Counting Non-Isomorphic Planar Maps: a General Approach via Rooted Quotient Maps

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Summary

A general method for the counting of *unrooted* planar maps is proposed. It reduces the problem to the enumeration of rooted maps of several classes of three kinds: planar, projective and "circular". The method is based upon Burnside's Lemma, a permutation triplet model of the maps and a rigorous classification of the periodic sphere homeomorphisms, including *sense-reversing* ones, into five types.

Résumé

Nous proposons une méthode générale pour le calcul des cartes planaires *non pointées*. On réduit le problème à l'énumération des cartes pointées de trois classes : planaires, projectives et "circulaires". La méthode s'appuie sur le Lemme de Burnside, un modèle de triplet combinatoire pour les cartes et une classification des homéomorphismes périodiques de la sphère, y compris ceux qui renversent l'orientation, en cinq types.

1 Introduction

In [7] we developed a general technique for the exact counting of unrooted planar maps up to *sense-preserving* sphere homeomorphisms. It is based upon Burnside's Lemma and, due to some properties of maps (cf. [8]), reduces the problem to that for *rooted* planar maps of the same and several auxiliary classes which depend heavily on the class under enumeration. This method proved to be rather effective for a number of particular classes of maps.

The aim of the present work is to extend this reductive approach (in a slightly improved form) to orientation-reversing homeomorphisms. We achieve it, though with considerable complications: the reduction leads not only to rooted *planar* maps but also to two different generalized kinds of rooted maps: projective and circular with singularities. For the latter type an effective enumerative technique has *not* yet been developed and hardly can turn out as simple as for planar maps. Nevertheless we think that in many cases this approach reveals the nature and intrinsic difficulties of the problem.

Our basic idea is to classify the symmetries of planar maps rigorously and then to introduce the corresponding quotient maps.

The main result (Theorem) is a general uniform formula; when applied to a particular class of planar maps, it needs to be supplemented with *ad hoc* means for studying and enumerating the rooted quotient maps of the classes arisen.

2 Maps

By a *map* we mean a finite cell dissection of a closed topological surface, i.e. a dissection of it into *open* 0-, 1- and 2-dimensional cells called *vertices*, *edges* and *faces* respectively. Other useful definitions and details can be found in [2, 7, 13].

Any edge may be doubly *oriented* along and across it, i.e. we can select directions towards one of its ends and towards one of its sides. Each edge (including a loop and an isthmus) has four possible orientations. Such a doubly oriented edge will be called a *flag*. It is well known (cf. [1, 5, 6, 13]) that maps allow a rigorous combinatorial description as triples of permutations acting on the flags. The simplest map model has the form of a triple of fixed-point-free involutions which generate a transitive group (cf. [6, 8]). In particular the following assertion is valid.

Lemma 1. *For $n > 2$ up to trivial homeomorphisms preserving all cells, each map automorphism is defined uniquely by its action on the flags. This action is regular, i.e. it consists of cycles of an equal length. Any map automorphism is represented as a permutation that commutes with the three model involutions, and vice versa.*

Let $\mathcal{M}(n)$ be a set of n-edged maps. We assume the validity of the closure condition [7] which ensures that $\mathcal{M}(n)$ is invariant with respect to (the induced action of) the symmetric group $\Sigma(\mathbf{Y}) = \Sigma_{4n}$ where $\mathbf{Y}$ denotes the set of (labelled) flags, $|\mathbf{Y}| = 4n$.

Proposition 1 ([8]). *Let $M(n)$ be the number of non-isomorphic (unlabelled) maps in $\mathcal{M}(n)$, $M^{(L)}(n)$ be the number of non-isomorphic* **L***-rooted maps and $\varphi(L)$ denote the Euler totient function. Then*

$$M(n) = \frac{1}{4n} \sum_{L \geq 1, L|4n} \varphi(L) M^{(L)}(n), \qquad n \geq 2.$$

Here an unlabelled **L**-*rooted map* is a map which has a selected cyclically ordered L-element set of root-flags (a cycle of $\mathbf{L}^o$), is invariant with respect to the selected regular permutation $\mathbf{L}^o$, and is considered up to symmetries preserving the group generated by the selected cycle. This definition is a natural generalization of the well-known notion of a rooted (i.e. 1-rooted) map. We will write $M'(n)$ instead of $M^{(1)}(n)$ for the number of rooted maps.

3 Classification of planar map automorphisms

In the case of the sphere **S** we will also use geometric presentations of maps. By a result of P. Mani [10], any planar 3-connected (polyhedral) graph may be represented on the geometrical sphere in such a way that *all* its automorphisms are induced by symmetries of the sphere. For our purposes the following assertion is therefore important.

