D} w_2} + T_{w_2}^{-1} L_{\tilde{D} w_2} T_{w_2}) T^{-1} y + T^{-1} \tilde{D} y \\ &= L_{\tilde{D} w_2 + T_{w_2}^{-1} \tilde{D} w_1} T^{-1} y + T^{-1} \tilde{D} y. \end{aligned} \tag{33}$$

We thus deduce that $T_{w_1} T_{w_2} = T_w$ where

$$\tilde{D} w = T_{w_2}^{-1} \tilde{D} w_1 + \tilde{D} w_2. \tag{34}$$

Composition of two Lie transformations appears clearly as a Lie transformation. Furthermore, the product may be expressed as Lie transformations by an iteration algorithm, in a polynomial time of Lie brackets evaluations. Using Lie operators or Lie series exponentials, we should have computed the so-called Hausdorff product of w_1 and w_2.

4.3. The Dragt-Finn transform

The Dragt-Finn transform is the infinite product of exponential maps (see [4]). Given $g = \sum_{n \geq 1} g_n$, we define M_g and M_g^{-1} as

$$M_g = \exp(-L_{g_1}) \cdots \exp(-L_{g_n}) \cdots, \quad M_g^{-1} = \cdots \exp(L_{g_n}) \cdots \exp(L_{g_1}). \tag{35}$$

4.4. Relations

The three above transformations are totally defined by generating series which satisfy the following

PROPOSITION 7. — *Given $w, k, g \in \tilde{L}(X)^+$, the series defined in proposition 6 satisfy*

$$\exp(L_{k'}) = T_w^{-1},\ T_{w'}^{-1} = M_g^{-1},\ M_{g'}^{-1} = \exp(L_k). \tag{36}$$

Remark. — We deduce in passing that the Lie transform is a Lie series automorphism close to identity and that any Lie transformation $T \in \mathrm{G}$ may be expressed as an exponential of a Lie operator or as an infinite product of single exponentials or as a proper Lie transform. The use of a representation depends deeply on the result we look for. For example, if we have to compose transformations, it is much easier to consider Lie transforms because their product is a Lie transform whose generating function appears easily from (34).

We will not explain in this paper how to compute explicitly the relations between these transformations, but that can be made, using the Lyndon basis and does not require to go through the associative algebra (see [6]).

Proposition 7 may be turned in

PROPOSITION 8. — *Given $w, g, k \in \tilde{L}(X)^+$, such that*

$$\exp(L_k) = T_w = M_g,$$

then for each $n \in \mathbb{N}$, we have

$$w_n - k_n \in \tilde{L}_{n-1}(X),\ w_n - g_n \in \tilde{L}_{n-1}(X),\ g_n - k_n \in \tilde{L}_{n-1}(X),$$

and

$$\tilde{L}_n(w_1, \ldots, w_n) = \tilde{L}_n(k_1, \ldots, k_n) = \tilde{L}_n(g_1, \ldots, g_n).$$

5. Free Lie algebras isomorphisms

Let us first remind the elimination theorem of M. Lazard [1].

THEOREM 9. — *Let $S \subset X$ and*

$$T = \{(s_1, \ldots, s_n, x),\ n \geq 0,\ s_1, \ldots, s_n \in S, x \in X - S\}.$$

- *$L(X)$ is the direct sum of $L(X - S)$ and of the ideal $\mathcal{S}$ generated by S.*
- *$L(T)$ and $\mathcal{S}$ are isomorphic through $(s_1, \ldots, s_n, x) \mapsto L_{s_1} \cdots L_{s_n} x$.*

By considering $X = \{a, b\}$ and $S = \{a\}$, we get the following isomorphism

$$L(\{a, b\}) = L(\{a\}) \oplus L(\{L_a^n b,\ n \geq 0\}) = K.a \oplus L(\{L_a^n b,\ n \geq 0\}). \tag{37}$$

By posing $X = \{L_a^k b,\ k \geq 0\}$ and $S = \{L_a^k b,\ k \geq p\}$, we deduce that

$$\begin{aligned} L(\{a,b\}) &= K.a \oplus L(\{L_a^k b,\ k \geq 0\}) \\ &= K.a \oplus L(\{L_a^k b, 0 \leq k \leq p-1\}) \oplus (L_a^k b,\ k \geq p). \end{aligned} \tag{38}$$

We therefore conclude that

$$\begin{aligned} L(\{a,b\})/(L_a^p b) &= L(\{a,b\})/(L_a^k b,\ k \geq p) \\ &= K.a \oplus L(\{L_a^k b, 0 \leq k \leq p-1\}). \end{aligned} \tag{39}$$

That proves that the algebra generated by $\{a, b, L_a^p b = 0\}$ is isomorphic to the weighted free Lie algebra $L(\{L_a^k b, 0 \leq k \leq p-1\})$ and the line generated by a.

We will show now that these isomorphisms are isomorphisms between homogeneous submodules.