Lemma 2 ([3], Theorem 7.4.1). *Every non-trivial periodic symmetry of the sphere is a rotation, a reflection, or the product of a (commuting) pair* *which consists of a reflection and a rotation around the axis perpendicular to the reflection plane.*

We need a more thorough classification which we found only implicitly in the literature (cf. [4]). There exist five types (classes) of non-trivial periodic sphere symmetries which we denote by mnemonic Greek letters.

I – rotations. Every $\rho \in \mathrm{I}$ is defined by (i) its order l, $l \geq 2$, (ii) the pair of poles fixed by it and lying on the rotation axis, and (iii) the rotation angle $2\pi d/l$ where $1 < d < l$, $(d,l) = 1$ (here (d,l) denotes the g.c.d. of numbers d and l).

O – reflections. Every $\omega \in \mathrm{O}$ has order 2 and a great circle of fixed points lying in the reflection plane. The set of points fixed by it forms a great circle lying in the reflection plane.

X – the central (antipodal) inversion. This class contains a unique symmetry χ. It is of order 2 and has no fixed points.

Θ – reflections combined with the corresponding rotations of *even* orders greater than 2. Every $\theta \in \Theta$ is defined by (i) its order l, $l > 4$, l is even, (ii) the pair of poles it interchanges, and (iii) the rotation angle $2\pi d/l$ where $1 < d < l$, $(d,l) = 1$. All points, except for the poles, have order l.

Φ – reflections combined with the corresponding rotations of *odd* orders greater than 1. Every $\phi \in \Phi$ is defined by (i) its order $2l$, $l > 3$, l is odd, (ii) the pair of poles it interchanges, and (iii) the rotation angle $2\pi d/l$ where $1 < d < l$, $(d,l) = 1$. Equatorial points have order l, while the other points, except for the poles, have order $2l$.

I-symmetries are orientation-preserving, the other types are orientation-reversing.

Let us write $\mathbf{K} = \{\mathrm{I}, \mathrm{O}, \mathrm{X}, \Theta, \Phi\}$. We will also refine these symmetry classes with various parameters. In particular, the rotation order will be written as a superscript in parentheses.

We apply the above classification to planar maps. The cells containing the poles are called *axial* and those intersecting the reflection plane are called *equatorial.*

Proposition 2. *Each non-trivial automorphism of a planar map belongs to a unique class of* $\mathbf{K}$. *It is defined by the pair of axial cells and the rotation angle in the cases of* I, Θ *and* Φ, *or by the set of equatorial cells in the case of* O.

4 Reduction to quotient maps

The central construction of the present work is the quotient map of a planar map by a symmetry, which turns out a certain generalized map.

A planar map is called *punctured* if two of its cells, other than edges, are distinguished as axial. A projective map (that is, a map on the projective plane) is called *punctured* if one of its vertices or faces is distinguished as axial. Any pendant (i.e. 1-valent) axial vertex of a punctured map may be declared to be *singular*. An edge with a singular end is also called *singular* (or a half-edge). It is considered to have only two flags oriented towards the non-singular end.

The equator of a map symmetry may be considered as a polygon with two types of points (polygon vertices) and two types of segments (polygon sides). Its points of intersections with edges of the map are called *singular*. Segments of intersections with faces are called *quasi-edges*, whereas the other map edges are called *ordinary*. Moreover, such polygons satisfy the following condition:

(C1) both segments incident with a singular point are quasi-edges (they may coincide in the case of a 1-gon, i.e. a loop).

Such a polygon will be called *generalized*.

Definition 1 (cf. [1, 12]). A *circular* map means a finite cell dissection of a closed disc that induces a generalized polygon on the boundary and possesses the following properties:

(C2) each face is incident with at most one boundary quasi-edge;

(C3) each boundary vertex is incident with at least one ordinary edge;

(C4) each singular boundary vertex is incident with exactly one internal (i.e. not boundary) edge;

(C5) each ordinary edge is incident with at most one singular vertex.

In a circular map we consider a quasi-edge to belong to the face incident with it. Such faces will be called *boundary*, whereas the other faces are called *internal*. Thus, a quasi-edge has no flags (i.e. no 'usual' sides and ends). On the contrary, an ordinary boundary edge has only one side and two flags. An ordinary internal edge incident with a singular vertex is also called *singular*. It has one end and two flags.

Internal non-singular edges are called *normal*. An internal edge may connect two boundary vertices.

A circular map is called *punctured* if one of its internal vertices or faces is distinguished as *axial.*

Planar, projective and circular maps, punctured or not, are all called *generalized* maps. The number of flags in a generalized map is always even. The number of edges in such a map is an integer or a *half-integer*. It is equal to the number of normal edges plus half of the number of singular internal edges and boundary edges; quasi-edges are not taken into account at all.