5.1. Dimension of the homogeneous components

Let us first remind some well-known identities. Given an indexed alphabet X, we consider the dimension $l(\alpha)$ of $L^\alpha(X)$. Using the following identity between formal series (see [1]), which results from the Poincaré-Birkhoff-Witt's theorem:

$$1 - \sum_{x \in X} T_x = \prod_{\alpha \in \mathbb{N}^{(X)} - \{0\}} (1 - T^\alpha)^{l(\alpha)}, \tag{40}$$

we deduce that

$$l(\alpha) = \frac{1}{|\alpha|} \sum_{d|\alpha} \mu(d) \frac{\left(\left|\frac{\alpha}{d}\right|\right)!}{\left(\frac{\alpha}{d}!\right)} \quad \text{or} \quad \sum_{\beta|\alpha} |\beta|\, l(\beta) = \frac{|\alpha|!}{\alpha!}. \tag{41}$$

Let us take now the gradation by the length and calculate $l_n = \sum_{|\alpha|=n} l(\alpha)$, the dimension of $L_n(X)$. In (40), let us substitute the same unknown U to T_x, we get for a finite alphabet of cardinality q:

$$1 - qU = \prod_{\alpha \in \mathbb{N}^X - \{0\}} (1 - U^{|\alpha|})^{l(\alpha)} = \prod_{r>0} (1 - U^r)^{l_r}. \tag{42}$$

that is to say, the Witt's formula ([1]): $\sum_{d|n} d l_d = q^n$.

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Let $X = \biguplus_{p \geq 1} X_p$ be a weighted alphabet where each letter of X_p has a weight p. Let $\tilde{l}_n = \sum_{||\alpha||=n} l(\alpha)$ be the dimension of $\tilde{L}_n(X)$. In (40), substituting U^i to T_x for $x \in X_i$, we get

$$1 - \sum_{i \geq 1} q_i U^i = \prod_{r>0} \prod_{||\alpha||=r} (1 - U^{||\alpha||})^{l(\alpha)} = \prod_{r>0} (1 - U^r)^{\tilde{l}_r}. \tag{43}$$

In the particular case where $q_i = p$ for each $i \in \mathbb{N}$, we thus deduce

$$\prod_{r>0}(1-U^r)^{\tilde{l}_r} = 1 - p\sum_{i>0} U^i = \frac{1-(p+1)U}{1-U}. \tag{44}$$

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From identities (42) and (44), we then obtain:

ISOMORPHISM 1.
Let $X = \{x_1, \ldots, x_q\}$ *and* $Y = \biguplus_{i\geq 1} Y_i$, *where* $\mathrm{Card} Y_i = q-1$. *We have*

$$\dim L_1(X) = q,\ \dim \tilde{L}_1(Y) = q-1,\ \dim L_n(X) = \dim \tilde{L}_n(Y),\ n \geq 2, \tag{45}$$

that can be also expressed as

$$\sum_{d|n} d \dim L_d(X) = q^n,\ \sum_{d|n} d \dim \tilde{L}_d(Y) = q^n - 1. \tag{46}$$

In the particular case where $q = 2$, we recover the isomorphism 9, by posing $Y = \{L_a^p b, p \geq 0\}$.

5.2. The Hausdorff series

Let us suppose now that $X = \{a, b\}$ and $\|a\| = \|b\| = 1$. Let $H(a,b)$ the Hausdorff series of a, b defined in theorem 1, page 3. We have

$$\exp(H(a,b)) = \exp(a)\exp(b).$$

From the definition (15)

$$((Ta)^{-1})_n = \sum_{p=1}^{n} \tfrac{p}{n} a_p((Ta)^{-1})_{n-p} = \tfrac{1}{n}((Ta)^{-1})_{n-1}$$

we deduce that $(Ta)^{-1} = \exp(a)$ and that $(Tb)^{-1} = \exp(b)$.

Let $G(a,b)$ be the solution of $(TG(a,b))^{-1} = \exp(a)\exp(b)$, we get using relation (34)

$$\tilde{D}G(a,b) = DG(a,b) = \exp(L_a)Db + Da,$$

that is to say

$$G(a,b) = a + b + \sum_{n\geq 1} \frac{1}{(n+1)!} L_a^n b.$$

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We can now prove the following result:

ISOMORPHISM 2. — *Let* $X = \{a, b\}$ *and* $H(a,b) = \sum_{n\geq 1} H_n$. *The subalgebra* $L(\{H_n,\ n \geq 0\})$ *is isomorphic to the free Lie algebra* $L(\{L_a^n b,\ n \geq 0\})$.

Proof. — Using proposition 8, we know that for $d \geq 1$

$$\tilde{L}_d(\{G_n(a,b),\ n \geq 1\}) = \tilde{L}_d(\{H_n(a,b),\ n \geq 1\}). \tag{47}$$

But $G_n(a,b) = \frac{1}{n!}L_a^{n-1}b$, and from isomorphism 1, we know that the subalgebra $\tilde{L}(\{L_a^n b,\ n \geq 0\})$ is free and that

$$\tilde{L}_d(\{L_a^n b,\ n \geq 0\}) = L_d(\{a,b\}),\ d \geq 2. \tag{48}$$

We thus deduce that the subalgebra generated by the homogeneous terms of the Hausdorff series is free and therefore isomorphic to the free Lie algebra $L(\{a,b\})$ without a line.

6. Conclusion

We have shown in this in paper how to express any transformation that belongs to the subgroup of Lie transformations in three different ways,. In hamiltonian mechanics this subgroup is exactly the group of Lie series automorphisms close to identity. These methods have many applications like the search of the so-called symplectic integrators that are numerical methods to integrate dynamical systems ([7]). Using this formalism, one can also compute formal first integral for perturbed hamiltonian systems ([3, 6, 11]). Regards to the computational cost, these methods have the advantage that all the series we manipulate are formal Lie series. It avoids calculations in the associative algebra ([12]) and the use of the Poincaré-Birkhoff-Witt basis ([10]).

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