The definition and lemmas given below are merely straightforward consequences of well-known facts from the theory of Riemann surfaces and covering spaces (cf. [11, 5]).

Definition 2. A *quotient* map (an Ω-quotient map) $B = A/\alpha$ of a planar map A with respect to an automorphism α, $\alpha \in \Omega$, $\Omega \in \mathbf{K}$, means the quotient space (or orbit space) $\mathbf{S}/<\alpha>$ together with its induced cell dissection which forms a generalized map.

Lemma 3. *The quotient map of a planar map with respect to a non-trivial automorphism is of one of the following forms:*

I-*quotient maps are punctured planar maps, which may contain singular axial vertices in the case of* $\mathrm{I}^{(2)}$*;*

O-*quotient maps are circular maps;*

X-*quotient maps are projective maps;*

Θ-*quotient maps are punctured projective maps, which may contain a singular axial vertex in the case of* $\Theta^{(4)}$*;*

Φ-*quotient maps are punctured circular maps.*

It is convenient to build quotient maps B geometrically with the help of Mani's Theorem. In particular the I-quotient map is obtained by selecting a sphere sector with angles $2\pi/l$ at the poles and then by identifying its boundary half-circles [7]. The O-quotient map is the half of the original map that lies above the equator and is endowed with the appropriate singular vertices and quasi-edges on the boundary. The standard 2-fold covering of the projective plane by the sphere induces the X-quotient map.

Lemma 4 (on lifting). *B is the $\Omega^{(l)}$-quotient map of a uniquely defined planar map A with respect to a given automorphism α of type $\Omega^{(l)}$ with the corresponding rotation axis and/or reflection plane.*

By Propositions 1 and 2,

$$M^{(L)}(n) = \sum_{\Omega \in \mathbf{K}} M^{(\Omega,L)}(n), \qquad n \geq 2,\ L \geq 2,$$

where $M^{(\Omega,L)}(n)$ is the number of L-rooted maps in $\mathcal{M}(n)$ for which the corresponding L-automorphism $\mathbf{L}^o$ is of type Ω.

Let $\mathcal{M}_{\Omega,L}(t)$ be the set of t-edged generalized maps that are quotient maps of the maps in $\mathcal{M}(n)$ with respect to Ω-automorphisms of order L. By Proposition 2 and Lemma 4

$$M^{(\Omega,L)}(n) = M'_{\Omega,L}(n/L), \qquad L|2n,\ \Omega \in \mathbf{K},$$

where $M'_{\Omega,L}(t)$ is the number of rooted maps in $\mathcal{M}_{\Omega,L}(t)$.

Theorem. *The number $M(n)$ of non-isomorphic planar maps in $\mathcal{M}(n)$ is given by the formula*

$$\begin{aligned} M(n) = \frac{1}{4n}\Bigg[& M'(n) + M'_{\mathrm{I},2}(n/2) + \sum_{l\geq 3,\, l|n} \varphi(l) M'_{\mathrm{I},l}(n/l) + M'_{\mathrm{O}}(n/2) \\ & + M'_{\mathrm{X}}(n/2) + 2M'_{\Theta,4}(n/4) + \sum_{l\geq 6,\, l|n,\, 2|l} \varphi(l) M'_{\Theta,l}(n/l) \\ & + \sum_{l\geq 3,\, l|n,\, l \text{ odd}} \varphi(l) M'_{\Phi,2l}(n/2l)\Bigg]. \end{aligned}$$

Moreover, here

$$\begin{aligned} M'_{\mathrm{I},2}(n/2) &= M'_{\mathrm{I},2,1}(n/2), && n \text{ odd}, \\ M'_{\mathrm{I},2}(n/2) &= M'_{\mathrm{I},2,0}(n/2) + M'_{\mathrm{I},2,2}(n/2), && n \text{ even}, \\ M'_{\mathrm{X}}(n/2) &= M'_{\Theta,4}(n/4) = 0, && n \text{ odd}, \\ M'_{\Theta,4}(n/4) &= M'_{\Theta,4,2}(n/4), && n \equiv 2 \pmod 4, \\ M'_{\Theta,4}(n/4) &= M'_{\Theta,4,0}(n/4), && 4|n, \end{aligned}$$

where the third subscript means the number of singular axial vertices.

This theorem reduces the enumeration of non-isomorphic planar maps to that of rooted quotient maps of six classes. In the case of all planar maps without restrictions (Step 7 of our program [9]) it is possible to advance the result considerably. This will be described elsewhere.

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